Int. J. Anal. Appl. (2024), 22:51

International Journal of Analysis and Applications

Qualitative Behavior for a Discretized Conformable Fractional-Order Lotka-Volterra Model With Harvesting Effects

Messaoud Berkal[1], Juan F. Navarro[1], M. B. Almatrafi[2,*]

[1]*Department of Applied Mathematics, University of Alicante, San Vicente del Raspeig, Alicante, Spain*
[2]*Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia*

[*]*Corresponding author:* mmutrafi@taibahu.edu.sa

Abstract. The predator-prey model is a widely mathematical structure that explains the dynamics between two interacting populations: predators and prey. The predator-prey interaction represents a fundamental dynamic in nature, influencing the stability and balance of ecosystems worldwide. The purpose of this article is to provide insight into the complex interactions and feedback mechanisms between predators and prey in ecological systems via mathematical tools such as stability and bifurcation. We investigate a fractional-order Lotka-Volterra model with a harvesting effect using stability and bifurcation theory. The equilibrium points and local stability of the purposed model are presented in this article. The bifurcation analysis, which is a potent approach used to analyse the qualitative behavior of the predator-prey system as the parameter values are varied, is also explored. In particular, a Neimark-Sacker bifurcation and a period-doubling bifurcation are theoretically and numerically examined. Furthermore, we illustrate some 2D figures to show the phase portriat and bifurcations of this model at various points.

1. Introduction

One of the most frequently discussed topics in biomathematics is population dynamics. Beginning with populations of only one species and progressing to more realistic models where several species compete and interact in an identical habitat, the investigation of population development has long been of significant interest. Some experts have successfully explored models in which populations communicate in ways such as competition, symbiosis, commensalism, or predator-prey relationships. Some predator-prey dynamical systems have been investigated with some effects, including the immigration effect, fear effect, cannibalism effect and Allee effect. Lotka-Volterra systems, which were first proposed by Volterra and Lotka [1,2] in the nineteenth century,

Received: Jan. 21, 2024.

2020 *Mathematics Subject Classification.* 37N25, 37N30, 34D35, 34D05, 34D08, 39A28, 39A30, 39A05.

Key words and phrases. stability; prey-predator model; Neimark-Sacker bifurcation; period-doubling bifurcation; fractional derivatives.

https://doi.org/10.28924/2291-8639-22-2024-51
ISSN: 2291-8639

are probably the most well-known predator-prey models. The predator-prey model is usually characterized as a system of ordinary differential equations, where the population sizes of predators and prey are expressed by variables that evolve over time. The model includes significant factors such as birth rates, death rates, predation rates, and competition for resources. In the predator-prey model, the prey is consumed by the predator. The predator may possibly eliminate all the prey, which would lead to the extinction of the latter. However, if that occurs, the predator will eventually become extinct. This cyclic may lead to the fact that the density of prey species depends on the density of predator species for some time.

The mechanism of a variety of biological phenomena has been satisfactorily explained using mathematics. In actuality, many experts have used the concepts of differential equations and difference equations to provide a clear clarification on such phenomena. This is observable in an extensive range of discrete and continuous dynamical systems, including the Lotka-Volterra model. For the purpose of offering a more logical explanation for the interaction between different kinds of animals, this nonlinear model has been thoroughly investigated by a massive number of scholars. For example, Uddin et al. [3] investigated a discrete prey–predator system with harvesting on the predator species by using the Caputo fractional derivative. They showed that this model undergoes a Neimark–Sacker and a Period-doubling bifurcations under some certain conditions. The chaos in the dynamical system was also obtained in [3]. Lee and Baek [4] analysed a predator–prey system with Beddington–DeAngelis functional response with harvesting. In [4], the authors proved that the considered model goes through some types of bifurcation including the subcritical and supercritical Hopf bifurcation, the saddle–node bifurcation, and the Bogdanov–Takens bifurcation. Moreover, Liu and Huang [5] investigated a harvested predator–prey model using the analysis of two-prey and one-predator system. The occurrence and the attractivity behavior of seven fixed points of these three equations were further presented. In [6], Sahoo et al. provided a detail study about a predator–prey system with a square root functional response. This model, in which an alternative resource was provided in the population predator, was successfully investigated with a predator harvesting. Furthermore, Moitri et al. [7] explored a harvested predator–prey model with alternative prey. Lin [8] investigated a model of two species with non-monotonic functional response and non-selective harvesting. The local and global stability of this model were also studied. Das and Pal [9] formulated and analysed a prey-predator model with a harvesting effect for both the prey and predator population. The stability criteria of this model was also examined for this model. Finally, Mohdeb [10] explored the stability of fixed points and the occurrence of periodic solutions for a prey-predator model including nonlinear harvesting effect. Transcritical and saddle node bifurcations were also discussed in [10]. More studies about predator-prey model can be found in refs. [11, 14, 15, 19].

The motivation behind writing this article stems from the profound significance and applicability of this ecological concept. By exploring this model, we gain insights into the complicated connections between the considered species, the influence of population dynamics, and the delicate equilibrium that exists in natural communities. Understanding the predator-prey model can shed light on the mechanisms driving population fluctuations, the impacts of predation on biodiversity, and the potential consequences of disturbances or interventions within ecosystems. Furthermore, elucidating this model can have broader implications, aiding in the development of effective conservation strategies, wildlife management practices, and even informing our understanding of human socio-ecological systems. This paper aims to investigate the following fractional order dynamical system:

$$\begin{cases} \mathrm{T}^{\alpha}u(t) = Ru(t)(1-u(t)) - Bu(t)v(t) - H_1u(t), \\ \mathrm{T}^{\alpha}v(t) = v(t)(eBu(t) - D) - H_2v(t). \end{cases} \tag{1.1}$$

Here, all parameters are positive. We present a brief description for each parameter in terms of its biological meaning in Table 1.

Parameter	Biological description
t	Time
$u(t)$	Population densities of prey at time t
$v(t)$	Population densities of predator at time t
R	Intrinsic growth rates of prey
B	Consumption rate of the prey
H_1	Harvesting effect on prey population
e	Conversion factor
D	Death rates of predator
H_2	Harvesting effect on predator population
α	Fractional-order parameter with $0 < \alpha \leq 1$
T^{α}	Fractional derivative of the conformable-type

Table 1. Biological description of parameters and variables present in model (1.1).

We also provide a detail investigation about the Neimark-Sacker and period-doubling bifurcations. Bifurcations occur when the dynamics of the system encounter qualitative changes, such as the emergence of new stable states, limit cycles, or chaotic behavior. These changes can have significant implications for the long-term stability and coexistence of predator and prey populations. In addition, we present numerical investigations for the obtained results.

We outline this paper as follows. Section 2 discusses the discretization method which we use to discretize the proposed model. In Section 3, the occurrence and stability of the fixed points are shown. Moreover, the bifurcation analysis is given in Section 4. Section 5 presents the chaos control. Section 6 is added to confirm the obtained results using numerical examples. Ultimately, we conclude this work in Section 7.

Definition 1.1 ([20,21]). *Let $H : (0,\infty) \to \mathrm{R}$ be a function. Then, the conformable fractional derivative of order $0 < \alpha \leq 1$ of H at $t > 0$ is defined by*

$$\mathrm{T}_S^{\alpha} H(t) = \lim_{\epsilon \to 0} \frac{H(t + \epsilon(t-S)^{1-\alpha}) - H(t)}{\epsilon}, \quad 0 < \alpha \leq 1. \tag{1.2}$$

Here, T_S^{α} is the conformable-type fractional derivative. The discretization parameter is $S > 0$. It was presented in [22], the derivative of Eq. (1.2) *is given by*

$$\mathrm{T}_S^{\alpha} H(t) = (t-S)^{1-\alpha} H'(t). \tag{1.3}$$

2. Discretization Strategy

This part will discretize the proposed model using the piecewise constant argument approach [23]. We also employ the concept of the conformable fractional derivative [11, 24] (Definition 1.1). Using this concept and simplify the obtained results, model (1.1) can be converted into the following system:

$$\begin{cases} u_{i+1} = u_i e^{(R(1-u_i) - Bv_i - H_1)\frac{S^{\alpha}}{\alpha}}, \\ v_{i+1} = v_i e^{(eBu_i - D - H_2)\frac{S^{\alpha}}{\alpha}}, \end{cases} \tag{2.1}$$

where S represents the discretization parameter.

3. Equilibrium Points and Stability Analysis

In this section, we study the existence and stability of equilibrium points of system (2.1). The equilibrium point (u, v) of system (2.1) satisfies

$$u = ue^{(R(1-u) - Bv - H_1)\frac{S^{\alpha}}{\alpha}}, \quad v = ve^{(eBu - D - H_2)\frac{S^{\alpha}}{\alpha}}.$$

Clearly, system (2.1) always has the boundary equilibrium points $O = (0, 0)$ and $B_E = \left(\frac{R-H_1}{R}, 0\right)$ if $R > H_1$. Moreover, if $ReB > RD + RH_2 + eBH_1$ then it also has a positive equilibrium point $P_E = \left(\frac{D+H_2}{eB}, \frac{ReB - RD - RH_2 - eBH_1}{eB^2}\right)$. In order to analyse the stability of the obtained equilibrium points, we introduce the following lemmas.

