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Solving a Nonlinear Fractional Integral Equation by Fixed Point Approaches Using Auxiliary Functions Under Measure of Noncompactness

Hasanen A. Hammad[1,2], Hassen Aydi[3,4,*], Manuel De la Sen[5]

[1]*Department of Mathematics, College of Sciences, Qassim University, Buraydah 52571, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3]*Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, 4000, Tunisia*
[4]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[5]*Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, 48940-Leioa (Bizkaia), Spain*

[*]*Corresponding author:* hassen.aydi@isima.rnu.tn

Abstract. This manuscript is devoted to ensure the existence of a solution to nonlinear fractional integral equations with three variables under a measure of noncompactness. In order to accomplish our main goal, we develop a new fixed point theorem that generalizes Darbo's fixed point theorem by utilizing a measure of noncompactness and a new contraction operator. A related tripled FP theorem is also obtained. Finally, we use this generalized Darbo's fixed point theorem to solve a nonlinear fractional integral equation involving three variables, and an example to demonstrate our results is presented.

1. Introduction

The study of derivatives and integrals of any order using the Gamma function is known as fractional calculus (FC). In applied mathematics and mathematical analysis, a derivative of any non-integer order, real or complex, is known as a fractional derivative. In a letter to Antoine de l'Hopital from G.W. Leibniz in the sixteenth century, the first instance is documented [1]. FC was used in one of N. H. Abel's early studies [2], where the following components can be taken into consideration: Integration and differentiation (ID) of fractional orders are defined; their relationship is strictly inverse; the ID of fractional orders can be perceived as part of the same

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generalized operation; and the ID of ambiguous real orders can be expressed coherently. For more details about the contributions of fixed point theory in many directions, see [3–11].

The theory and applications of FC have advanced significantly during the course of the nineteenth and early twentieth centuries, and innumerable authors have contributed their interpretations of fractional derivatives and integrals. Numerous areas of mathematics, including porous media, viscoelasticity, and electrochemistry, utilize the Erdélyi-Kober fractional integrals; for more information, see [12, 13].

Numerous IEs can be solved using fixed point (FP) theory and the measure of noncompactness (MNC) to address a variety of real-world issues. See, for example, [14–23]. It is crucial to learn these types of equations because of the significance of integral equations (IEs) of fractional order.

Many scholars [17,21,24,25] have used Darbo's fixed point theorem (FPT) and its generalizations that incorporate the idea of MNC to examine both differential equations and IEs. Several academics have recently generalized Darbo's FPT, as seen in [15, 21, 26], by using various types of operator contraction. Through the use of weak JS-contractions in Banach spaces (BSs), Işik et al. [27] have expanded Darbo's FPT. They have also derived the coupled FP theorem and used it to investigate the existence of solutions for a set of IEs. Prompted by these works, we generalize Darbo's FPT using a new operator that is defined with the aid of a function used in [28] and apply it to a generalized fractional integral equation (FIE) of three variables to check the resolvability.

2. Basic concepts

In this section, we provide notations, definitions, and other information to aid in discussion of our main findings. Let $(\Theta, \|.\|)$ be a real BS. From now on, we denote $\Xi\left[\omega, z_0\right]$, $\overline{\mathfrak{I}}$, $Con\ \mathfrak{I}$, $\mathbb{R}_+$, $\mathbb{N}^*$, $\emptyset$, χ_Θ and $\wp_\Theta$ by the closed ball with center ω and radius z_0 in Θ, the closure of a subset $\mathfrak{I}$ of Θ, the convex hull of a subset $\mathfrak{I}$, the set of all positive real numbers, the set of all natural numbers without zero, the empty set, the class of all non-empty bounded subsets of Θ, and the subfamily of all relatively compact subsets, respectively.

