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$F-$Modular $b-$Metric Spaces and Some Analogies of Classical Fixed Point Theorems

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Abstract. The main aim of this study is to provide a novel concept of $F-$modular $b-$metric spaces. Within this comprehensive framework, we establish three well-known fixed point theorems for self-maps. The results we have obtained broaden and enrich prior findings in the field of fixed point theory. To support our arguments, we provide four concrete examples along with graphical representations.

1. Introduction

Over the last few years, the fixed-point theory has evolved as one of the most interesting topics for researchers. It has a broad set of applications and is a very popular and effective tool used in solving problems in mathematical analysis.

M. Frechet [1] developed the well-known idea of metric space as an extension to conventional distance. In the theory of metric space, particularly in non-linear analysis, number of authors studied non-contraction mappings. It is proficient that physical problems generally involve nonlinear differential and integral equations.

Banach contraction principle [2] plays a vital role to deal with such kind of physical problems and provide a powerful tool for obtaining the solutions of these equations. It is a most essential result in the metric fixed point theory. Since 1922, this theory has been improved and extended in several ways and has been used widely.

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In 1968, Kannan [3] proved an extension of Banach [2] without assuming the continuity condition of the map. Since then, there exist several extensions and generalization of contraction principle, some of them are refer to [4–8].

In literature, in addition to the contraction mappings, the idea of metric spaces is further explored, expanded and diversified through numerous diverse approaches. One of the well known generalizations of metric spaces are $b-$metric spaces. The idea of $b-$ metric was initiated from the works of Bourbaki [9] and Bakhtin [10]. Later, Czerwik [11] introduced and formally defined the notion of $b-$metric space.

Another new method in the area of metric extension pertains to the application of the geometric characteristics shown by three points. The concept of $2-$ metric (Gähler) [12], D - metric (Dhage) [13] and G - metric (Mustafa and Sims) [14], are the most famous examples of this trend. Branciari [15] proposed a novel extension of the metric concept by substituting the triangle inequality with a more comprehensive inequality that includes four points.

In contrast, Nakano [16] established the idea of modularity in relation to the theory of order spaces. The concept of modular metric space on a general set was first developed by Chistyakov [17, 18]. Abdou and Khamsi introduced a novel approach in this particular field and established a fixed point theorem in such spaces. Jleli and Samet [19] proposed an innovative idea of a metric space by introducing an altered metric distance that spans many types of metric spaces, including metric spaces, $b-$ metric spaces, dislocated metric spaces, and modular vector spaces. In 2018, Jleli and Samet [20] further investigated $F-$ metric space as an extension of metric space. They also defined topology on these spaces and explored their features. During the same time period, Ege and Alaca [21] introduced an expansion of the modular metric space, which they called as the modular $b-$metric space. Recently, the notion of weighted $b-$metric space and its topological features were introduced by Nurwahyu and Firman [22]. This concept serves as an extension of both $b-$metric and F-metric space. For more elaboration on the aforementioned notions and other extensions of the metric concepts, we refer to [23–30, 33, 34].

Prior to presenting the primary findings of this work, it is necessary to review several fundamental definitions, instances, and essential lemmas that will greatly assist in establishing our main theorem.

2. Foundational Concepts and Relevant Literature

Authors in [10, 11] defined $b-$ metric space as follows:

Definition 2.1. *Let Δ be a space, and let R^+ denotes the set of all non-negative numbers. A function $\rho : \Delta \times \Delta \to R^+$ is said to be a $b-$ metric on Δ if for all u, v, q in Δ and $b \geq 1$, following conditions are satisfied:*

(1) $\rho(u, v) = 0$ *if and only if* $u = v$
(2) $\rho(u, v) = \rho(v, u)$
(3) $\rho(u, q) \leq b[\rho(u, v) + \rho(v, q)]$

The pair (Δ, ρ) *is called a* $b-$ *metric space.*

Example 2.1. *Let* (Δ, ρ) *be a metric space and* $\rho(u, v) = (d(u, v))^{\kappa}$, *where* $\kappa > 1$ *is a real number and* d *is a usual metric. Then* ρ *is a* $b-$*metric with* $b = 2^{\kappa-1}$.

Definition 2.2. *[31] Let* Δ *be a real linear space.* Δ *is said to be a modular if a functional* $\kappa : \Delta \longrightarrow [0, +\infty]$ *satisfies the following conditions:*

(1) $\kappa(0) = 0$;
(2) *If* $m \in \Delta$ *and* $\kappa(um) = 0$ *for all numbers* $u > 0$, *then* $m = 0$;
(3) $\kappa(-m) = \kappa(m)$ *for all* $m \in \Delta$;
(4) $\kappa(um + vn) \leq \kappa(m) + \kappa(n)$ *for all* $u, v \geq 0$ *with* $u + v = 1$ *and* $m, n \in \Delta$.

Consider a set $\Delta \neq \emptyset$ and $\mu \in (0, +\infty)$. In the rest of the paper, for all $\mu > 0$ and $u, v \in \Delta, \beta_{\mu}(u, v) = \beta(\mu, u, v)$ denotes the map $\beta : (0, +\infty) \times \Delta \times \Delta \longrightarrow [0, +\infty]$.

Definition 2.3. *[17] For any set* $\Delta \neq \emptyset$, *assume that the map* $\beta : (0, +\infty) \times \Delta \times \Delta \longrightarrow [0, +\infty]$ *satisfies the following conditions for all* $u, v, q \in \Delta$:

(1) $\beta_{\mu}(u, v) = 0$ *for all* $\mu > 0 \Leftrightarrow u = v$;
(2) $\beta_{\mu}(u, v) = \beta_{\mu}(v, u)$ *for all* $\mu > 0$;
(3) $\beta_{\mu_1+\mu_2}(u, v) \leq \beta_{\mu_1}(u, q) + \beta_{\mu_2}(q, v)$ *for all* $\mu_1, \mu_2 > 0$.

Then we say that β *is a metric modular on* Δ.

Following are few examples of metric modular on set $\Delta = \mathbb{R}$

Example 2.2. *Let* $\Delta = \mathbb{R}$, *and let* $\beta : (0, \infty) \times \Delta \times \Delta \to \mathbb{R}$ *be a function defined as*

(1) $\beta_{\mu}(u, v) = \rho(u, v)$.
(2) $\beta_{\mu}(u, v) = \rho(u, v)/\zeta(\mu)$.

where $\rho(u, v)$ *is an ordinary metric on* Δ *and* $\zeta : (0, \infty) \to (0, \infty)$ *is a non decreasing continuous function. Then* β *is metric modular on* Δ *and the pair* (Δ, β) *is a modular metric space for all* $\mu > 0$.

Definition 2.4. *[21] Let* Δ *be a non-empty set and let* $b \geq 1$ *be a real number. A map* $\beta : (0, +\infty) \times \Delta \times \Delta \longrightarrow [0, +\infty]$ *is called a modular* $b-$*metric, if the following statements hold for all* $u, v, q \in \Delta$,

(A) $\beta_{\mu}(u, v) = 0$ *for all* $\mu > 0 \Leftrightarrow u = v$;
(B) $\beta_{\mu}(u, v) = \beta_{\mu}(v, u)$ *for all* $\mu > 0$;
(C) $\beta_{\mu_1+\mu_2}(u, v) \leq b\left[\beta_{\mu_1}(u, q) + \beta_{\mu_2}(q, v)\right]$ *for all* $\mu_1, \mu_2 > 0$.