Lemma 3.1 ([23,24]). *Suppose that (u, v) is a equilibrium point for system (2.1) with multipliers γ_1 and γ_2. Then,*

(1) *The point (u, v) is a sink if $|\gamma_1| < 1$ and $|\gamma_2| < 1$.*
(2) *The point (u, v) is a source if $|\gamma_1| > 1$ and $|\gamma_2| > 1$.*
(3) *The point (u, v) is a saddle if $|\gamma_1| < 1$ and $|\gamma_2| > 1$ (or if $|\gamma_1| > 1$ and $|\gamma_2| < 1$).*
(4) *The point (u, v) is a non-hyperbolic if $|\gamma_1| = 1$ or $|\gamma_2| = 1$.*

Lemma 3.2 ([14,23,25]). *Suppose that the polynomial $\rho(\gamma) = \gamma^2 - P\gamma + Q$, where $\rho(1) > 0$, and γ_1 and γ_2 are the two roots of $\rho(\gamma) = 0$. Then,*

(1) *$|\gamma_1| < 1$ and $|\gamma_2| < 1$ if and only if $\rho(-1) > 0$ and $\rho(0) < 1$.*
(2) *$|\gamma_1| > 1$ and $|\gamma_2| > 1$ if and only if $\rho(-1) > 0$ and $\rho(0) > 1$.*

(3) $|\gamma_1| < 1$ *and* $|\gamma_2| > 1$ *(or* $|\gamma_1| > 1$ *and* $|\gamma_2| < 1$*) if and only if* $\rho(-1) < 0$.
(4) $\gamma_1 = -1$ *and* $|\gamma_2| \neq 1$ *if and only if* $\rho(-1) = 0$ *and* $P \neq 0, 2$.
(5) γ_1 *and* γ_2 *are complex numbers and* $|\gamma_1| = |\gamma_2| = 1$ *if and only if* $|P| < 2$ *and* $\rho(0) = 1$.

Note that the Jacobian matrix of system (2.1) calculated at any point (u, v) is written as

$$\mathcal{M}_J((u,v)) = \begin{pmatrix} \left(1 - \dfrac{RS^\alpha u}{\alpha}\right) e^{(R(1-u)-Bv-H_1)\frac{S^\alpha}{\alpha}} & -\dfrac{BS^\alpha u}{\alpha} e^{(R(1-u)-Bv-H_1)\frac{S^\alpha}{\alpha}} \\ \dfrac{eBS^\alpha v}{\alpha} e^{(eBu-D-H_2)\frac{S^\alpha}{\alpha}} & e^{(eBu-D-H_2)\frac{S^\alpha}{\alpha}} \end{pmatrix}. \tag{3.1}$$

Hence, evaluating matrix (3.1) at the point O gives

$$\mathcal{M}_J(O) = \begin{pmatrix} e^{(R-H_1)\frac{S^\alpha}{\alpha}} & 0 \\ 0 & e^{(-D-H_2)\frac{S^\alpha}{\alpha}} \end{pmatrix}. \tag{3.2}$$

As a result, the eigenvalues of matrix (3.2) are given by $\gamma_1 = e^{(R-H_1)\frac{S^\alpha}{\alpha}}$ and $\gamma_2 = e^{(-D-H_2)\frac{S^\alpha}{\alpha}} < 0$. Furthermore, the Jacobian matrix at the equilibrium point B_E is given by

$$\mathcal{M}_J(B_E) = \begin{pmatrix} \left(1 - \dfrac{(R-H_1)S^\alpha}{\alpha}\right) & -\dfrac{B(R-H_1)S^\alpha}{R\alpha} \\ 0 & e^{\left(\frac{eB(R-H_1)}{R} - D - H_2\right)\frac{S^\alpha}{\alpha}} \end{pmatrix}, \tag{3.3}$$

whose eigenvalues are $\gamma_1 = \left(1 - \frac{R(R-H_1)S^\alpha}{\alpha}\right)$ and $\gamma_2 = e^{\left(\frac{eB(R-H_1)}{R} - D - H_2\right)\frac{S^\alpha}{\alpha}}$.
Hence, using Lemma (3.1), we attain the local stability of the points O and B_E, as shown in Lemma (3.3) and (3.4), respectively.

Lemma 3.3. *The trivial equilibrium* $O = (0, 0)$ *is a*

1 *sink if* $R < H_1$;
2 *saddle if* $R > H_1$;
3 *non-hyperbolic if* $R = H_1$.

Lemma 3.4. *When* $R > H_1$ *is satisfied, the boundary equilibrium* $B_E = \left(\frac{R-H_1}{R}, 0\right)$ *is a*

1 *sink if* $0 < S < \left(\frac{2\alpha}{R(R-H_1)}\right)^{1/\alpha}$ *and* $\frac{eB(R-H_1)}{R} < D + H_1$;
2 *source if* $S > \left(\frac{2\alpha}{R(R-H_1)}\right)$ *and* $\frac{eB(R-H_1)}{R} > D + H_1$;
3 *saddle if* $S > \left(\frac{2\alpha}{R(R-H_1)}\right)^{1/\alpha}$ *and* $\frac{eB(R-H_1)}{R} < D + H_1$ *(or* $0 < S < \left(\frac{2\alpha}{R(R-H_1)}\right)$ *and* $\frac{eB(R-H_1)}{R} > D + H_1$*)*;
4 *non-hyperbolic if* $S = \left(\frac{2\alpha}{R(R-H_1)}\right)^{1/\alpha}$ *or* $\frac{eB(R-H_1)}{R} = D + H_1$.

Now, we calculate the Jacobian matrix of system (2.1) at the positive equilibrium point P_E as follows:

$$\mathcal{M}_J(P_E) = \begin{pmatrix} \left(1 - \dfrac{R(D+H_2)S^\alpha}{eB\alpha}\right) & -\dfrac{(D+H_2)S^\alpha}{e\alpha} \\ \dfrac{(ReB - RD - RH_2 - eBH_1)S^\alpha}{\alpha B} & 1 \end{pmatrix}. \tag{3.4}$$

The characteristic polynomial of matrix (3.4) is shown as follows:

$$\rho(\gamma) = \gamma^2 - P\gamma + Q, \tag{3.5}$$

where

$$P = 2 - \left(\frac{R(D+H_2)}{eB\alpha}\right)S^\alpha,$$
$$Q = 1 - \left(\frac{R(D+H_2)}{eB\alpha}\right)S^\alpha + \left(\frac{(D+H_2)(ReB - RD - RH_2 - eBH_1)}{eB\alpha^2}\right)S^{2\alpha}.$$

Hence, we have

$$\rho(-1) = 4 - 2\left(\frac{R(D+H_2)}{eB\alpha}\right)S^\alpha + \left(\frac{(D+H_2)(ReB - RD - RH_2 - eBH_1)}{eB\alpha^2}\right)S^{2\alpha},$$
$$\rho(0) = 1 - \left(\frac{R(D+H_2)}{eB\alpha}\right)S^\alpha + \left(\frac{(D+H_2)(ReB - RD - RH_2 - eBH_1)}{eB\alpha^2}\right)S^{2\alpha},$$
$$\rho(1) = \left(\frac{(D+H_2)(ReB - RD - RH_2 - eBH_1)}{eB\alpha^2}\right)S^{2\alpha} > 0.$$

According to the above results and Lemma (3.2), we can state and prove the following Lemma.

Lemma 3.5. *Suppose that* $ReB > (RD + RH_2 + eBH_1)$ *and let*

$$\delta = R^2(D+H_2) - 4eB(ReB - RD - RH_2 - eBH_1),\ S_0 = \left(\frac{\alpha R}{(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha},$$

$$S_1 = \left(\frac{\alpha(R(D+H_2) - \sqrt{R^2(D+H_2)^2 - 4eB(D+H_2)(ReB - RD - RH_2 - eBH_1)})}{(D+H_2)(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha},$$

$$\text{and } S_1 = \left(\frac{\alpha(R(D+H_2) + \sqrt{R^2(D+H_2)^2 - 4eB(D+H_2)(ReB - RD - RH_2 - eBH_1)})}{(D+H_2)(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha}.$$

Then, there exist different topological types of the stability of the positive equilibrium point P_E *for all possible parameters.*

(1) *The point* P_E *is a sink (locally asymptotic stable) if one of the following conditions hold:*
 (i) $\delta < 0$ *and* $0 < S < S_0$,
 (ii) $\delta \geq 0$ *and* $0 < S < S_1$.
(2) *The point* P_E *is source if one of the following conditions hold:*
 (i) $\delta < 0$ *and* $S > S_0$,
 (ii) $\delta \geq 0$ *and* $S > S_2$.

(3) *The point P_E is saddle if $\delta \geq 0$ and $S_1 < S < S_2$.*

(4) *The point P_E is non-hyperbolic if one of the following conditions hold:*

(i) $\delta < 0$ *and* $S = S_0$,

(ii) $\delta \geq 0$ *and* $S = S_{1,2}$ *and* $S \neq \left(\frac{2\alpha eB}{R(D+H_2)}\right)^{1/\alpha}$.