Definition 2.1. *[29] A function $\Upsilon : \chi_\Theta \to \mathbb{R}_+$ is said to be a MNC in Θ if the assertions below are true:*

(i) *for all $\mathfrak{I} \in \chi_\Theta$, we get $\Upsilon(\mathfrak{I}) = 0$, which yields $\mathfrak{I}$ is relatively compact;*
(ii) *$ker(\Upsilon) = \{\mathfrak{I} \in \chi_\Theta : \Upsilon(\mathfrak{I}) = 0\} \neq \emptyset$ and $ker(\Upsilon) \subset \wp_\Theta$;*
(iii) *$\mathfrak{I} \subseteq \mathfrak{I}_1$ implies $\Upsilon(\mathfrak{I}) \leq \Upsilon(\mathfrak{I}_1)$;*
(iv) *$\Upsilon\left(\overline{\mathfrak{I}}\right) = \Upsilon(\mathfrak{I})$;*
(v) *$\Upsilon(con\ \mathfrak{I}) = \Upsilon(\mathfrak{I})$;*
(vi) *for all $\rho \in [0, 1]$, $\Upsilon(\rho\mathfrak{I} + (1-\rho)\mathfrak{I}_1) \leq \rho\Upsilon(\mathfrak{I}) + (1-\rho)\Upsilon(\mathfrak{I}_1)$;*
(vii) *if $\mathfrak{I}_m \in \chi_\Theta$, $\Upsilon\left(\overline{\mathfrak{I}}\right) = \Upsilon(\mathfrak{I})$, $\mathfrak{I}_{m+1} \subseteq \mathfrak{I}_m$, $m = 1, 2, \ldots$, and that $\lim_{m\to\infty}\Upsilon(\mathfrak{I}_m) = 0$, then $\mathfrak{I}_\infty = \cap_{m=1}^{\infty}\mathfrak{I}_m \neq \emptyset$.*

Remark 2.1. *The kernel of a MNC Υ is denoted by $ker(\Upsilon)$. Further, $\mathfrak{I}_\infty \in ker(\Upsilon)$ and $\Upsilon(\mathfrak{I}_\infty) \leq \Upsilon(\mathfrak{I}_m)$ for $m \geq 1$, we get $\Upsilon(\mathfrak{I}_\infty) = 0$. Thus, $\mathfrak{I}_\infty \in ker(\Upsilon)$.*

Theorem 2.1. *[30] (Schauder) Let Θ be a BS and $\Omega \subset \Theta$ be nonempty, bounded, closed, and convex (NBCC). If the mapping $\xi : \Omega \to \Omega$ is compact continuous, then it owns at least one FP.*

Darbo's FPT, which is a generalization of the previous theorem, is stated as follows:

[31] (Darbo theorem) Assume that Υ is a MNC and Ω is a NBCC subset of a BS Θ. Let $\xi : \Omega \to \Omega$ be a continuous map. Then ξ possesses a FP in Ω if the inequality below holds:

$$\Upsilon(\xi Z) \leq \nu \Upsilon(Z), \ Z \subset \Omega, \text{ for all } \nu \in [0,1).$$

In order to determine Darbo's FPT extension, the following associated ideas should be remembered:

Definition 2.2. *[28] Assume that Φ is the set of all functions $\aleph : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(a) *for $\zeta, \sigma \geq 0$, $\max\{\zeta, \sigma\} \leq \aleph(\zeta, \sigma)$;*
(b) *$\aleph$ is nondecreasing and continuous;*
(c) *for $\zeta, \sigma, \zeta^*, \sigma^* \geq 0$, $\aleph(\zeta + \sigma, \zeta^* + \sigma^*) \leq \aleph(\zeta, \zeta^*) + \aleph(\sigma, \sigma^*)$.*

For example, take $\aleph(\zeta, \sigma) = \zeta + \sigma$, then $\aleph \in \Phi$.

Definition 2.3. *[26] Assume that Ψ is the set of all functions $\beta : \mathbb{R}_+ \to [1, \infty)$ such that*

(i) $\lim_{m\to\infty} \beta(\zeta_m) = 1$ *iff* $\lim_{m\to\infty} \zeta_m = 0$, *for every* $\{\zeta_m\} \subset \mathbb{R}_+$,
(ii) *the function β is strictly increasing and continuous.*

For example, consider $\beta(\zeta) = e^{\zeta}$, then $\beta \in \Psi$.

Definition 2.4. *Assume that Π is the set of all functions $\pi : [1, \infty) \to \mathbb{R}_+$ so that*

(1) $\lim_{m\to\infty} \pi(\zeta_m) = 0$ *iff* $\lim_{m\to\infty} \zeta_m = 1$, *for each* $\{\zeta_m\} \subset [1, \infty)$,
(2) $\pi(1) = 0$;
(3) *π is continuous.*

For example, suppose the following:

- $\pi_1(\zeta) = \ln(\zeta)$,
- $\pi_2(\zeta) = \zeta - \zeta^{\frac{1}{m}}$, $m \geq 1$,
- $\pi_3(\zeta) = e^{\zeta - 1} - 1$.