Then we say that (Δ, β) *is a modular* $b-$*metric space.*

Example 2.3. *[21] Consider the space*

$$l_p = \left\{(u_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |u_n|^p < \infty\right\}, 0 < p < 1,$$

Define for all $\mu \in (0,+\infty)$,

$$\beta_\mu(u,v) = \frac{\rho(u,v)}{\mu},$$

where

$$\rho(u,v) = \left(\sum_{n=1}^{\infty} |u_n - v_n|^p\right)^{\frac{1}{p}} \quad (u = u_n, v = v_n \in l_p).$$

Then (Δ,β) is a modular $b-$metric space.

Definition 2.5. *[19] Consider $\Pi = \{\pi | \pi : (0,+\infty) \to \mathbb{R}\}$ such that for all $s,t \in (0,+\infty)$*

F 1-: $0 < s < t$ implies $\pi(s) \leq \pi(t)$ (π is non decreasing function),

F 2-: $\forall$ sequence $\{t_n\} \subset (0,+\infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} \pi(t_n) = -\infty.$$

Then function π is said to be logarithmic like function.

Some examples of logarithmic like functions are $\pi(u) = -\frac{1}{u} + u$, $\pi(u) = \log u$, $\pi(u) = -e^{-u} + u$.

Definition 2.6. *[20] Let Δ be a nonempty set, and let $P : \Delta \times \Delta \to [0,+\infty)$ be a given mapping. Suppose that there exists $(\pi,\lambda) \in \Pi \times [0,+\infty)$ such that for all $(u,v) \in \Delta \times \Delta$,*

(1) $P(u,v) = 0 \Longleftrightarrow u = v$.
(2) $P(u,v) = P(v,u)$.
(3) for every $N \in \mathbb{N}, N \geq 2$, and for every $(u_i)_{i=1}^{N} \subset \Delta$ with $(u_1,u_N) = (u,v)$, we have

$$P(u,v) > 0 \Longrightarrow \pi(P(u,v)) \leq \pi\left(\sum_{i=1}^{N-1} P\left(u_i,u_{i+1}\right)\right) + \lambda.$$

Then P is said to be an $F-$metric on Δ, and the pair (Δ,P) is said to be an $F-$metric space.

Following is an example of $F-$metric space.

Example 2.4. *[20] Let $\Delta = \mathbb{N}$, and let $P : \Delta \times \Delta \to [0,+\infty)$ be the mapping such that for all $(u,v) \in \Delta \times \Delta$, we have*

$$P(u,v) = \begin{cases} (u-v)^2, & \text{if} \quad (u,v) \in [0,3] \times [0,3], \\ |u-v|, & \text{if} \quad (u,v) \notin [0,3] \times [0,3], \end{cases}$$

Then P is an $F-$metric on Δ with $\pi(t) = \ln t, t > 0$, and $\lambda = \ln 3$, but P is not a metric on Δ (doesn't satisfy the triangle inequality).

Definition 2.7. *[32] Let Δ be a nonempty set and $P_\mu : (0,+\infty) \times \Delta^2 \to [0,+\infty)$ be a function. If there exists $(\pi,\lambda) \in \Pi \times [0,+\infty)$ such that $\forall\, u,v \in \Delta$, it satisfies*

(1) $P_\mu(u,v) = 0$ if and only if $u = v$
(2) $P_\mu(u,v) = P_\mu(v,u)$

(3) for all $n \in N$, with $n \geqslant 2$ and $\{v_1, v_2, v_3, \dots v_n\} \subset \Delta$ with $(v_1, v_n) = (u, v)$, we have

$$P_\mu(u,v) > 0 \text{ implies } \pi\left(P_\mu(u,v)\right) \leqslant \pi\left(\sum_{j=1}^{n-1} P_{\frac{\mu}{j}}\left(v_j, v_{j+1}\right)\right) + \lambda.$$

Then P_μ is called an modular F- metric on Δ. The pair $\left(\Delta, P_\mu\right)$ is called modular F-metric space.

Definition 2.8. *[22] Let Δ be a non-empty set and let $b \geq 1$(be a real number). Let $(\pi, \lambda) \in \Pi \times [0, +\infty)$. A mapping $\rho : \Delta \times \Delta \to [0, +\infty)$ is called a function weighted $b-$ metric ($P - b$ -metric), if for any $(u, v) \in (\Delta, \rho)$ satisfies the following conditions:*

(1) $\rho(u,v) = 0$, if and only if $u = v$,
(2) $\rho(u,v) = \rho(v,u)$,
(3) $\rho(u,v) > 0$ then

$$\pi(\rho(u,v)) \leq \pi\left(\sum_{j=1}^{N-1} b^j \rho\left(a_j, a_{j+1}\right)\right) + \lambda,$$

for every $\{a_1 = u, a_2, a_3, \dots, a_N = v\} \subset \Delta$ and $N \in \mathbb{N}, N \geq 2$.

The pair (Δ, ρ) is called a function weighted $b-$metric space ($F - b-$metric space).

In this article, we introduce a new kind of metric space called the $F-$modular $b-$metric space, which builds upon the concepts of modular metric spaces, $b-$metric spaces, and the F-metric space. Section 3 presents the definition of a $F-$modular $b-$metric space, along with its associated topological features. In section 4, we have derived three basic theorems of fixed point theory that serve as analogies to well-known classical fixed point theorems. Section 5 presents many instances of the aforementioned theorems in $F-$modular $b-$metric spaces, which are well shown by graphical representation.

3. $F-$Modular $b-$Metric

We will begin this section with the definition of $F-$modular $b-$metric spaces that underpin our main outcomes.

Definition 3.1. *Let Δ be a nonempty set and let $P_\mu : (0, +\infty) \times \Delta^2 \to [0, +\infty)$ be a continuous mapping. If there exists $(\pi, \lambda) \in \Pi \times [0, +\infty)$ such that $\forall\, u, v \in \Delta$, it satisfies*

P 1-: $P_\mu(u,v) = 0 \Leftrightarrow u = v$,
P 2-: $P_\mu(u,v) = P_\mu(v,u)$
P 3-: *for all $n \in \mathbb{N}, n \geqslant 2$ $\{v_i\}_{i=1}^n \subset \Delta$ such that $v_l = u$ & $v_n = v$*

$$P_\mu(u,v) > 0 \Rightarrow \pi\left(P_\mu(u,v)\right) \leqslant \pi\left(\sum_{j=1}^{n-1} b^j P_{\mu_j}\left(v_j, v_{j+1}\right)\right) + \lambda,$$

where $\mu = \sum_{j=1}^{n-1} \mu_j$ and $b \geqslant 1$. Then P_μ is said to be $F-$modular $b-$metric and the pair (Δ, P_μ) is said to be $F-$modular $b-$metric space.