Lemma 3.6. *The positive equilibrium point P_E of system (2.1) loses its stability in two cases.*

(1) *The point P_E loses its stability via a period-doubling bifurcation if*

$$R^2(D+H_2) \geq 4eB(ReB - RD - RH_2 - eBH_1), \quad S \neq \left(\frac{2\alpha eB}{R(D+H_2)}\right)^{1/\alpha},$$

and

$$S = S_{1,2} = \left(\frac{\alpha(R(D+H_2) \mp \sqrt{R^2(D+H_2)^2 - 4eB(D+H_2)(ReB - RD - RH_2 - eBH_1)})}{(D+H_2)(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha}.$$

(2) *The point P_E loses its stability via a Neimark-Sacker bifurcation if*

$$R^2(D+H_2) < 4eB(ReB - RD - RH_2 - eBH_1), \text{ and } S = S_0 = \left(\frac{\alpha R}{(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha}.$$

4. Bifurcation Analysis

Our main purpose in this section is to extensively examine whether the considered populations exhibit cyclic dynamics over time. We now prove that system (2.1) can either undergo a Neimark-Sacker bifurcation or a period-doubling bifurcation when the positive equilibrium point

$$P_E = \left(\frac{D+H_2}{eB}, \frac{ReB - RD - RH_2 - eBH_1}{eB^2}\right),$$

loses its stability.

4.1. **Neimark-Sacker bifurcation.** The Neimark-Sacker bifurcation of system (2.1) is discussed in this part. This type of bifurcation takes place when a closed invariant curve arises from a fixed point in a discrete dynamical system. Then, the stability of the point changes via a pair of complex eigenvalues with unit modulus [24–28]. In this discussion, we study the considered system around the point P_E. Note that this point loses its stability via a Neimark-Sacker bifurcation if the parameters $(R, B, H_1, D, e, H_2, S, \alpha)$ vary in a neighborhood of the set:

$$\mathcal{K}_0 = \left\{ \begin{array}{c} (R, B, H_1, D, e, H_2, S, \alpha) \in \mathrm{R}^8 \left| R^2(D+H_2) < 4eB(ReB - RD - RH_2 - eBH_1), \right. \\ S = S_0 = \left(\frac{\alpha R}{(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha}, \ \alpha \in (0,1] \end{array} \right\}.$$

Next, assume that $(R, B, H_1, D, e, H_2, S_0, \alpha) \in \mathcal{K}_0$ and $\bar{S}$ is a small perturbation of S_0 where $|\bar{S}| \ll 1$. Then, system (2.1) can be expressed as follows:

$$\begin{cases} u_{i+1} = u_i e^{(R(1-u_i) - Bv_i - H_1)\frac{(S_0 - \bar{S})^\alpha}{\alpha}} = \mathcal{F}_1(u_i, v_i, \bar{S}), \\ v_{i+1} = v_i e^{(eBu_i - D - H_2)\frac{(S_0 - \bar{S})^\alpha}{\alpha}} \mathcal{F}_2(u_i, v_i, \bar{S}). \end{cases} \tag{4.1}$$

If we consider the shift $X_i = u_i - \dfrac{D+H_2}{eB}$ and $Y_i = v_i - \dfrac{ReB - RD - RH_2 - eBH_1}{eB^2}$, then the equilibrium point P_E becomes the origin. Using Taylor series at origin and expanding $\mathcal{F}_1$ and $\mathcal{F}_2$ to the 3-order, system (4.1) gives

$$\begin{cases} X_{i+1} &= A_{11}X_i + A_{12}Y_i + A_{13}X_i^2 + A_{14}X_iY_i + A_{15}Y_i^2 + A_{16}X_i^3 + \\ & \quad A_{17}X_i^2Y_i + A_{18}X_iY_i^2 + A_{19}Y_i^3 + O_1(|X_i|,|Y_i|)^4, \\ Y_{i+1} &= A_{21}X_i + A_{22}Y_i + A_{23}X_i^2 + A_{24}X_iY_i + A_{25}Y_i^2 + A_{26}X_i^3 + \\ & \quad A_{27}X_i^2Y_i + A_{28}X_iY_i^2 + A_{29}Y_i^3 + O_2(|X_i|,|Y_i|)^4. \end{cases} \tag{4.2}$$

Here,

$$\begin{aligned}
&A_{11} = \left(1 - \frac{R(D+H_2)(S_0-\bar{S})^\alpha}{eB\alpha}\right), \quad A_{12} = -\frac{(D+H_2)(S_0-\bar{S})^\alpha}{e\alpha}, \\
&A_{21} = \tfrac{(ReB-RD-RH_2-eBH_1)(S_0-\bar{S})^\alpha}{\alpha B}, \quad A_{22} = 1, \ A_{13} = -\tfrac{R(S_0-\bar{S})^\alpha}{\alpha}\left(2 - \tfrac{R(S_0-\bar{S})u^*}{\alpha}\right), \\
&A_{14} = -\tfrac{B(S_0-\bar{S})^\alpha}{\alpha}\left(1 - \tfrac{B(S_0-\bar{S})u^*}{\alpha}\right), \ A_{15} == \tfrac{B^2(S_0-\bar{S})^\alpha u^*}{\alpha}, \ A_{16} = \left(\tfrac{R(S_0-\bar{S})^\alpha}{\alpha}\right)^2\left(3 - \tfrac{R(S_0-\bar{S})u^*}{\alpha}\right), \\
&A_{17} = \tfrac{B(S_0-\bar{S})^{2\alpha}}{\alpha^2}\left(B + R\left(1 - \tfrac{B(S_0-\bar{S})u^*}{\alpha}\right)\right), \ A_{18} = -\tfrac{B^2(S_0-\bar{S})^\alpha}{\alpha}\left(1 - \tfrac{R(S_0-\bar{S})u^*}{\alpha}\right), \\
&A_{19} = \tfrac{-B^3u^*(S_0-\bar{S})^\alpha}{\alpha}, \ A_{23} = \left(\tfrac{eB(S_0-\bar{S})^\alpha}{\alpha}\right)^2 v*, \ A_{24} = \left(\tfrac{eB(S_0-\bar{S})^\alpha}{\alpha}\right), \ A_{25} = A_{28} = A_{29} = 0, \\
&A_{26} = \left(\tfrac{eB(S_0-\bar{S})^\alpha}{\alpha}\right)^3 v^*, \ A_{27} = \left(\tfrac{eB(S_0-\bar{S})^\alpha}{\alpha}\right)^2,
\end{aligned}$$

with

$$u^* = \frac{D+H_2}{eB} \quad \textit{and} \quad v^* = \frac{ReB - RD - RH_2 - eBH_1}{eB^2}.$$

The Jacobian matrix $\mathcal{M}_J(P_E)$ of system (4.2) about the point P_E is

$$\mathcal{M}_J(P_E) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Here, the characteristic equation of the linearization of system (4.2) at the origin is given by

$$\gamma^2 - P(\bar{S})\gamma + Q(\bar{S}) = 0\,, \tag{4.3}$$

where

$$\begin{aligned}
P(\bar{S}) &= 2 - \left(\tfrac{R(D+H_2)}{eB\alpha}\right)(S_0-\bar{S})^\alpha \quad \textit{and}, \\
Q(\bar{S}) &= 1 - \left(\tfrac{R(D+H_2)}{eB\alpha}\right)(S_0-\bar{S})^\alpha + \left(\tfrac{(D+H_2)(ReB-RD-RH_2-eBH_1)}{eB\alpha^2}\right)(S_0-\bar{S})^{2\alpha}.
\end{aligned}$$

We have $(R, B, H_1, D, e, H_2, S_0, \alpha) \in \mathcal{K}_0$. Therefore, the complex conjugate roots with unit modulus of Eq. (4.3) are

$$\gamma_{1,2}(\bar{S}) = \frac{P(\bar{S})}{2} \pm \frac{i}{2}\sqrt{4Q(\bar{S}) - P^2(\bar{S})}\,.$$

Hence, $|\gamma_{1,2}(\bar{S})| = \sqrt{Q(\bar{S})}$ and $Q(0) = 1$ from which we have

$$\left(\frac{d|\gamma_{1,2}|}{d\bar{S}}\right)_{\bar{S}=0} = \left(\frac{d\sqrt{Q(\bar{S})}}{d\bar{S}}\right)_{\bar{S}=0} = \frac{R(D+H_2)}{2eB}\left(\frac{\alpha R}{(ReB-RD-RH_2-eBH_1)}\right)^{\frac{\alpha-1}{\alpha}} > 0.$$

The following conditions must be satisfied to ensure the existence of a Neimark-Sacker bifurcation:

$$\left(\frac{d|\gamma_{1,2}(\bar{S})|}{d\bar{S}}\right)_{\bar{S}=0} \neq 0, \tag{4.4}$$

and

$$\gamma_{1,2}(0)^n \neq 1, \quad n = 1,2,3,4. \tag{4.5}$$

The constraint $P(0) \neq 0, 1$ implies that

$$S_0 \neq \left(\frac{2\alpha eB}{R(D+H_2)}\right)^{\frac{1}{\alpha}}, \quad \left(\frac{\alpha eB}{R(D+H_2)}\right)^{\frac{1}{\alpha}}.$$