Clearly, $\pi_1, \pi_2, \pi_3 \in \Pi$.

Definition 2.5. *Suppose that Δ is a completes BS and $\ell(\Delta)$ is a Banach algebra of all linear continuous mappings. The mapping $Q : [0, \infty) \to \ell(\Delta)$ is called a strongly continuous semi-group on Δ if the assertions below are true:*

(a_1) *for all $\zeta \in \Delta$, $Q(.)\zeta$ is continuous on $[0, \infty)$;*
(a_2) *for each $\zeta, \zeta^* \geq 0$, $S(0) = I$ (where I is the identity mapping) and $S(\zeta + \zeta^*) = S(\zeta)S(\zeta^*)$.*

Definition 2.6. *[29] Let $\hbar$ be a non-empty set and $P : \hbar \times \hbar \times \hbar \to \hbar$ be a given mapping. A trio $\left(\zeta, \widetilde{\zeta}, \widehat{\zeta}\right) \in \hbar \times \hbar \times \hbar$ is called a tripled fixed point (TFP) of P if $\zeta = P\left(\zeta, \widetilde{\zeta}, \widehat{\zeta}\right)$, $\widetilde{\zeta} = P\left(\widetilde{\zeta}, \widehat{\zeta}, \zeta\right)$, and $\widehat{\zeta} = P\left(\widehat{\zeta}, \zeta, \widetilde{\zeta}\right)$.*

3. Main results

We begin this part with the following theorem:

Theorem 3.1. *Assume that Ω is a NBCC subset of a BS Θ and $\eth : \Omega \to \Omega$ is a continuous mapping satisfying*

$$\beta\left[\aleph\left(\Upsilon\left(\eth U\right), \sigma\left(\Upsilon\left(\eth U\right)\right)\right)\right] \leq \frac{\beta\left[\aleph\left(\Upsilon\left(U\right), \sigma\left(\Upsilon\left(U\right)\right)\right)\right]}{1+\beta\left[\aleph\left(\Upsilon\left(U\right), \sigma\left(\Upsilon\left(U\right)\right)\right)\right]} - \pi\left[\beta\left[\aleph\left(\Upsilon\left(U\right), \sigma\left(\Upsilon\left(U\right)\right)\right)\right]\right], \quad (3.1)$$

for all $U \subseteq \Omega$, $\beta \in \Psi$, $\pi \in \Pi$, $\aleph \in \Phi$, where $\Upsilon : \chi_\Omega \to \mathbb{R}_+$ is an arbitrary MNC and $\sigma : [0,\infty) \to [0,\infty)$ is a continuous mapping. Then $\eth$ owns at least one FP in Ω.

Proof. Consider a sequence $\{\Omega_m\}$ with $\Omega_0 = \Omega$ and $\Omega_{m+1} = con\left(\eth\Omega_m\right)$, for all $m \geq 0$.

Additionally,

$$\eth\Omega_0 = \eth\Omega \subseteq \Omega = \Omega_0 \text{ and } \Omega_1 = con\left(\eth\Omega_0\right) \subseteq \Omega = \Omega_0.$$

Continuing with the same approach we find that

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_m \supseteq \Omega_{m+1} \supseteq \cdots .$$

If $\Upsilon(\Omega_i) = 0$, for all $i \in \mathbb{N}$, then Ω_i is compact and by Schauder (Theorem 2.1), $\eth$ has a FP, and the proof is finished. So, let $\Upsilon(\Omega_m) > 0$, for some $m \in \mathbb{N}$. Obviously, the nonnegative sequence $\{\Upsilon(\Omega_m)\}$ is bounded below and decreasing, hence it is convergent to s (say), i.e., $\lim_{m\to\infty} \Upsilon(\Omega_m) = s \geq 0$. Also, $\Upsilon(\Omega_{m+1}) = \Upsilon\left(con\left(\eth\Omega_m\right)\right) = \Upsilon\left(\eth\Omega_m\right)$. From (3.1), one has