Example 3.1. *Let $\Delta = \mathbb{R}$ be a set of all real numbers. Define a function*

$$P_\mu(u,v) = \frac{|u-v|^\kappa}{\mu}$$

with $\kappa \geq 2$ and $u, v \in \Delta$.
Since $|u-v|^\kappa > 0$ with $\kappa \geq 2$,and given that $\mu > 0$ implies that $P_\mu(u,v) > 0$ and $P_\mu(u,v) = 0$ if and only if $u = v$. Also, $|u-v|^\kappa = |v-u|^\kappa$ for all $\kappa \geq 2$. Therefore, $P_\mu(u,v) = P_\mu(v,u)$. Hence property (P1) and (P2) of Definition 3.1 holds good.
Next we prove that property (P3) holds good.
Let $n \in \mathbb{N}$, $n \geq 2$, and consider a set $\{a_1 = u, a_2, a_3, \ldots, a_n = v\} \subset \Delta$. Then from definition and by using Jenson inequality, we have

$$\begin{aligned}
P_\mu(u,v) &= \frac{|u-v|^\kappa}{\mu} \\
&= \frac{|a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \ldots + a_{n-1} - a_n|^\kappa}{\mu} \\
&\leq \frac{b|a_1 - a_2|^\kappa + (b)^2|a_2 - a_3|^\kappa + \ldots + (b)^{n-1}|a_{n-1} - a_n|^\kappa}{\mu} \\
&\leq b\frac{|a_1 - a_2|^\kappa}{\mu_1} + b^2\frac{|a_2 - a_3|^\kappa}{\mu_2} + b^3\frac{|a_3 - a_4|^\kappa}{\mu_3} + \cdots + b^{n-1}\frac{|a_{n-1} - a_n|^\kappa}{\mu_{n-1}} \\
&\leq \left(\sum_{j=1}^{n-1} (b)^j \frac{|a_j - a_{j+1}|^\kappa}{\mu_j}\right),
\end{aligned}$$

where $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_{n-1} = \frac{\mu}{n-1}$. Thus, we obtain.

$$P_\mu(u,v) \leq \sum_{j=1}^{n-1} (b)^j P_{\mu j}(a_j, a_{j+1}).$$

Define a function $\pi(t) = \log t$ and $\lambda = 0$. Then $(\pi, \lambda) \in \Pi \times [0, +\infty)$ for any $t \in (0, +\infty)$.
Therefore, we have

$$\pi(P_\mu(u,v)) \leq 0 + \pi(\sum_{j=1}^{n-1} (b)^j P_{\mu j}(a_j, a_{j+1}).$$

Thus all the properties are satisfied. Therefore, P_μ is a F−modular b−metric space with $b = 2^{\kappa-1}$.

Proposition 3.1. *The family of all F- modular b open subsets of Δ, denoted by τ_{P_μ} induces a topology on Δ, and known as P_μ topology.*

Proposition 3.2. *Let $\left(\Delta, P_\mu\right)$ is a F- modular b−metric space, then for any non-empty subset A of Δ, the following statements are equivalent:*

(1) *A is modular P_μ closed.*
(2) *for any sequence $\{v_n\}_{n \in \mathbb{N}} \subset A$ and for $u \in \Delta$, we have*

$$\lim_{n \to \infty} P_\mu\left(v_n, u\right) = 0,$$

then $u \in A$.

Definition 3.2. *Let $\left(\Delta, P_\mu\right)$ is a F- modular b−metric space and let $\zeta \neq A \subset \Delta$. We denote $\bar{A}$ the closure of set A w.r.t. topology τ_{P_μ}, and define as intersection of all F- modular b−metric closed subsets Δ containing A.*

Remark 3.1. *Clearly, A is the smallest F- modular b−metric closed subset contains itself.*

Proposition 3.3. *Let $\left(\Delta, P_\mu\right)$ be a F- modular b−metric space, and*

(1) *let $\{v_n\} \subset \Delta$ modular $F - b$ convergent to v.*
(2) $\mathrm{Lt}_{n\to\infty} P_\mu\left(v_n, v\right) = 0$

Definition 3.3. *Let $\left(\Delta, P_\mu\right)$ is a F- modular b−metric space and $\{v_n\} \subset \Delta$ be a sequence in Δ. Then*

(1) *$\{v_n\}$ is said to be F- modular b convergent to v, if*
$$\lim_{n\to\infty} P_\mu\left(v_n, v\right) = 0.$$
(2) *$\{v_n\}$ is said to be F- modular b Cauchy sequence, if*
$$\lim_{m,n\to\infty} P_\mu\left(v_n, v_m\right) = 0.$$
(3) *the space $\left(\Delta, P_\mu\right)$ is said to be F- modular b complete, if every F- modular b Cauchy sequence in Δ is F- modular b convergent to a certain element in Δ.*

Proposition 3.4. *Let $\left(\Delta, P_\mu\right)$ be a F−modular b−metric space. If $\{v_n\} \subset \Delta$ is a F−modular b convergent then it is F−modular b Cauchy sequence.*

Proof. Let $(\pi, \lambda) \in \Pi \times [0, +\infty)$ be such that (P3) is satisfied. Let $u \in \Delta$ be such that
$$\lim_{n\to+\infty} P_\mu(u_n, u) = 0.$$
Let $\epsilon > 0$ be fixed. By (F2), we know that there exists some $\delta > 0$ such that
$$0 < t < \delta \implies \pi(t) < \pi(\epsilon) - \lambda.$$
On the other hand, there exists some $N \in \mathbb{N}$ such that
$$P_{\frac{\mu}{2}}(u_n, u) + P_{\frac{\mu}{2}}(u_m, u) < \delta, \quad n, m \geq N.$$
Let $n, m \geq N$. We discuss two cases.
Case 1: If $u_m = u_n$. In this case, by (P1), we have
$$P_\mu(u_n, u_m) = 0 < \epsilon.$$
Case 2: If $u_m \neq u_n$. In this case, we have
$$0 < bP_{\frac{\mu}{2}}(u_n, u) + bP_{\frac{\mu}{2}}(u_m, u) < \delta.$$
Therefore, we have
$$\pi(P_{\frac{\mu}{2}}(u_n, u) + P_{\frac{\mu}{2}}(u_m, u)) < \pi(\epsilon) - \lambda.$$

Now, using (P3), we obtain

$$\pi(P_{\frac{\mu}{2}}(u_n, u_m)) \leq \pi(P_{\frac{\mu}{2}}(u_n, u) + P_{\frac{\mu}{2}}(u_m, u)) + \lambda < \pi(\epsilon),$$

which implies from (F1) that

$$P_{\frac{\mu}{2}}(u_n, u_m) < \epsilon.$$

As a consequence, we have

$$P_{\frac{\mu}{2}}(u_n, u_m) < \epsilon, \quad n, m \geq N,$$

which yields

$$\lim_{n,m\to+\infty} P_{\frac{\mu}{2}}(u_n, u_m) = 0,$$

i.e., $\{u_n\}$ is a F- modular b Cauchy sequence. □

4. Fixed Point Theorems in $F-$Modular b Metric Spaces

In this part, we will prove three renowned fixed point theorems in $F-$modular b metric spaces. To begin, let us provide a precise definition of a $F-$modular b contraction mapping. Subsequently, we shall formulate a corresponding version of the Banach fixed point theorem within the context of $F-$modular b metric spaces, without requiring the assumption of map continuity.