This is exactly equivalent to $\gamma_{1,2}^n \neq 1$ with $n = 1,2,3,4$. Moreover, the normal form of system (4.2) at $\bar{S} = 0$ is obtained by defining the following transformation:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} A_{12} & 0 \\ \mathcal{R} - A_{11} & -\mathcal{I} \end{pmatrix}\begin{pmatrix} \bar{u}_i \\ \bar{v}_i \end{pmatrix},$$

with $\mathcal{R} = \frac{P(0)}{2}$ and $\mathcal{I} = \frac{\sqrt{4-P(0)^2}}{2}$. Using the previous transformation, system (4.2) becomes

$$\begin{aligned} \bar{u}_{i+1} &= \mathcal{R}\bar{u}_i - \mathcal{I}\bar{v}_i + \tilde{\mathcal{F}}_1(\bar{u}_i, \bar{v}_i), \\ \bar{v}_{i+1} &= \mathcal{I}\bar{u}_i + \mathcal{R}\bar{v}_i + \tilde{\mathcal{F}}_2(\bar{u}_i, \bar{v}_i), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \tilde{\mathcal{F}}_1(\bar{u}_i, \bar{v}_i, S_0) &= \frac{A_{13}}{A_{12}}X_i^2 + \frac{A_{14}}{A_{12}}X_iY_i + \frac{A_{15}}{A_{12}}Y_i^2 + \frac{A_{16}}{A_{12}}X_i^3 + \frac{A_{17}}{A_{12}}X_i^2Y_i + \frac{A_{18}}{A_{12}}X_iY_i^2 + \\ &\quad \frac{A_{19}}{A_{12}}Y_i^3 + \tilde{O}_1(|X_i|, |Y_i|)^4, \\ \tilde{\mathcal{F}}_2(\bar{u}_i, \bar{v}_i, S_0) &= \left(\frac{A_{13}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{23}}{\mathcal{I}}\right)X_i^2 + \left(\frac{A_{14}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{24}}{\mathcal{I}}\right)X_iY_i + \\ &\quad \left(\frac{A_{15}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{25}}{\mathcal{I}}\right)Y_i^2 + \left(\frac{A_{16}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{26}}{\mathcal{I}}\right)X_i^3 + \\ &\quad \left(\frac{A_{17}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{27}}{\mathcal{I}}\right)X_i^2Y_i + \left(\frac{A_{18}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{28}}{\mathcal{I}}\right)X_iY_i^2 + \\ &\quad \left(\frac{A_{19}(\mathcal{R}-A_{11})}{\mathcal{I}A_{12}} - \frac{A_{29}}{\mathcal{I}}\right)Y_i^3 + \tilde{O}_2(|X_i|, |Y_i|)^4, \end{aligned}$$

with

$$X_i = A_{12}\bar{u}_i, \quad Y_i = (\mathcal{R} - A_{11})\bar{u}_i - \mathcal{I}\bar{v}_i.$$

Ultimately, to make sure that the Neimark-Sacker bifurcation at $(0,0,S_0)$ of system (4.6) exists, the following discriminatory quantity must not be zero:

$$\mathcal{L} = \Re(\gamma_2 \mathrm{n}_{21}) - \Re\left(\frac{(1-2\gamma_1)\gamma_2^2}{1-\gamma_1}\mathrm{n}_{20}\mathrm{n}_{11}\right) - \frac{1}{2}|\mathrm{n}_{11}|^2 - |\mathrm{n}_{02}|^2 , \tag{4.7}$$

where

$$\mathrm{n}_{20} = \frac{1}{8}\left[\frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{u}^2} - \frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{v}^2} + 2\frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{u}\partial\bar{v}} + i\left(\frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{u}^2} - \frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{v}^2} - 2\frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{u}\partial\bar{v}}\right)\right]\Bigg|_{\bar{S}=0},$$

$$\mathrm{n}_{11} = \frac{1}{4}\left[\frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{u}^2} + \frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{v}^2} + i\left(\frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{u}^2} + \frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{v}^2}\right)\right]\Bigg|_{\bar{S}=0},$$

$$\mathrm{n}_{02} = \frac{1}{8}\left[\frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{u}^2} - \frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{v}^2} - 2\frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{u}\partial\bar{v}} + i\left(\frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{u}^2} - \frac{\partial^2\tilde{\mathcal{F}}_2}{\partial\bar{v}^2} + 2\frac{\partial^2\tilde{\mathcal{F}}_1}{\partial\bar{u}\partial\bar{v}}\right)\right]\Bigg|_{\bar{S}=0},$$

$$\mathrm{n}_{21} = \frac{1}{16}\left[\frac{\partial^3\tilde{\mathcal{F}}_1}{\partial\bar{u}^3} + \frac{\partial^3\tilde{\mathcal{F}}_1}{\partial\bar{u}\partial\bar{v}^2} + \frac{\partial^3\tilde{\mathcal{F}}_2}{\partial\bar{u}^2\partial\bar{v}} + \frac{\partial^3\tilde{\mathcal{F}}_2}{\partial\bar{v}^3} + i\left(\frac{\partial^3\tilde{\mathcal{F}}_2}{\partial\bar{u}^3} + \frac{\partial^3\tilde{\mathcal{F}}_2}{\partial\bar{u}\partial\bar{v}^2} - \frac{\partial^3\tilde{\mathcal{F}}_1}{\partial\bar{u}^2\partial\bar{v}} - \frac{\partial^3\tilde{\mathcal{F}}_1}{\partial\bar{v}^3}\right)\right]\Bigg|_{\bar{S}=0}.$$

Theorem 4.1. *Assume that conditions (4.4), (4.5) are satisfied, and let $(R,B,H_1,D,e,H_2,S,\alpha) \in \mathcal{K}_0$ with $\mathcal{L} \neq 0$. Then, system (2.1) goes through a Neimark-Sacker bifurcation at the equilibrium point P_E when the bifurcation parameter S varies in a small neighbourhood of*

$$S_0 = \left(\frac{\alpha R}{(ReB - RD - RH_2 - eBH_1)}\right)^{1/\alpha}.$$

If $\mathcal{L} < 0$, the equilibrium point bifurcates in an attracting invariant closed curve for $\bar{S} > 0$. If $\mathcal{L} > 0$, a repelling invariant closed curve bifurcates from the equilibrium point for $\bar{S} < 0$.

4.2. **Period-doubling bifurcation.** The following lemma is utilized to demonstrate the period-doubling bifurcation of system (2.1) at the point $P_E = \left(\frac{D+H_2}{eB}, \frac{ReB - RD - RH_2 - eBH_1}{eB^2}\right)$ with $ReB > RD + RH_2 + eBH_1$.

Lemma 4.1 ([29,30]). *[Existence of period-doubling bifurcation] Assume that $U_{k+1} = G_s(U_k)$ is a n-dimensional discrete dynamical system where $s \in R$ is a bifurcation parameter. Let U^* be an equilibrium point of G_s and suppose that the characteristic equation of the Jacobian matrix $M_J(U^*) = (a_{ij})_{n\times n}$ of n-dimensional map $G_s(U_k)$ is expressed as*

$$\rho_s(\gamma) = \gamma^n + a_1\gamma^{n-1} + \cdots + a_{n-1}\gamma + a_n. \tag{4.8}$$

Here, $a_i = a_i(s,u)$, $i = 1,2,3,\cdots,n$ and u is a control parameter. Suppose that $\Delta_0^{\pm}(s,u) = 1$, $\Delta_1^{\pm}(s,u),\cdots,\Delta_n^{\pm}(s,u)$ are a sequence of the determinants described by

$$\Delta_i^{\pm}(s,u) = \det(N_1 \pm N_2), \quad i = 1,2,\cdots,n, \tag{4.9}$$

where

$$N_1=\begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{i-1}\\ 0 & 1 & a_1 & \cdots & a_{i-2}\\ 0 & 0 & 1 & \cdots & a_{i-3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},\ N_2=\begin{pmatrix} a_{n-i+1} & a_{n-i+2} & \cdots & a_{n-1} & a_n\\ a_{n-i+2} & a_{n-i+3} & \cdots & a_n & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{n-1} & a_n & \cdots & 0 & 0\\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.10}$$

Furthermore, suppose that the following statements hold.