$$\begin{aligned} &\beta\left(\aleph\left(\Upsilon\left(\Omega_{m+1}\right), \sigma\left(\Upsilon\left(\Omega_{m+1}\right)\right)\right)\right) = \beta\left(\aleph\left(\Upsilon\left(\eth\Omega_m\right), \sigma\left(\Upsilon\left(\eth\Omega_m\right)\right)\right)\right) \\ \leq\ & \frac{\beta\left[\aleph\left(\Upsilon\left(\Omega_m\right), \sigma\left(\Upsilon\left(\Omega_m\right)\right)\right)\right]}{1+\beta\left[\aleph\left(\Upsilon\left(\Omega_m\right), \sigma\left(\Upsilon\left(\Omega_m\right)\right)\right)\right]} - \pi\left[\beta\left[\aleph\left(\Upsilon\left(\Omega_m\right), \sigma\left(\Upsilon\left(\Omega_m\right)\right)\right)\right]\right]. \end{aligned}$$

Letting $m \to \infty$, and assume that $s > 0$ (if possible), we get

$$\begin{aligned} \beta\left(\aleph\left(s, \sigma\left(s\right)\right)\right) &\leq \frac{\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right]}{1+\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right]} - \pi\left[\aleph\left(s, \sigma\left(s\right)\right)\right] \\ &\leq \beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right] - \pi\left[\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right]\right], \end{aligned}$$

which implies that $\pi\left[\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right]\right] \leq 0$. Hence, $\pi\left[\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right]\right] = 0$, from the definition of π, we have $\beta\left[\aleph\left(s, \sigma\left(s\right)\right)\right] = 1$. Using the definition of β, we conclude that $\aleph\left(s, \sigma\left(s\right)\right) = 0$, which yields $\lim_{m\to\infty} \Upsilon(\Omega_m) = 0$. As $\Omega_m \supseteq \Omega_{m+1}$, thanks to Definition 2.1, $\Omega_\infty = \cap_{m=1}^{\infty} \Omega_m$ is a NBCC subset of Ω and Ω_∞ is $\eth-$invariant. Thus, from Theorem 2.1, we conclude that $\eth$ owns at least one FP in $\Omega_\infty \subseteq \Omega$. □

Remark 3.1. *By employing a novel contraction operator that uses MNC to analyze operators with features that fall somewhere between those of contraction and compact mappings, we have expanded the scope of Darbo's FPT. The primary benefit of this generalization utilizing MNC is the relaxation of the compactness of the operator's domain, which is crucial for Shauder's theorem.*

Corollary 3.1. *Suppose that Ω is a NBCC subset of a BS Θ and $\Game : \Omega \to \Omega$ is a continuous mapping verifying*

$$\beta\left[\Upsilon\left(\Game U\right)+\sigma\left(\Upsilon\left(\Game U\right)\right)\right] \leq \frac{\beta\left[\Upsilon\left(U\right)+\sigma\left(\Upsilon\left(U\right)\right)\right]}{1+\beta\left[\Upsilon\left(U\right)+\sigma\left(\Upsilon\left(U\right)\right)\right]}-\pi\left[\beta\left[\Upsilon\left(U\right)+\sigma\left(\Upsilon\left(U\right)\right)\right]\right],$$

for all $U \subseteq \Omega$, $\beta \in \Psi$, $\pi \in \Pi$, where $\Upsilon : \chi_\Omega \to \mathbb{R}_+$ is an arbitrary MNC and $\sigma : [0,\infty) \to [0,\infty)$ is a continuous mapping. Then $\Game$ owns at least one FP in Ω.

Proof. The result follows immediately, if we take $\aleph(a,b) = a+b$ in Theorem 3.1. □

Corollary 3.2. *Let Ω be a NBCC subset of a BS Θ and $\Game : \Omega \to \Omega$ be a continuous mapping fulfilling*

$$\beta\left(\Upsilon\left(\Game U\right)\right) \leq \frac{\beta\left(\Upsilon\left(U\right)\right)}{1+\beta\left(\Upsilon\left(U\right)\right)}-\pi\left(\beta\left(\Upsilon\left(U\right)\right)\right),$$

for all $U \subseteq \Omega$, $\beta \in \Psi$, $\pi \in \Pi$, where $\Upsilon : \chi_\Omega \to \mathbb{R}_+$ is an arbitrary MNC. Then $\Game$ owns at least one FP in Ω.