Definition 4.1. *Consider $R : \Delta \to \Delta$ be a map and and P_μ is a $F-$modular $b-$metric on Δ. If $\forall, u, v \in \Delta$, and for all $\mu > 0$, there exists $0 < k < 1$ such that*

$$P_\mu(Rv, Rw) \leq kP_\mu(v, w). \tag{4.1}$$

Then the map R is said to be $F-$modular b contraction.

Theorem 4.1. *Suppose Δ is any arbitrary set and P_μ is a $F-$modular $b-$metric defined on Δ such that $\left(\Delta, P_\mu\right)$ be a $F-$modular b complete metric space. If a map $R : \Delta \to \Delta$ be a $F-$modular b contraction with $bk < 1$, then R has a unique fixed point in Δ.*

Proof. Let $v_0 \in \Delta$ be any arbitrary element. Construct a sequence $\{v_n\} \subset \Delta$ such that for $n \in \mathbb{N} \cup \{0\}$

$$R^n(v_0) = R(v_n) = v_{n+1}. \tag{4.2}$$

Since R is a $F-$modular b contraction, therefore on substituting $v = v_n$ and $w = v_{n+1}$ in (4.1), we have

$$\begin{aligned} P_\mu(v_n, v_{n+1}) &= P_\mu(Rv_{n-1}, Rv_n) \\ &\leq kP_\mu(v_{n-1}, v_n) \\ &\leq k^2 P_\mu(v_{n-2}, v_{n-1}) \\ &\vdots \\ &\leq k^n P_\mu(v_0, v_1). \end{aligned} \tag{4.3}$$

Next we claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence.
Without loss of generality assume that, for all $n, m(\geq 2) \in N$ we have sequences v_n and v_m with $v_n \neq v_m$ for any $n \neq m$. Clearly, $P_\mu(v_n, v_m) > 0$ and so for all $m > n$, we have

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant bP_{\mu_1}(v_n, v_{n+1}) + b^2P_{\mu_2}(v_{n+1}, v_{n+2}) + \cdots + b^{m-n}P_{\mu_{m-n}}(v_{m-1}, v_m) \\ &\leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\left(v_j, v_{j+1}.\right) \end{aligned}$$

On using (4.3) and the fact $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \cdots = \mu_{m-n} = \frac{\mu}{m-n}$ in above inequality, we obtain

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant bk^n P_{\mu_1}(v_0, v_1) + b^2k^{n+1}P_{\mu_2}(v_0, v_1) + \cdots + b^{m-n}k^{m-1}P_{\mu(m-n)}(v_0, v_1) \\ &= \left(1 + kb + (kb)^2 + (kb)^3 + \cdots\cdots (bk)^{m-n-1}\right) bk^n P_{\frac{\mu}{m-n}}(v_0, v_1) \\ &\leq \left(\frac{bk^n}{1 - bk}\right) P_{\frac{\mu}{m-n}}(v_0, v_1), \end{aligned}$$

which yields

$$P_\mu(v_n, v_m) \leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\left(v_j, v_{j+1}\right) \leqslant \left(\frac{bk^n}{1 - bk}\right) P_{\frac{\mu}{m-n}}(v_0, v_1). \tag{4.4}$$

Since $\lim_{n\to\infty} \left(\frac{bk^n}{1-bk}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) = 0$, therefore we can find $\epsilon > 0$ such that

$$0 \leqslant \left(\frac{bk^n}{1 - bk}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) < \epsilon \tag{4.5}$$

Since $(\pi, \lambda) \in \Pi \times [0, +\infty)$, therefore π is a non-decreasing function. Hence for all $\epsilon > 0$ and a $q > 0$, we can find a $\kappa > 0$ such that $0 < \kappa < q$ implies $\pi(\kappa) < \pi(q) < \pi(\epsilon) - \lambda$. Therefore for all $m > n$ and from eq (4.4) and eq (4.5), we have

$$\pi\left(P_\mu(v_n, v_m)\right) \leqslant \pi\left(\sum_{j=n}^{m-1} P_\mu\left(v_j, v_{j+1}\right)\right) \leq \pi\left(\frac{bk^n}{1 - bk} P_{\frac{\mu}{m-n}}(v_0, v_i)\right) < \pi(\epsilon) - \lambda$$

and so

$$\pi\left(P_\mu(v_n, v_m)\right) \leqslant \pi\left(\sum_{j=n}^{m-1} P_\mu\left(v_j, v_{j+1}\right)\right) + \lambda < \pi(\epsilon).$$

Condition on π implies that $P_\mu(v_n, v_m) < \epsilon$.
This proves our claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence. As $\left(\Delta, P_\mu\right)$ is $F-$modular b complete, implies that $\exists\, \bar{v} \in \Delta$ such that the sequence $\{v_n\}$ is $F-$modular b convergent to $\bar{v}$ i.e.

$$\lim_{n\to\infty} P_\mu(v_n, \bar{v}) = 0.$$

Now, we claim $\bar{v}$ is fixed point of R. If $R\bar{v} = \bar{v}$ nothing to prove.
Assume that $R\bar{v} \neq \bar{v}$. Thus $P_\mu(R\bar{v}, \bar{v}) \neq 0$ and so $P_\mu(R\bar{v}, \bar{v}) > 0$.

Therefore, by (P3) of Definition 3.1, we get

$$\begin{aligned}\pi\left(P_\mu(R\bar{v},\bar{v})\right) &\leqslant \pi\left(bP_{\frac{\mu}{2}}\left(R\bar{v},Rv_n\right)+bP_{\frac{\mu}{2}}(Rv_n,\bar{v})\right)+\lambda\\ &\leq \pi\left(bkP_{\frac{\mu}{2}}\left(\bar{v},v_n\right)+bP_{\frac{\mu}{2}}\left(v_{n+1},\bar{v}\right)\right)+\lambda.\end{aligned}$$

For all $n\in\mathbb{N}$, and $n\geqslant 2$, on letting $n\to\infty$, we have

$$\pi\left(P_\mu(R\bar{v},\bar{v})\right)\leqslant \pi\left(bkP_{\frac{\mu}{2}}\left(\bar{v},v_n\right)+bP_{\frac{\mu}{2}}\left(v_{n+1},\bar{v}\right)\right)+\lambda=-\infty.$$

This is a contradiction and hence $R\bar{v}=\bar{v}$.

Uniqueness: Assume that $\exists$ two fixed points of R say u and v, with $u\neq v$ such that $Ru=u$ and $Rv=v$.

Since $u\neq v$ implies that $P_\mu(u,v)>0$, therefore from (4.1), we get

$$\begin{aligned}P_\mu(u,v)&=P_\mu(Ru,Rv)\\ &\leqslant kP_\mu(u,v)\\ &< P_\mu(u,v).\end{aligned}$$

This is contradiction and contradicts to our assumptions that $u\neq v$. This proves that fixed point is unique. Also completes the proof of the theorem. □

The following theorem presents a refined version of the Kannan contraction principle, specifically in the context of $F-$modular $b-$metric spaces.