- **C1-** *Eigenvalue criterion:* $\rho_{s_0}(-1)=0$, $\Delta^{\pm}_{n-1}(s_0,u)>0$, $\rho_{s_0}(1)>0$, $\Delta^{\pm}_{i}(s_0,u)>0$, $i=n-2,n-4,\cdots,1$ *(or* 1*), when* n *is even (or odd), respectively.*
- **C2-** *Transversality criterion:* $\dfrac{\sum_{i=1}^{n}(-1)^{n-i}a'_i}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)a_{i-1}}\neq 0$*, where* a'_i *represents the derivative of* $a(s)$ *at* $s=s_0$*. Then, a period-doubling bifurcation exists at critical value* s_0*.*

Theorem 4.2. *System (2.1) goes through a P-D bifurcation at the positive equilibrium point* P_E*, if the following conditions hold:*

$$1+Q>0,$$
$$1+P+Q=0,$$
$$1-P+Q>0.$$

Hence, the period-doubling bifurcation takes place at S *if the parameters* $(R,B,H_1,D,e,H_2,S,\alpha)$ *vary in a neighborhood of the set*

$$\mathcal{K}_1=\left\{\begin{array}{c}(R,B,H_1,D,e,H_2,S,\alpha)\in \mathrm{R}^8\,\Big|R^2(D+H_2)\geq 4eB(ReB-RD-RH_2-eBH_1),\ \ \alpha\in(0,1]\\ S=S_1=\left(\frac{\alpha(R(D+H_2)-\sqrt{R^2(D+H_2)^2-4eB(D+H_2)(ReB-RD-RH_2-eBH_1)})}{(D+H_2)(ReB-RD-RH_2-eBH_1)}\right)^{\frac{1}{\alpha}},\ S\neq\left(\frac{2\alpha eB}{R(D+H_2)}\right)^{\frac{1}{\alpha}}\end{array}\right\}.$$

Or,

$$\mathcal{K}_2=\left\{\begin{array}{c}(R,B,H_1,D,e,H_2,S,\alpha)\in \mathrm{R}^8\,\Big|R^2(D+H_2)\geq 4eB(ReB-RD-RH_2-eBH_1),\ \ \alpha\in(0,1]\\ S=S_2=\left(\frac{\alpha(R(D+H_2)+\sqrt{R^2(D+H_2)^2-4eB(D+H_2)(ReB-RD-RH_2-eBH_1)})}{(D+H_2)(ReB-RD-RH_2-eBH_1)}\right)^{\frac{1}{\alpha}},\ S\neq\left(\frac{2\alpha eB}{R(D+H_2)}\right)^{\frac{1}{\alpha}}\end{array}\right\}.$$

Proof. Using Lemmas 3.6, and 4.1, and from the evaluation of Eq. (3.5) of system (2.1) at P_E, we have

$$\Delta_0^{\mp}(S)=1>0,$$
$$\Delta_1^{+}(S)=1+Q>0,$$
$$(-1)^2\rho(-1)=1+P+Q=0,$$
$$\rho(1)=1-P+Q>0,$$

if and only if

$$S=S_{1,2}=\left(\frac{\alpha(R(D+H_2)\mp\sqrt{R^2(D+H_2)^2-4eB(D+H_2)(ReB-RD-RH_2-eBH_1)})}{(D+H_2)(ReB-RD-RH_2-eBH_1)}\right)^{\frac{1}{\alpha}},$$

$$and \quad R^2(D+H_2) \geq 4eB(ReB - RD - RH_2 - eBH_1).$$

Furthermore, the transversality condition is

$$\frac{P'+Q'}{P+2} = \frac{2\sqrt{R^2(D+H_2)^2 - 4eB(D+H_2)(ReB-RD-RH_2-eBH_1)})}{(ReB-RD-RH_2-eBH_1)} \times \left(\frac{\alpha(R(D+H_2) \mp \sqrt{R^2(D+H_2)^2-4eB(D+H_2)(ReB-RD-RH_2-eBH_1)})}{(D+H_2)(ReB-RD-RH_2-eBH_1)}\right)^{\frac{\alpha-1}{\alpha}} \neq 0,$$

with $P' = \frac{dP}{dS}\Big|_{S=S_{1,2}}$ and $Q' = \frac{dQ}{dS}\Big|_{S=S_{1,2}}$.

As a result, the period-doubling bifurcation occurs at S. □

5. Chaos Control

It is worth noting that the chaos consists of many periodic points and orbits that depend massively on the initial conditions. Thus, the outcome of a chaotic system is unpredictable and uncertain. A control tool is necessary. Several techniques, such as the state feedback method, the hybrid control method, and the pole-placement technique [31–34] can be used to control a chaotic behavior of dynamic models. This part utilizes the state feedback process to control the chaos behavior of system (2.1). System (2.1) can be easily expressed as

$$\begin{cases} u_{i+1} = u_i e^{(R(1-u_i)-Bv_i-H_1)\frac{S^\alpha}{\alpha} - \mathcal{X}_n(u_n,v_n)}, \\ v_{i+1} = v_i e^{(eBu_i-D-H_2)\frac{S^\alpha}{\alpha}}, \end{cases} \tag{5.1}$$

where $\mathcal{X}_n(u_n,v_n) = \beta_1(x_n - x^*) + \beta_2(y_n - y^*)$ is the feedback controlling force and β_1, β_2 present feedback gains. The Jacobian matrix of system (5.1) is given by

$$C_{J(P_E)} = \begin{pmatrix} A_{11}-\beta_1 & A_{12}-\beta_2 \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \left(1 - \frac{R(D+H_2)S^\alpha}{eB\alpha}\right), \quad A_{12} = -\frac{(D+H_2)S^\alpha}{e\alpha},$$
$$A_{21} = \frac{(ReB-RD-RH_2-eBH_1)S^\alpha}{\alpha B}, \quad A_{22} = 1, \; A_{13} = -\frac{RS^\alpha}{\alpha}\left(2 - \frac{RSu^*}{\alpha}\right).$$

It is worth noting that the characteristic equation of Jacobian matrix $C_{J(P_E)}$ is

$$\bar{\gamma}^2 - (A_{11}+A_{22}-\beta_1)\bar{\gamma} + A_{22}(A_{11}-\beta_1) - A_{21}(A_{12}-\beta_2) = 0. \tag{5.2}$$

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be the roots of Eq. (5.2). Then, we have

$$\bar{\gamma}_1 + \bar{\gamma}_2 = A_{11} + A_{22} - 1, \text{ and } \bar{\gamma}_1\bar{\gamma}_2 = A_{22}(A_{11}-\beta_1) - A_{21}(A_{12}-\beta_2). \tag{5.3}$$

The lines of the marginal stability L_m^1, L_m^2 and L_m^3 are derived by solving $\bar{\gamma}_1\bar{\gamma}_2 = 1$, $\bar{\gamma}_1 = 1$ and $\bar{\gamma}_1 = -1$, respectively. These conditions guarantee that $|\bar{\gamma}_{1,2}| = 1$. Then, we derive the marginal stability lines as follows:

$$L_m^1: \quad A_{22}\beta_1 - A_{21}\beta_2 = A_{22}A_{11} - A_{12}A_{21} - 1,$$
$$L_m^2: \quad \beta_1(1 - A_{22}) + \beta_2 A_{21} = -1 - A_{22}A_{11} + A_{21}A_{12} + A_{11} + A_{22},$$
$$L_m^3: \quad \beta_1(1 + A_{22}) - \beta_2 A_{21} = 1 + A_{22}A_{11} - A_{21}A_{12} + A_{11} + A_{22}.$$

Therefore, L_m^1, L_m^2, and L_m^3 in (β_1, β_2)-plane give a triangular region which leads to $|\bar{\gamma}_{1,2}| < 1$.

6. Numerical Simulation

This section is added to present some critically numerical examples to show the theoretical results we have shown in this paper. The following fixed parameter values and initial conditions are used in the given figures.

Cases	**Fixed parameter**	**Initial conditions**	**Bifurcation parameters**
Case (1)	$R = 14.5$, $B = 15.26$, $H_1 = 1.14$, $D = 0.45$, $e = 0.6$, $H_2 = 1.08$ and $\alpha = 0.95$	$u_0 = 0.1671$ and $v_0 = 0.7167$	$S_0 = 0.1239$ and $0 \le S \le 0.24$
Case (2)	$R = 8.2$, $B = 6.5$, $H_1 = 1.79$, $D = 0.75$, $e = 0.49$, $H_2 = 1.32$ and $\alpha = 0.95$	$u_0 = 0.6499$ and $v_0 = 0.1663$	$S_1 = 0.3713$ $0 \le S \le 0.65$

Table 2. Parameter values and initial conditions.

Example 6.1. *This example uses the parameter values and initial conditions given in Case (1). Using these values, system* (2.1) *has a unique positive equilibrium point P_E. Therefore, the Jacobian matrix of system* (2.1) *calculated at P_E is giving by*

$$\mathcal{M}_J(P_E) = \begin{pmatrix} 0.6492 & -0.3692 \\ 0.9502 & 1.0000 \end{pmatrix}.$$

Here, the characteristic equation of the previous matrix is shown as follows:

$$\rho(\gamma) = \gamma^2 - 1.6492\gamma + 1 = 0\,,$$

whose eigenvalues are

$$\gamma_{1,2} = 0.8246 \mp 0.5658i\,,$$

with $|\gamma_{1,2}| = 1$. *This makes sure that model* (2.1) *faces a Neimark-Sacker bifurcation at P_E as bifurcation parameter S passes through $S_0 = 0.1239$. The bifurcation behaviors for both prey and predator populations are presented in Figs 1a and 1b, respectively, while the maximum Lyapunov exponent is depicted in Fig 1c. From the bifurcation plots, it is obvious that the equilibrium point is locally asymptotically stable when $S < 0.1239$, and loses its stability when $S = S_0 = 0.1239$. As a result, a closed invariant curve is formed* *around the equilibrium point. When the considered model passes the value* $S = S_0$*, the point* P_E *becomes unstable and periodic orbits appear at* $0.148 < S < 0.174$ *(see Figs 2). Figs. 3 and 4 represent some phase portraits of system (2.1). Other phase portraits are shown in 3a and 3c for the values* $S = 0.021$ *and* $S = 0.121$*, respectively. In this figures, the equilibrium point* P_E *is stable. At* $S = 0.1239$*, a closed invariant curve establishes around the point* P_E *as can be seen in Fig. 4a. The point* P_E *becomes unstable at* $S = 0.12408$*, as all orbits (with initial conditions inside and outside the closed invariant curve) travel toward the closed invariant curve (see Fig. 4c). Lastly, chaotic attractors are plotted in Figs. 5a and 5b when* $S = 0.19$ *and* $S = 0.24$*, respectively.*