Proof. Setting $\sigma = 0$ in Theorem 3.1, we get the proof. □

Corollary 3.3. *Let Ω be a NBCC subset of a BS Θ and $\Game : \Omega \to \Omega$ be a continuous mapping such that*

$$\Upsilon\left(\Game U\right) \leq \tau\Upsilon\left(U\right),$$

for all $U \subseteq \Omega$, $\tau \in [0,1)$, $\pi \in \Pi$, where $\Upsilon : \chi_\Omega \to \mathbb{R}_+$ is an arbitrary MNC. Then $\Game$ owns at least one FP in Ω.

Proof. Letting $\beta(s) = e^s$, and $\pi(s) = s - s^\tau$, for all $s \geq 0$ and $\tau \in [0,1)$ in Theorem 3.1, we have the result. □

Remark 3.2. *Since Corollary 3.4 can be seen to be Darbo's FPT, it follows that Theorem 3.1 is a generalization of Corollary 3.4.*

Now, to obtain a generalization of Darbo's FPT in the tripled variables, we need the following result.

Theorem 3.2. *[32] Let $\Upsilon_1, \Upsilon_2, ..., \Upsilon_m$ be a MNC in $\Theta_1, \Theta_2, ..., \Theta_m$, respectively. Further, assume that $\nabla : \mathbb{R}_+^m \to \mathbb{R}_+$ is a convex function such that $\nabla(q_1, q_2, ..., q_m) = 0$ iff $q_k = 0$ for all $k \in \mathbb{N}$. Then, $\Upsilon\left(\mathfrak{I}\right) = \nabla\left(\Upsilon_1\left(\mathfrak{I}_1\right), \Upsilon_2\left(\mathfrak{I}_2\right), ..., \Upsilon_m\left(\mathfrak{I}_m\right)\right)$ defines a MNC $\Theta_1 \times \Theta_2 \times ... \times \Theta_m$, where Θ_k is the natural projection (NP) of Θ into Θ_k, for $k = 1, 2, ..., m$.*

Example 3.1. *Let Υ be a MNC in Θ. Describe $\nabla(q_1, q_2, q_3) = q_1 + q_2 + q_3$, $q_1, q_2, q_3 \in \mathbb{R}_+$. Then ∇ fulfills all conditions of Theorem 3.2. Hence, $\Upsilon^{TFP}\left(\mathfrak{I}\right) = \Upsilon\left(\mathfrak{I}_1\right) + \Upsilon\left(\mathfrak{I}_2\right) + \Upsilon\left(\mathfrak{I}_3\right)$ is a MNC in $\Theta_1 \times \Theta_2 \times \Theta_3$, where Θ_k is the NP of Θ into Θ_k, for $k = 1, 2, ..., m$.*

Theorem 3.3. *Let Ω be a NBCC subset of a BS Θ and $\eth : \Omega \times \Omega \times \Omega \to \Omega$ be a continuous map fulfilling*

$$\begin{aligned} & \beta\left[\aleph\left(\Upsilon\left(\eth\left(U_1, U_2, U_3\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_1, U_2, U_3\right)\right)\right)\right)\right] \\ \leq\; & \frac{1}{3}\beta\left[\aleph\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right), \sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right)\right] \\ & -\frac{1}{3}\pi\left[\beta\left[\aleph\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right), \sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right)\right]\right], \end{aligned} \tag{3.2}$$

for all $U_1, U_2, U_3 \subseteq \Omega$, β, π, $\aleph$, σ are as in Theorem 3.1, where $\Upsilon : \chi_\Omega \to \mathbb{R}_+$ is an arbitrary MNC. Then $\eth$ owns a TFP in Ω, provided that

$$\beta\left(q_1, q_2, q_3\right) \leq \beta\left(q_1\right)+\beta\left(q_2\right)+\beta\left(q_3\right), \text{ and } \sigma\left(q_1, q_2, q_3\right) \leq \sigma\left(q_1\right)+\sigma\left(q_2\right)+\sigma\left(q_3\right),$$

for all $q_1, q_2, q_3 \geq 0$.