Theorem 4.2. *Suppose Δ is any arbitrary set and P_μ is a $F-$modular $b-$metric defined on Δ such that $\left(\Delta,P_\mu\right)$ be a $F-$modular b complete metric space. If a continuous map $R:\Delta\to\Delta$ satisfying,*

$$P_\mu(Ru,Rv)\leqslant k\left[P_\mu(u,Ru)+P_\mu(v,Rv)\right]\quad u,v\in\Delta \tag{4.6}$$

where $0\leq k<\frac{1}{2}$ and $b\geqslant 1$ such that $0<bk<1$. Then R has a unique fixed point in Δ.

Proof. Let $v_0\in\Delta$ be any arbitrary element. Construct a sequence $\{v_n\}\subset\Delta$ such that for $n\in\mathbb{N}\cup\{0\}$

$$R^n(v_0)=R\left(v_n\right)=v_{n+1}. \tag{4.7}$$

On substituting $v=v_n$ and $w=v_{n+1}$ in (4.6), we have

$$\begin{aligned}P_\mu\left(v_n,v_{n+1}\right)&=P_\mu\left(Rv_{n-1},Rv_n\right)\\ &\leqslant k\left[P_\mu\left(v_{n-1},Rv_{n-1}\right)+P_\mu\left(v_n,Rv_n\right)\right]\\ &\leqslant k\left[P_\mu\left(v_{n-1},v_n\right)+P_\mu\left(v_n,v_{n+1}\right)\right]\\ \Rightarrow P_\mu\left(v_n,v_{n+1}\right)&\leqslant \frac{k}{1-k}P_\mu\left(v_{n-1},v_n\right).\end{aligned}$$

Since $k\in\left[0,\frac{1}{2}\right)$, therefore $\beta=\frac{k}{1-k}\in(0,1)$ and hence by induction

$$P_\mu\left(v_n,v_{n+1}\right)\leqslant \beta^n P_\mu\left(v_0,v_1\right). \tag{4.8}$$

Next we claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence.
Without loss of generality assume that, for all $n, m\ (\geq 2) \in N$ we have sequences v_n and v_m with $v_n \neq v_m$ for any $n \neq m$. Clearly, $P_\mu(v_n, v_m) > 0$ and so for all $m > n$, we have

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant bP_{\mu_1}(v_n, v_{n+1}) + b^2 P_{\mu_2}(v_{n+1}, v_{n+2}) + \cdots + b^{m-n} P_{\mu_{m-n}}(v_{m-1}, v_m) \\ &\leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\left(v_j, v_{j+1}\right). \end{aligned}$$

On using (4.8) and the fact $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \cdots = \mu_{m-n} = \frac{\mu}{m-n}$ in above inequality, we obtain

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant b\beta^n P_{\mu_1}(v_0, v_1) + b^2\beta^{n+1} P_{\mu_2}(v_0, v_1) + \cdots + b^{m-n}\beta^{m-1} P_{\mu(m-n)}(v_0, v_1) \\ &= \left(1 + \beta b + (\beta b)^2 + (\beta b)^3 + \cdots\cdots (b\beta)^{m-n-1}\right) b\beta^n P_{\frac{\mu}{m-n}}(v_0, v_1) \\ &\leq \left(\frac{b\beta^n}{1 - b\beta}\right) P_{\frac{\mu}{m-n}}(v_0, v_1), \end{aligned}$$

which yields

$$P_\mu(v_n, v_m) \leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\left(v_j, v_{j+1}\right) \leqslant \left(\frac{b\beta^n}{1 - b\beta}\right) P_{\frac{\mu}{m-n}}(v_0, v_1). \tag{4.9}$$

Since $\lim_{n\to\infty}\left(\frac{b\beta^n}{1-b\beta}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) = 0$, therefore we can find $\epsilon > 0$ such that

$$0 \leqslant \left(\frac{b\beta^n}{1 - b\beta}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) < \epsilon. \tag{4.10}$$

Since $(\pi, \lambda) \in \Pi \times [0, +\infty)$, therefore π is a non-decreasing function. Hence for all $\epsilon > 0$ and a $q > 0$, we can find a $\kappa > 0$ such that $0 < \kappa < q$ implies $\pi(\kappa) < \pi(q) < \pi(\epsilon) - \lambda$. Therefore for all $m > n$ and from eq (4.5), we have

$$\pi\left(P_\mu(v_n, v_m)\right) \leqslant \pi\left(\sum_{j=n}^{m-1} P_{\mu_j}\left(v_j, v_{j+1}\right)\right) \leq \pi\left(\frac{b\beta^n}{1 - b\beta} P_{\frac{\mu}{m-n}}(v_0, v_i)\right) < \pi(\epsilon) - \lambda$$

and so

$$\pi\left(P_\mu(v_n, v_m)\right) \leqslant \pi\left(\sum_{j=n}^{m-1} P_{\mu_j}\left(v_j, v_{j+1}\right)\right) + \lambda < \pi(\epsilon).$$

Condition on π implies that $P_\mu(v_n, v_m) < \epsilon$.
This proves our claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence. Completeness of space (Δ, P_μ) implies that $\exists\, \bar{v} \in \Delta$ such that the sequence $\{v_n\}$ is $F-$modular b convergent to $\bar{v}$ i.e.

$$\lim_{n\to\infty} P_\mu(v_n, \bar{v}) = 0.$$

Now, we claim $\bar{v}$ is fixed point of R. If $R\bar{v} = \bar{v}$ nothing to prove. Otherwise,
Assume that $R\bar{v} \neq \bar{v}$. Thus $P_\mu(R\bar{v}, \bar{v}) \neq 0$ and so $P_\mu(R\bar{v}, \bar{v}) > 0$.

On using (4.6), we get

$$\begin{aligned}P_\mu(\bar{v},R\bar{v}) &\leqslant bP_\mu\left(\bar{v},v_n\right)+bP_\mu\left(v_n,R\bar{v}\right)\\&=bP_\mu\left(\bar{v},v_n\right)+bP_\mu\left(Rv_{n-1},R\bar{v}\right)\\&\leq bP_\mu\left(\bar{v},v_n\right)+bkP_\mu\left(v_{n-1},Rv_{n-1}\right)+bkP_\mu(\bar{v},R\bar{v})\\&=\left(\frac{b}{1-bk}\right)P_\mu\left(\bar{v},v_n\right)+\left(\frac{bk}{1-bk}\right)P_\mu\left(v_{n-1},v_n\right)\\&\leq\left(\frac{b}{1-bk}\right)P_\mu\left(\bar{v},v_n\right)+\left(\frac{bk}{1-bk}\right)\beta^{n-1}P_\mu\left(v_0,v_1\right).\end{aligned}$$

Therefore, by (P3) of Definition 3.1, we have

$$\pi\left(P_\mu(\bar{v},R\bar{v})\right)\leq\pi\left(\left(\frac{b}{1-bk}\right)P_\mu\left(\bar{v},v_n\right)+\left(\frac{bk}{1-bk}\right)\beta^{n-1}P_\mu\left(v_0,v_1\right)\right).$$

On letting $n\to\infty$ and using that π is logarithmic like function

$$\lim_{n\to\infty}P_\mu(\bar{v},R\bar{v})+\lambda=-\infty,$$

which is a contradiction to our assumptions. Hence $\bar{v}$ is a fixed point of R.
Assume that $\exists$ two fixed points of R say u and v, with $u\neq v$ such that $Ru=u$ and $Rv=v$.
Since $u\neq v$ implies that $P_\mu(u,v)>0$, and thus

$$\begin{aligned}0<P_\mu(u,v)&=P_\mu(Ru,Rv)\\&\leq kP_\mu(u,Ru)+kP_\mu(v,Rv)\\&=kP_\mu(u,u)+kP_\mu(v,v)=0.\end{aligned}$$

This is possible only if $P_\mu(u,v)=0$. Hence $u=v$. This completes the proof of the theorem 4.2. □

The subsequent theorem we are going to explore a generalization of the Chatterjee contraction principle in $F-$modular $b-$metric spaces.