Example 6.2. *We take the parameter values and initial condition of Case (2). For these parametric values the system* (2.1) *has a unique positive equilibrium point* P_E*. The Jacobian matrix of system* (2.1) *evaluated at* P_E *is giving by:*

$$\mathcal{M}_J(P_E) = \begin{pmatrix} -1.1886 & -1.7349 \\ \\ 0.2175 & 1.0000 \end{pmatrix}.$$

whose characteristic equation is

$$\rho(\gamma) = \gamma^2 + 0.1886\gamma - 0.8114 = 0\,.$$

The eigenvalues are given by

$$\gamma_1 = 1, \quad \gamma_2 = 0.8114\,,$$

and $|\gamma_2| \neq 1$*. Furthermore*

$$\begin{aligned} &\Delta_0^{\mp}(S) = 1 > 0, \\ &\Delta_1^{+}(S) = 1 + \mathcal{D} = 0.1886 > 0, \\ &\rho(-1) = 1 + T + D = 0, \\ &\rho(1) = 1 - P + Q = 0.3773 > 0, \end{aligned}$$

that confirms that the system (2.1) *experiencing period-doubling bifurcation at positive equilibrium point* P_E *as bifurcation parameter* S *passes through* $S_1 = 0.3713$*. It can be noted from the bifurcation diagrams of* u_n *and* v_n *(shown in Fig. 6a and Fig. 6b, respectively) that the positive equilibrium point* P_E *of system* (2.1) *is stable for* $0 < S < 0.3713$ *while this point loses its stability through a period doubling bifurcation when* $S \geq 0.3713$*. The maximum Lyapunov exponents associated with bifurcation diagrams is drawn in Fig. 6c. This surely verifies the existence of the chaotic behavior and period orbits in the parametric space. From Fig. 6c we note that some "Maximal LE" values are positive and some of them are negative. Thus, there exists a stable equilibrium point or stable period orbits in the chaotic region. Moreover, there is a period doubling cascade in orbits of periods-*$2, 4, 8, 16$ *(see Figs. 7b, 7c, 7d).*

Example 6.3. *In this example, we apply a state feedback control method by choosing the parameter values* $R = 14.5$, $B = 15.26$, $H_1 = 1.14$, $D = 0.45$, $e = 0.6$, $H_2 = 1.08$, $\alpha = 0.95$*, and* $S = 0.126$ *and the initial* *conditions* $(u_0, v_0) = (0.167, 0.716)$. *Hence, the corresponding controlled form is given by*

$$\begin{cases} u_{i+1} = u_i Exp\left((14.5(1-u_i) - 15.26v_i - 1.14)\,0.1471 - \beta_1(u_i - 0.167) - \beta_2\left(v_i - 0.716\right)\right), \\ v_{i+1} = v_i Exp\left((9.15u_i - 1.53)\,0.1471\right). \end{cases} \tag{6.1}$$

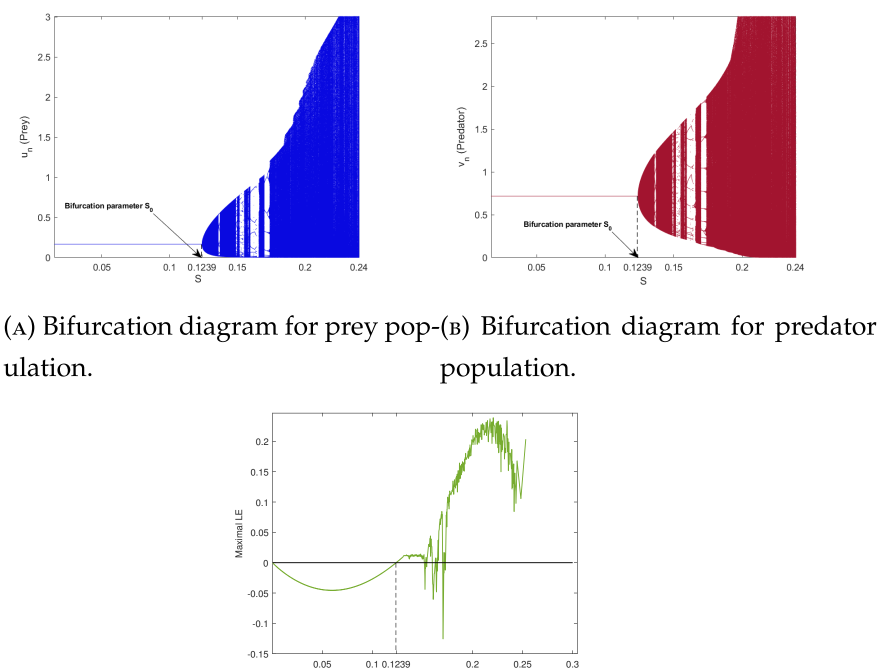
The Jacobian matrix for system (6.1) *is given by*

$$C_{J(P_E)} = \begin{pmatrix} 0.6492 - \beta_1 & -0.3692 - \beta_2 \\ 0.9502 & 1.0000 \end{pmatrix},$$

Furthermore, the lines of marginal stability L_m^1, L_m^2, *and* L_m^3 *are provided by*

$$\begin{aligned} L_m^1 :&\quad \beta_1 - 0.9502\xi_2 = -4.4409 \times 10^{-9}, \\ L_m^2 :&\quad 0.9502\beta_2 = -0.3508, \\ L_m^3 :&\quad 2\beta_1 - 0.9502\beta_2 = 3.6492, \end{aligned}$$

The marginal lines L_m^1, L_m^2 *and* L_m^3 *specify a triangular region (called the region of stability) in the* (β_1, β_2)*-plane. This region is bounded by these lines of the controlled system (6.1) which are plotted in Fig. 7a. When* $\beta_1 = 1.9$ *and* $\beta_2 = 1.36$*, the equilibrium point* P_E *is locally asymptotically stable (see Fig. 8b) for the controlled system (6.1). However, the point* P_E *becomes unstable in the original system (2.1) (see Figure 9).*

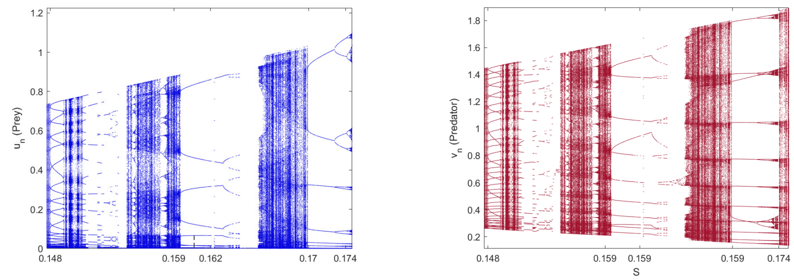


(A) Bifurcation diagram for prey population.

(B) Bifurcation diagram for predator population.

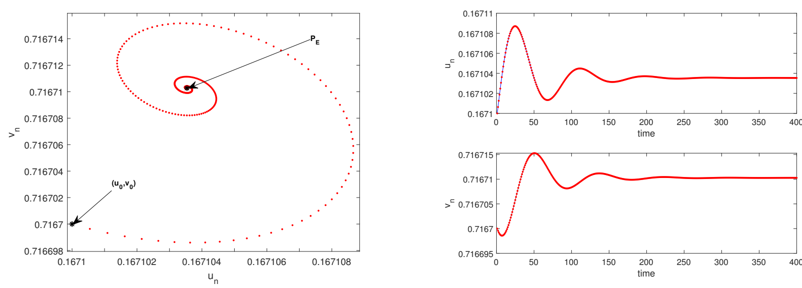
(C) Maximum Lyapunov exponent.

FIGURE 1. Graphs (a) and (b) illustrate the Neimark-Sacker bifurcation of system (2.1) while graph (c) presents the maximum Lyapunov exponent for system (2.1) at the parameter values and condition of Case (1) in Table 2.

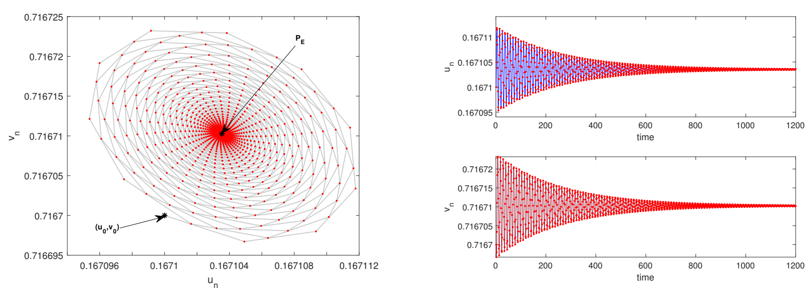


(A) (B)

Figure 2. Graphs (a) and (b) illustrate the Neimark-Sacker bifurcation of system (2.1) at the parameter values and condition of Case (1) in Table 2 and $S \in [0.148, 0.174]$.

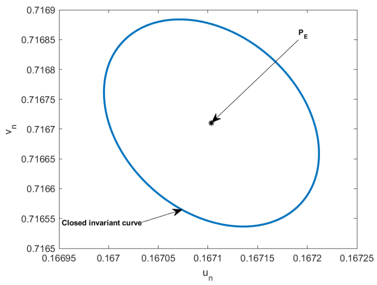


(A) Phase portrait of model (2.1) when $S = 0.012$. (B) Time evolution of u_n and v_n when $S = 0.012$.