Proof. Define the mapping $\eth^{TFP} : \Omega \times \Omega \times \Omega \to \Omega \times \Omega \times \Omega$ by

$$\eth^{TFP}(u, v, w) = \left(\eth\left(u, v, w\right), \eth\left(v, w, u\right), \eth\left(w, u, v\right)\right).$$

Clearly, $\eth^{TFP}$ is continuous. Let $U \subset \Omega \times \Omega \times \Omega$ be a non-empty set and we have $\Upsilon^{TFP}(U) = \Upsilon(U_1) + \Upsilon(U_2) + \Upsilon(U_3)$ is a MNC where U_1, U_2, U_3 are NPs of U onto Θ.

Now, we have

$$\begin{aligned} & \beta\Big[\aleph\Big(\Upsilon^{TFP}\left(\eth\left(U\right)\right), \sigma\Big(\Upsilon^{TFP}\left(\eth\left(U\right)\right)\Big)\Big)\Big] \\ \leq\; & \beta\left[\aleph\left(\begin{array}{c} \Upsilon^{TFP}\left(\eth\left(U_1 \times U_2 \times U_3\right) \times \eth\left(U_2 \times U_3 \times U_1\right) \times \eth\left(U_3 \times U_1 \times U_2\right)\right), \\ \sigma\Big(\Upsilon^{TFP}\left(\eth\left(U_1 \times U_2 \times U_3\right) \times \eth\left(U_2 \times U_3 \times U_1\right) \times \eth\left(U_3 \times U_1 \times U_2\right)\right)\Big) \end{array}\right)\right] \\ =\; & \beta\left[\aleph\left(\begin{array}{c} \Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right)+\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)+\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right), \\ \sigma\left(\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)+\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)+\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right)\right)\right) \end{array}\right)\right] \\ \leq\; & \beta\left[\aleph\left(\begin{array}{c} \Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right)+\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)+\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right), \\ \sigma\left(\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right)\right)+\sigma\left(\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)\right)+\left(\sigma\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right)\right) \end{array}\right)\right], \end{aligned}$$

which implies that

$$\begin{aligned} & \beta\Big[\aleph\Big(\Upsilon^{TFP}\left(\eth\left(U\right)\right), \sigma\Big(\Upsilon^{TFP}\left(\eth\left(U\right)\right)\Big)\Big)\Big] \\ \leq\; & \beta\left[\aleph\left\{\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right)\right)\right\}\right. \\ & +\aleph\left\{\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)\right)\right\} \\ & \left.+\aleph\left\{\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right)\right)\right\}\right] \\ =\; & \beta\left[\aleph\left\{\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_1 \times U_2 \times U_3\right)\right)\right)\right\}\right] \\ & +\beta\left[\aleph\left\{\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_2 \times U_3 \times U_1\right)\right)\right)\right\}\right] \\ & +\beta\left[\aleph\left\{\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right), \sigma\left(\Upsilon\left(\eth\left(U_3 \times U_1 \times U_2\right)\right)\right)\right\}\right]. \end{aligned}$$

Applying (3.2), one has

$$\begin{aligned}
&\beta\left[\aleph\left(\Upsilon^{TFP}\left(\eth\left(U\right)\right),\sigma\left(\Upsilon^{TFP}\left(\eth\left(U\right)\right)\right)\right)\right]\\
\leq\;& \frac{1}{3}\beta\left[\aleph\left\{\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right),\sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right\}\right]\\
&-\frac{1}{3}\pi\left[\beta\left[\aleph\left\{\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right),\sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right\}\right]\right]\\
&+\frac{1}{3}\beta\left[\aleph\left\{\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right),\sigma\left(\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)\right)\right\}\right]\\
&-\frac{1}{3}\pi\left[\beta\left[\aleph\left\{\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right),\sigma\left(\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)\right)\right\}\right]\right]\\
&+\frac{1}{3}\beta\left[\aleph\left\{\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right),\sigma\left(\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)\right)\right\}\right]\\
&-\frac{1}{3}\pi\left[\beta\left[\aleph\left\{\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right),\sigma\left(\Upsilon\left(U_3\right)+\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)\right)\right\}\right]\right]\\
=\;& \beta\left[\aleph\left\{\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right),\sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right\}\right]\\
&-\pi\left[\beta\left[\aleph\left\{\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right),\sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right\}\right]\right]\\
=\;& \beta\left[\aleph\left\{\Upsilon^{TFP}\left(U\right),\sigma\left(\Upsilon^{TFP}\left(U\right)\right)\right\}\right]-\pi\left[\beta\left[\aleph\left\{\Upsilon^{TFP}\left(U\right),\sigma\left(\Upsilon^{TFP}\left(U\right)\right)\right\}\right]\right].
\end{aligned}$$