Theorem 4.3. *Suppose Δ is any arbitrary set and P_μ is a $F-$modular $b-$metric defined on Δ such that $\left(\Delta,P_\mu\right)$ be a $F-$modular b complete metric space. If a continuous map $R:\Delta\to\Delta$ satisfying,*

$$P_\mu(Ru,Rv)\leqslant k\left[P_\mu(u,Rv)+P_\mu(v,Ru)\right]\quad u,v\in\Delta, \tag{4.11}$$

where $0\leq k<\frac{1}{2}$ and $b\geqslant1$ such that $0<bk<1$. Then R has a unique fixed point in Δ.

Proof. Let $v_0\in\Delta$ be any arbitrary element. Construct a sequence $\{v_n\}\subset\Delta$ such that for $n\in\mathbb{N}\cup\{0\}$

$$R^n(v_0)=R\left(v_n\right)=v_{n+1}. \tag{4.12}$$

On substituting $v = v_n$ and $w = v_{n+1}$ in (4.6), we have

$$\begin{aligned} P_\mu(v_n, v_{n+1}) &= P_\mu(Rv_{n-1}, Rv_n) \\ &\leqslant k\Big[P_\mu(v_{n-1}, Rv_n) + P_\mu(v_n, Rv_{n-1})\Big] \\ &\leqslant k\Big[P_\mu(v_{n-1}, v_{n+1}) + P_\mu(v_n, v_n)\Big] \\ &\leqslant kbP_\mu(v_{n-1}, v_n) + kbP_\mu(v_n, v_{n+1}) \\ &\leqslant \frac{kb}{1-kb} P_\mu(v_{n-1}, v_n). \end{aligned}$$

Since $0 < bk < 1$ implies that $\gamma = \frac{bk}{1-kb} \in (0,1)$ and hence by induction

$$P_\mu(v_n, v_{n+1}) \leqslant \gamma^n P_\mu(v_0, v_1). \tag{4.13}$$

Next we claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence.
Without loss of generality assume that, for all $n, m(\geq 2) \in N$ we have sequences v_n and v_m with $v_n \neq v_m$ for any $n \neq m$. Clearly, $P_\mu(v_n, v_m) > 0$ and so for all $m > n$, we have

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant bP_{\mu_1}(v_n, v_{n+1}) + b^2 P_{\mu_2}(v_{n+1}, v_{n+2}) + \cdots + b^{m-n} P_{\mu_{m-n}}(v_{m-1}, v_m) \\ &\leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\Big(v_j, v_{j+1}\Big). \end{aligned}$$

On using (4.8) and the fact $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \cdots = \mu_{m-n} = \frac{\mu}{m-n}$ in above inequality, we obtain

$$\begin{aligned} P_\mu(v_n, v_m) &\leqslant b\gamma^n P_{\mu_1}(v_0, v_1) + b^2\gamma^{n+1} P_{\mu_2}(v_0, v_1) + \cdots + b^{m-n}\gamma^{m-1} P_{\mu(m-n)}(v_0, v_1) \\ &= \Big(1 + \gamma b + (\gamma b)^2 + (\gamma b)^3 + \cdots\cdots (b\gamma)^{m-n-1}\Big) b\gamma^n P_{\frac{\mu}{m-n}}(v_0, v_1) \\ &\leq \left(\frac{b\gamma^n}{1-b\gamma}\right) P_{\frac{\mu}{m-n}}(v_0, v_1). \end{aligned}$$

which yields

$$P_\mu(v_n, v_m) \leqslant \sum_{j=n}^{m-1} b^j P_{\mu_j}\Big(v_j, v_{j+1}\Big) \leqslant \left(\frac{b\gamma^n}{1-b\gamma}\right) P_{\frac{\mu}{m-n}}(v_0, v_1). \tag{4.14}$$

Since $\lim_{n\to\infty}\left(\frac{b\gamma^n}{1-b\gamma}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) = 0$, therefore we can find $\epsilon > 0$ such that

$$0 \leqslant \left(\frac{b\gamma^n}{1-b\gamma}\right) P_{\frac{\mu}{m-n}}(v_0, v_1) < \epsilon. \tag{4.15}$$

Since $(\pi, \lambda) \in \Pi \times [0, +\infty)$, therefore π is a non-decreasing function. Hence for all $\epsilon > 0$ and a $q > 0$, we can find a $\kappa > 0$ such that $0 < \kappa < q$ implies $\pi(\kappa) < \pi(q) < \pi(\epsilon) - \lambda$. Therefore for all $m > n$ and from eq (4.14) and eq (4.15), we have

$$\pi\Big(P_\mu(v_n, v_m)\Big) \leqslant \pi\left(\sum_{j=n}^{m-1} P_{\mu_j}\Big(v_j, v_{j+1}\Big)\right) \leq \pi\left(\frac{b\gamma^n}{1-b\gamma} P_{\frac{\mu}{m-n}}(v_0, v_i)\right) < \pi(\epsilon) - \lambda,$$

and so

$$\pi\left(P_{\mu}\left(v_{n}, v_{m}\right)\right) \leqslant \pi\left(\sum_{j=n}^{m-1} P_{\mu_{j}}\left(v_{j}, v_{j+1}\right)\right)+\lambda<\pi(\epsilon).$$

Condition on π implies that $P_{\mu}\left(v_{n}, v_{m}\right)<\epsilon$.
This proves our claim that the sequence $\{v_n\}$ is a $F-$modular b Cauchy sequence. Completeness of space (Δ, P_{μ}) implies that $\exists\ \bar{v} \in \Delta$ such that the sequence $\{v_n\}$ is $F-$modular b convergent to $\bar{v}$ i.e.