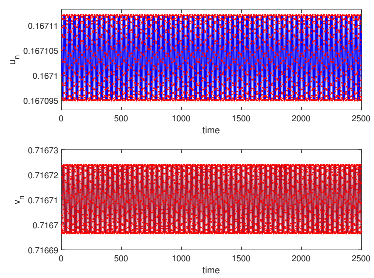


(C) Phase portrait of model (2.1) when $S = 0.121$. (D) Time evolution of u_n and v_n when $S = 0.121$.

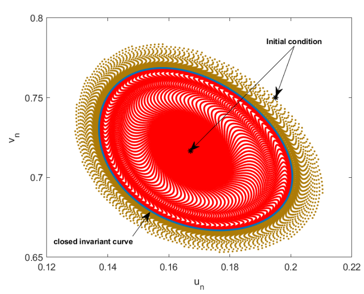
Figure 3. Phase portraits for $S = 0.012, 0.121$ of system (2.1) and time evolution of prey and predator at the parameter values and condition of Case (1) in Table 2.



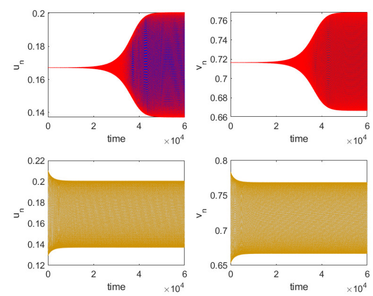
(A) Phase portrait of model (2.1) when $S = 0.1239$.



(B) Time evolution of u_n and v_n when $S = 0.1239$.

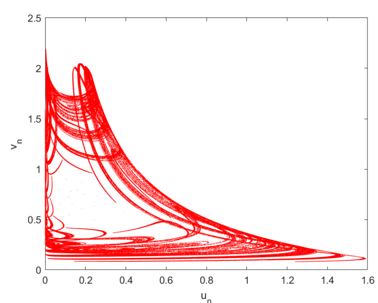


(C) Phase portrait of model (2.1) when $S = 0.12408$.

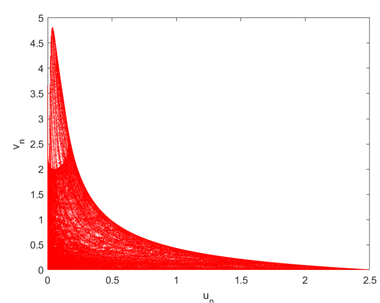


(D) Time evolution of u_n and v_n when $S = 0.12408$.

FIGURE 4. Phase portraits for $S = 0.1239, 0.12408$ of system (2.1) and time evolution of prey and predator at the parameter values and the conditions of Case (1) in Table 2.

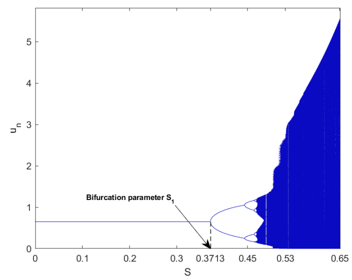


(A)

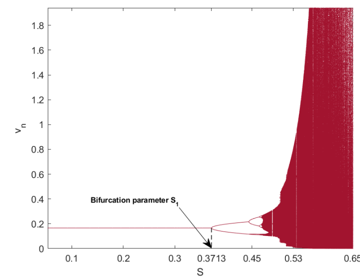


(B)

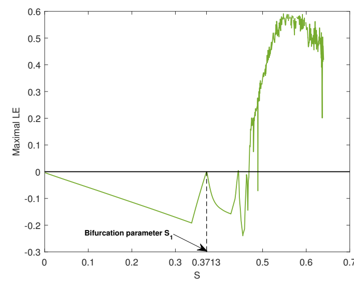
FIGURE 5. Phase portraits for $S = 0.203, 0.24$ of system (2.1) at the parameter values and the conditions of Case (1) in Table 2.



(a) Bifurcation diagram for prey population.



(b) Bifurcation diagram for predator population.



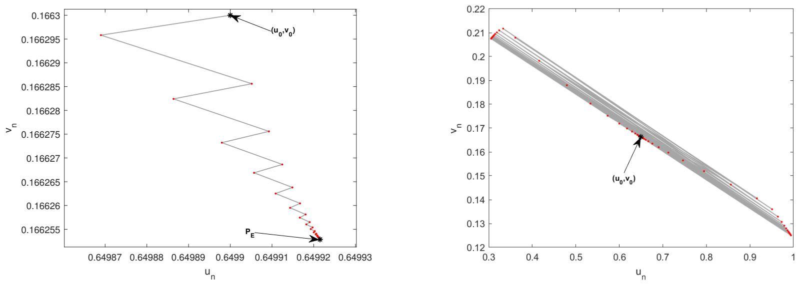
(c) Maximum Lyapunov exponent.

Figure 6. Graphs (a) and (b) illustrate the period-doubling bifurcation of system (2.1) while graph (c) presents the maximum Lyapunov exponent for system (2.1) at the parameter values and the conditions of Case (2) in Table 2.

7. Conclusion

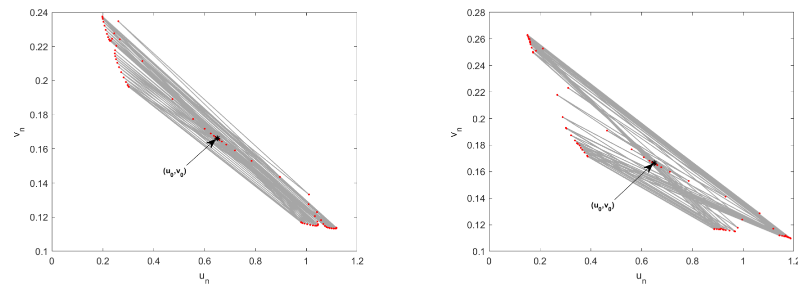
To sum up, this paper has investigated the dynamical behaviors of fractional-order Lotka-Volterra model with a harvesting effect using stability and bifurcation theory. In particular, we have obtained the fixed points, local stability, bifurcation and some numerical schemes for system (1.1). We have found that the considered system has three main equilibrium points in which one of them is positive. In Lemmas 3.3 and 3.4, we have analysed the stability of the first two equilibrium points while Lemma 3.5 shows the stability of the positive fixed point. In Theorem 4.1, we have proved the existence of the Neimark-Sacker bifurcation of system (2.1) which is occurred at the point P_E under certain conditions given in this theorem. For instance, Example 6.1 has used the values of the parameters given in Case (1) to verify that model (2.1) goes through a Neimark-Sacker bifurcation at P_E as the bifurcation parameter S passes through $S_0 = 0.1239$. This can be clearly seen in in Figs 1a and 1b. In Fig 1c, we have plotted the maximum Lyapunov exponent. In Figs. 3 and 4, we have demonstrated some phase portraits of system (2.1) for various values of the parameter S. It can be observed that the trajectories form a limit cycle when the parameter S increases. Theorem (4.2) has illustrated that system (2.1) faces a period-doubling bifurcation at the positive equilibrium point P_E when some conditions, given in Theorem (4.2), are satisfied. Moreover, the phase portraits of system (2.1) and its time evolution are plotted in under the coefficients $R = 14.5$, $B = 15.26$,

$H_1 = 1.14$, $D = 0.45$, $e = 0.6$, $H_2 = 1.08$, $\alpha = 0.95$, and $S = 0.126$. The used techniques can be utilized to be applied on other fractional-order models with some effects.



(A) Phase portrait of model (2.1) when $S = 0.33$.

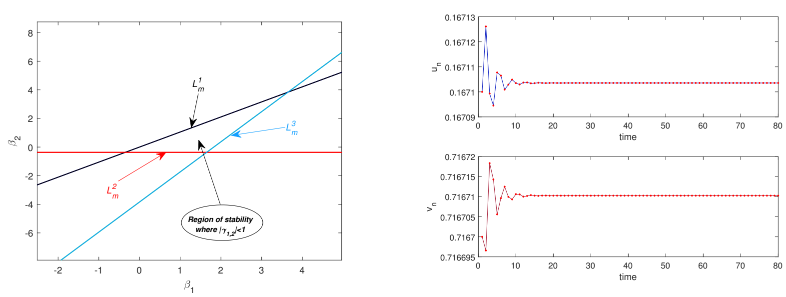
(B) Phase portrait of model (2.1) when $S = 0.42$.



(C) Phase portrait of model (2.1) when $S = 0.456$.

(D) Phase portrait of model (2.1) when $S = 0,4673$.

FIGURE 7. Phase portraits for $S = 0.33$ (Stability), 0.42 (Period-2), 0.456 (Period-4) and 0.4673 (Period-8) of the system (2.1) and time evolution of prey at the parameter values and the conditions of Case (2) in Table 2.



(A) The region of stability in (β_1, β_2)-plane.

(B) Time evolution of u_n and v_n for $(\beta_1, \beta_2) = (1.9, 1.36)$

.

FIGURE 8. Triangular stability region bounded by L_m^1, L_m^2 and L_m^3 of the controlled system (6.1) and time evolution of prey and predator for $(\beta_1, \beta_2) = (1.9, 1.36)$.

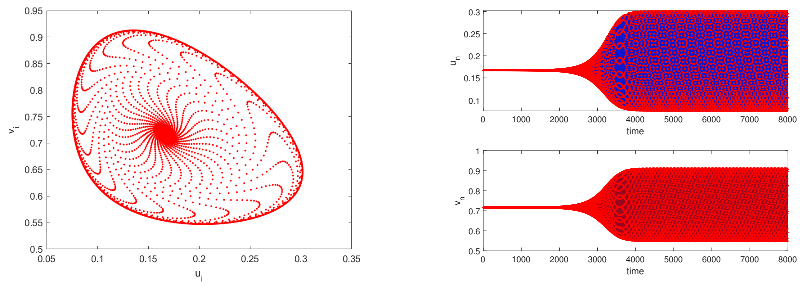


Figure 9. Phase portraits for $R = 14.5$, $B = 15.26$, $H_1 = 1.14$, $D = 0.45$, $e = 0.6$, $H_2 = 1.08$, $\alpha = 0.95$, and $S = 0.126$ of system (2.1), and time evolution of prey and predator for system (2.1).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.J. Lotka, Elements of Physical Biology, Williams & Wilkins, 1925.

[2] V. Volterra, Variazioni e fluttuazioni del numero di individui in specie animali conviventi, Mem. Acad. Lincei. 2 (1926), 31–113.

[3] M.J. Uddin, S.Md.S. Rana, S. Işık, F. Kangalgil, On the Qualitative Study of a Discrete Fractional Order Prey–predator Model With the Effects of Harvesting on Predator Population, Chaos Solitons Fractals. 175 (2023), 113932. https://doi.org/10.1016/j.chaos.2023.113932.

[4] J. Lee, H. Baek, Dynamics of a Beddington-Deangelis Type Predator-Prey System With Constant Rate Harvesting, Electron. J. Qual. Theory Differ. Equ. 2017 (2017), 1–20. https://doi.org/10.14232/ejqtde.2017.1.1.

[5] X. Liu, Q. Huang, Comparison and Analysis of Two Forms of Harvesting Functions in the Two-Prey and One-Predator Model, J. Inequal. Appl. 2019 (2019), 307. https://doi.org/10.1186/s13660-019-2260-y.

[6] B. Sahoo, B. Das, S. Samanta, Dynamics of Harvested-Predator-Prey Model: Role of Alternative Resources, Model. Earth Syst. Environ. 2 (2016), 140. https://doi.org/10.1007/s40808-016-0191-x.

[7] M. Sen, P.D.N. Srinivasu, M. Banerjee, Global Dynamics of an Additional Food Provided Predator–prey System With Constant Harvest in Predators, Appl. Math. Comput. 250 (2015), 193–211. https://doi.org/10.1016/j.amc.2014.10.085.

[8] Q. Lin, Dynamic Behaviors of a Commensal Symbiosis Model With Non-Monotonic Functional Response and Non-Selective Harvesting in a Partial Closure, Commun. Math. Biol. Neurosci. 2018 (2018), 4. https://doi.org/10.28919/cmbn/3652.

[9] A. Das, M. Pal, Theoretical Analysis of an Imprecise Prey-Predator Model with Harvesting and Optimal Control, J. Optim. 2019 (2019), 9512879. https://doi.org/10.1155/2019/9512879.

[10] N. Mohdeb, A Dynamic Analysis of a Prey–predator Population Model With a Nonlinear Harvesting Rate, Arab J. Math. Sci. (2023). https://doi.org/10.1108/ajms-03-2022-0052.

[11] M.B. Almatrafi, M. Berkal, Bifurcation Analysis and Chaos Control for Prey-Predator Model With Allee Effect, Int. J. Anal. Appl. 21 (2023), 131. https://doi.org/10.28924/2291-8639-21-2023-131.

[12] M.B. Almatrafi, Construction of Closed Form Soliton Solutions to the Space-Time Fractional Symmetric Regularized Long Wave Equation Using Two Reliable Methods, Fractals. 31 (2023), 2340160. https://doi.org/10.1142/s0218348x23401606.

[13] M.B. Almatrafi, Solitary Wave Solutions to a Fractional Model Using the Improved Modified Extended Tanh-Function Method, Fractal Fract. 7 (2023), 252. https://doi.org/10.3390/fractalfract7030252.

[14] M. Berkal, J.F. Navarro, Qualitative Behavior of a Two-dimensional Discrete-time Prey-predator Model, Comput. Math. Methods. 3 (2021), e1193. https://doi.org/10.1002/cmm4.1193.

[15] A.Q. Khan, M.B. Almatrafi, Two-Dimensional Discrete-Time Laser Model With Chaos and Bifurcations, AIMS Math. 8 (2023), 6804–6828. https://doi.org/10.3934/math.2023346.

[16] A.Q. Khan, S.A.H. Bukhari, M.B. Almatrafi, Global Dynamics, Neimark-Sacker Bifurcation and Hybrid Control in a Leslie's Prey-Predator Model, Alexandria Eng. J. 61 (2022), 11391–11404. https://doi.org/10.1016/j.aej.2022.04.042.

[17] A.Q. Khan, F. Nazir, M.B. Almatrafi, Bifurcation Analysis of a Discrete Phytoplankton-Zooplankton Model With Linear Predational Response Function and Toxic Substance Distribution, Int. J. Biomath. 16 (2022), 2250095. https://doi.org/10.1142/s1793524522500954.

[18] A.Q. Khan, M. Tasneem, M.B. Almatrafi, Discrete-Time COVID-19 Epidemic Model With Bifurcation and Control, Math. Biosci. Eng. 19 (2021), 1944–1969. https://doi.org/10.3934/mbe.2022092.

[19] R. Ahmed, M.B. Almatrafi, Complex Dynamics of a Predator-Prey System With Gompertz Growth and Herd Behavior, Int. J. Anal. Appl. 21 (2023), 100. https://doi.org/10.28924/2291-8639-21-2023-100.

[20] S. Kartal, F. Gurcan, Discretization of Conformable Fractional Differential Equations by a Piecewise Constant Approximation, Int. J. Comput. Math. 96 (2019), 1849–1860. https://doi.org/10.1080/00207160.2018.1536782.

[21] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A New Definition of Fractional Derivative, J. Comput. Appl. Math. 264 (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002.

[22] T. Abdeljawad, On Conformable Fractional Calculus, J. Comput. Appl. Math. 279 (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016.

[23] Y. Liu, X. Li, Dynamics of a Discrete Predator-Prey Model With Holling-II Functional Response, Int. J. Biomath. 14 (2021), 2150068. https://doi.org/10.1142/S1793524521500686.

[24] M. Berkal, M.B. Almatrafi, Bifurcation and Stability of Two-Dimensional Activator-Inhibitor Model With Fractional-Order Derivative, Fractal Fract. 7 (2023), 344. https://doi.org/10.3390/fractalfract7050344.

[25] M. Berkal, J.F. Navarro, Qualitative Study of a Second Order Difference Equation, Turk. J. Math. 47 (2023), 516–527. https://doi.org/10.55730/1300-0098.3375.

[26] Q. Din, M.A. Zulfiqar, Qualitative Behavior of a Discrete Predator-Prey System Under Fear Effects, Z. Naturforsch. A. 77 (2022), 1023–1043. https://doi.org/10.1515/zna-2022-0129.

[27] Q. Din, Controlling Chaos in a Discrete-Time Prey-Predator Model With Allee Effects, Int. J. Dyn. Control. 6 (2018), 858–872. https://doi.org/10.1007/s40435-017-0347-1.

[28] Q. Din, Neimark-Sacker Bifurcation and Chaos Control in Hassell-Varley Model, J. Differ. Equ. Appl. 23 (2017), 741–762. https://doi.org/10.1080/10236198.2016.1277213.

[29] R. Ma, Y. Bai, F. Wang, Dynamical Behavior Analysis of a Two-Dimensional Discrete Predator-Prey Model With Prey Refuge and Fear Factor, J. Appl. Anal. Comput. 10 (2020), 1683–1697. https://doi.org/10.11948/20190426.

[30] S.Md.S. Rana, Md.J. Uddin, Dynamics of a Discrete-Time Chaotic Lü System, Pan-Amer. J. Math. 1 (2022), 7. https://doi.org/10.28919/cpr-pajm/1-7.

[31] G. Chen, X. Dong, From Chaos to Order: Methodologies, Perspectives and Applications, World Scientific, 1998.

[32] S.N. Elaydi, An Introduction to Difference Equations, Springer, New York, 1996.

[33] S. Lynch, Dynamical Systems With Applications Using Mathematica, Springer, 2007.

[34] S. Vinoth, R. Sivasamy, K. Sathiyanathan, B. Unyong, R. Vadivel, N. Gunasekaran, A Novel Discrete-Time Leslie-Gower Model with the Impact of Allee Effect in Predator Population, Complexity. 2022 (2022), 6931354. https://doi.org/10.1155/2022/6931354.

[35] M. Imran, M.B. Almatrafi, R. Ahmed, Stability and Bifurcation Analysis of a Discrete Predator-prey System of Ricker Type With Harvesting Effect, Commun. Math. Biol. Neurosci. 2024 (2024), 11. https://doi.org/10.28919/cmbn/8313.