$$\lim_{n \rightarrow \infty} P_{\mu}\left(v_{n}, \bar{v}\right)=0.$$

Now, we claim $\bar{v}$ is fixed point of R. If $R\bar{v}=\bar{v}$ nothing to prove.
Assume that $R\bar{v} \neq \bar{v}$. Thus $P_{\mu}(R\bar{v}, \bar{v}) \neq 0$ and so $P_{\mu}(R\bar{v}, \bar{v})>0$.
On using (4.6), we get

$$\begin{aligned}
P_{\mu}(\bar{v}, R\bar{v}) & \leqslant b P_{\mu}\left(\bar{v}, v_{n+1}\right)+b P_{\mu}\left(v_{n+1}, R\bar{v}\right) \\
& =b P_{\mu}\left(\bar{v}, v_{n}\right)+b P_{\mu}\left(R v_{n}, R\bar{v}\right) \\
& \leq b P_{\mu}\left(\bar{v}, v_{n}\right)+b k P_{\mu}\left(v_{n}, R\bar{v}\right)+b k P_{\mu}\left(\bar{v}, R v_{n}\right) \\
& \leq b P_{\mu}\left(\bar{v}, v_{n}\right)+b k P_{\mu}\left(v_{n}, R\bar{v}\right)+b k P_{\mu}\left(\bar{v}, v_{n+1}\right).
\end{aligned}$$

Therefore, by (P3) of Definition 3.1, we have

$$\pi\left(P_{\mu}(\bar{v}, R\bar{v})\right) \leq \pi\left(b P_{\mu}\left(\bar{v}, v_{n}\right)+b k P_{\mu}\left(v_{n}, R\bar{v}\right)+b k P_{\mu}\left(\bar{v}, v_{n+1}\right)\right).$$

On letting $n \rightarrow \infty$ and using that π is logarithmic like function, we get a contradiction to our assumptions. Hence $\bar{v}$ is a fixed point of R.
Assume that $\exists$ two fixed points of R say u and v, with $u \neq v$ such that $Ru=u$ and $Rv=v$.
Since $u \neq v$ implies that $P_{\mu}(u, v)>0$, and thus

$$\begin{aligned}
0<P_{\mu}(u, v) & =P_{\mu}(R u, R v) \\
& \leq k P_{\mu}(u, R v)+k P_{\mu}(v, R u) \\
& =k P_{\mu}(u, v)+k P_{\mu}(v, u)
\end{aligned}$$

gives that

$$0<(1-2 k) P_{\mu}(u, v)=0.$$

This is possible only if $P_{\mu}(u, v)=0$. Hence $u=v$. This completes the proof of the Theorem 4.3. □

In the next section, we will provide illustrative examples that demonstrate the established outcomes outlined of presented work. Graphical depictions of inequalities are additionally provided for each relevant case.

5. Numerical Illustration with Graphical Behaviors of Contractions

Example 5.1. *Let $\Delta = [1, \infty)$ with a metric defined as*

$$P_{\mu 2} = \frac{(u-v)^2}{\mu}, \quad \mu > 0.$$

Define a function $\pi(t) = \log t$ and $\lambda = 0$. Then $(\pi, \lambda) \in \Pi \times [0, +\infty)$ for any $t \in (0, +\infty)$. Clearly, $(\Delta, P_{\mu 2})$ is a $F-$modular b complete metric spaces.

Let us define a map $R : \Delta \to \Delta$ by

$$R(u) = \frac{1}{u}.$$

From Eq. (4.1), we have

$$P_{\mu 2}(Rx, Ry) = \frac{1}{u^2 v^2}\left(\frac{(u-v)^2}{\mu}\right).$$

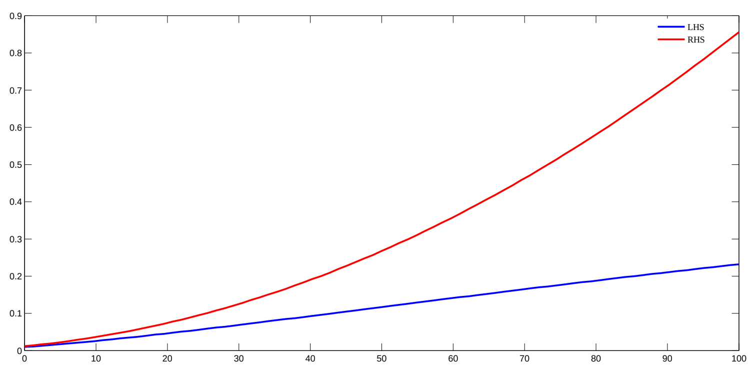


Figure 1. Graphical Behavior of the inequality (4.1) of the Example 5.1

Case- L_1: Let us start with trivial case, that is, if $u = v$, then

$$P_{\mu 2}(Rx, Ry) = 0 = P_{\mu 2} = \frac{(u-v)^2}{\mu}, \mu > 0.$$

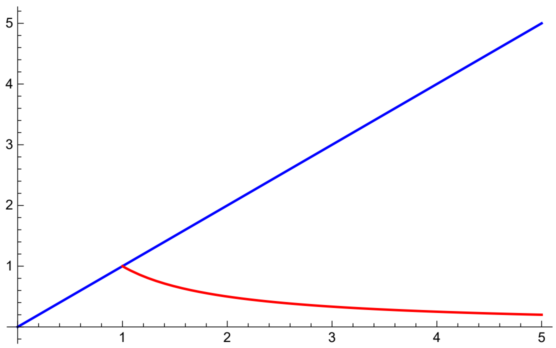
Thus Condition (4.1) of Theorem 4.1 is satisfied.

Case- L_2: If $u \neq v$, then for any $\mu > 0$, we have

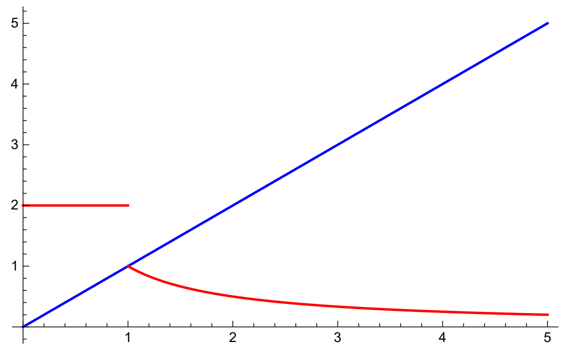
$$P_{\mu 2}(Rx, Ry) = \frac{1}{u^2 v^2}\left(\frac{u-v)^2}{\mu}\right) < kP_{\mu 2}(u, v),$$

where $k = \frac{99}{100}$

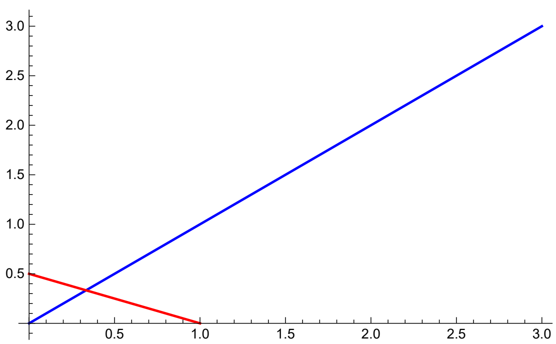
Thus all the conditions of Theorem 4.1 are satisfied. Also 1 is the unique fixed point of the map R.



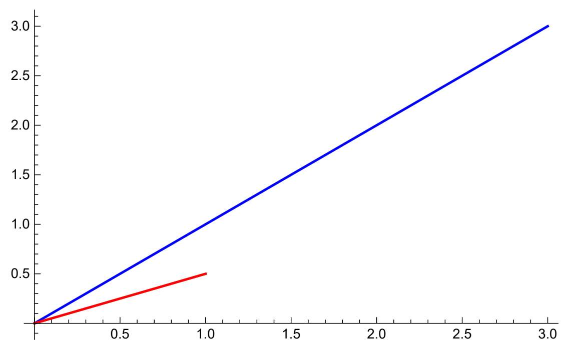
(A) Graph of the function defined in Example 5.1



(B) Graph of the function defined in Example 5.2



(C) Graph of the function defined in Example 5.3



(D) Graph of the function defined in Example 5.4

FIGURE 2. Graphical behavior of functions having fixed point

Remark 5.1. *It should be noted that in Theorem 4.1, continuity of map R is not necessarily required to get the existence and uniqueness of fixed point.*

Let us consider the following example:

Example 5.2. *Let $\Delta = [0, \infty)$ with a metric $P_{\mu 2}$, function $\pi(t)$ and λ same as in Example 4.2. Then $(\Delta, P_{\mu 2})$ is a $F-$modular b complete metric spaces.*
Let us define a map $R : \Delta \to \Delta$ by

$$R(u) = \begin{cases} 2, & u \in [0,1) \\ \frac{1}{u}, & u \in [1, \infty) \end{cases}$$

It is obvious that R is discontinuous function at $u = 1$.
Now we show that R satisfies Eq. (4.1).

We consider different cases as follows:

Case- B_1: Let $u, v \in [0,1)$, then from Eq. (4.1) for all $\mu > 0$, we have

$$P_{\mu 2}(Rx, Ry) = 0 \le k P_{\mu 2}(u, v)$$

Thus R is a F−modular b contraction.

Case- B_2: Let $u, v \in [1, \infty)$. Then two different cases arises:

(a) *If $u = v$, then from Eq. (4.1) for all $\mu > 0$, L.H.S = R.H.S = 0.*

(b) *If $u \neq v$, then for any $\mu > 0$, From Eq. (4.1), we have*

$$P_{\mu 2}(Rx, Ry) < kP_{\mu 2}(u, v),$$

where $k = \frac{99}{100}$

Thus R is a F−modular b contraction.

Therefore, all the conditions of Theorem 4.1 are satisfied. Moreover, 1 is the unique fixed point of the map R.

Example 5.3. *Let $\Delta = [0, 1]$ with a metric defined as*

$$P_{\mu 2} = \frac{(u-v)^2}{\mu}, \quad \mu > 0.$$

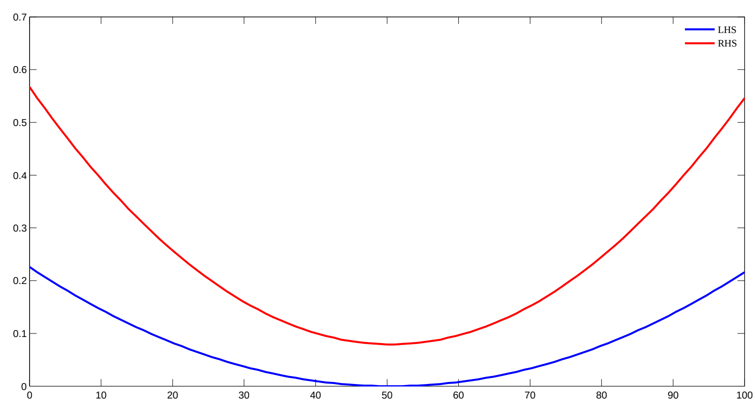


FIGURE 3. Graphical Behavior of the inequality (4.1) of the Example 5.3

Define a function $\pi(t) = \log t$ and $\lambda = 0$. Then $(\pi, \lambda) \in \Pi \times [0, +\infty)$ for any $t \in (0, +\infty)$. Clearly, $(\Delta, P_{\mu 2})$ is a F−modular b complete metric spaces.

Let us define a map $R : \Delta \to \Delta$ by

$$R(u) = \frac{1}{2}(1-u)$$

From Eq. (4.6), we have

$$\begin{aligned} L.H.S. = P_{\mu 2}(Rx, Ry) &= \frac{1}{4\mu}(u-v)^2 \\ &\leq \frac{1}{4\mu}\left[(u-Ru)^2 + (v-Rv)^2\right] \\ &= R.H.S \end{aligned}$$

Thus R is a F−modular b Kannan type contraction.

Therefore, all the conditions of Theorem 4.2 are satisfied. Moreover, $u = \frac{1}{3}$ *is the unique fixed point of the map* R.

Example 5.4. *Let* $\Delta = [0,1]$ *with a metric defined as*

$$P_{\mu 2} = \frac{(u-v)^2}{\mu}, \quad \mu > 0.$$

Define a function $\pi(t) = \log t$ *and* $\lambda = 0$. *Then* $(\pi, \lambda) \in \Pi \times [0, +\infty)$ *for any* $t \in (0, +\infty)$. *Clearly,* $(\Delta, P_{\mu 2})$ *is a* $F-$*modular* b *complete metric spaces.*
Let us define a map $R : \Delta \to \Delta$ *by*

$$R(u) = \frac{u}{2}$$

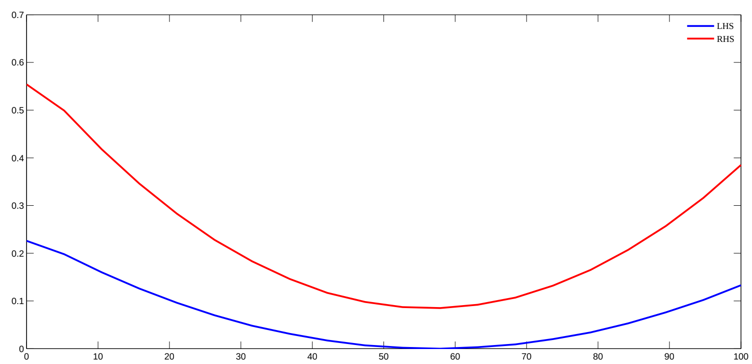


Figure 4. Graphical Behavior of the inequality (4.1) of the Example 5.4

From Eq. (4.11), we have

$$\begin{aligned} R.H.S. &= k\Big[P_\mu(u, Rv) + P_\mu(v, Ru)\Big] \\ &= \frac{k}{\mu}[u^2 + \frac{v^2}{4} + v^2 + \frac{u^2}{4} - 2uv] \\ &= \frac{k}{\mu}[(u-v)^2 + \frac{v^2}{4} + \frac{u^2}{4}] \\ &\geq \frac{1}{4\mu}(u-v)^2 = L.H.S \text{ for all } 0 \leq k < \frac{1}{2} \end{aligned}$$

Thus R *is a* $F-$*modular* b *Chaterjee type contraction.*
Therefore, all the conditions of Theorem 4.3 are satisfied.
Moreover, $u = 0$ *is the unique fixed point of the map* R.

6. Conclusion

In this paper, we began by introducing the innovative notion of a $F-$*modular* $b-$*metric space, accompanied by appropriate illustrations. In addition, we have formulated three theorems Theorem 4.1, Theorem 4.2 and Theorem 4.3 that are based upon the Banach contraction principle, Kannan contraction principle, and*

Chaterjee contraction principle. At last, we have provided many illustrative examples together with graphic representations to showcase the practicality of our primary findings.

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