According to Theorem 3.1, the mapping $\eth^{TFP}$ possesses at least one FP in $\Omega\times\Omega\times\Omega$, i.e., $\eth$ owns at least one TFP. □

Corollary 3.4. *Let Ω be a NBCC subset of a BS Θ and $\eth:\Omega\times\Omega\times\Omega\to\Omega$ be a continuous mapping such that*

$$\begin{aligned}
&\beta\left[\aleph\left(\Upsilon\left(\eth\left(U_1,U_2,U_3\right)\right),\sigma\left(\Upsilon\left(\eth\left(U_1,U_2,U_3\right)\right)\right)\right)\right]\\
\leq\;& \frac{1}{3}\left\{\beta\left[\aleph\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right),\sigma\left(\Upsilon\left(U_1\right)+\Upsilon\left(U_2\right)+\Upsilon\left(U_3\right)\right)\right)\right]\right\}^{\tau},
\end{aligned}$$

for all $U_1,U_2,U_3\subseteq\Omega$, β, π, $\aleph$, σ are as in Theorem 3.1 and $\tau\in[0,1)$, where $\Upsilon:\chi_\Omega\to\mathbb{R}_+$ is an arbitrary MNC. Then $\eth$ owns a TFP in Ω, provided that

$$\beta\left(q_1,q_2,q_3\right)\leq\beta\left(q_1\right)+\beta\left(q_2\right)+\beta\left(q_3\right),\ \text{and}\ \sigma\left(q_1,q_2,q_3\right)\leq\sigma\left(q_1\right)+\sigma\left(q_2\right)+\sigma\left(q_3\right),$$

for all $q_1,q_2,q_3\geq 0$.

Proof. Putting $\pi(s)=s-s^{\tau}$, for all $s\geq 0$ and $\tau\in[0,1)$ in Theorem 3.3, we get the result. □

4. A fractional integral equation of three variables

In 1993, Samko et al. [33] introduced the following FIE:

$$I^{\varpi}_{c^+,m}z(\varkappa)=\frac{1}{\Gamma\left(\varpi\right)}\int_c^{\varkappa}\frac{m'(s)z(s)}{\left(m(\varkappa)-m(s)\right)^{1-\varpi}}ds,\ \varpi>0,\ -\infty\leq c<d\leq\infty, \tag{4.1}$$

where $z(s)$ is a continuous function and $m(\varkappa)$ is a monotone function having a continuous derivative. Similar to Equation (4.1), in this part, we try to solve the following FIE:

$$I^{\varpi}_{c^+,m,k,l}z(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) = \frac{1}{(\Gamma(\varpi))^3}\int_c^{\varkappa}\int_c^{\widetilde{\varkappa}}\int_c^{\widehat{\varkappa}}\frac{m'(s)k'(r)l'(u)z(s,r,u)}{(m(\varkappa)-m(s))^{1-\varpi}(k(\widetilde{\varkappa})-k(r))^{1-\varpi}\left(l(\widehat{\varkappa})-l(u)\right)^{1-\varpi}}dudrds, \tag{4.2}$$

which is finite, where $-\infty \leq c < d \leq \infty$, $\Gamma(.)$ is the Euler's Gamma function, $\varkappa,\widetilde{\varkappa},\widehat{\varkappa} \in [c,d]$, $z(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})$ is a continuous function on $[c,d]\times[c,d]\times[c,d]$ and m,k,l are monotone functions of order ϖ.

We will now determine whether or not the operator (4.2) is a strongly continuous semi-group (SCS) on $\beth = C([c,d]\times[c,d]\times[c,d],\mathbb{R})$. The continuity of the operator (4.2) is trivial. For $z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}), z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}), z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) \in \beth$ and $\zeta_1,\zeta_2,\zeta_3 \in \mathbb{R}$, we get

$$\begin{aligned}
& I^{\varpi}_{c^+,m,k,l}\left(\zeta_1 z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_2 z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_3 z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right) \\
= \ & \frac{1}{(\Gamma(\varpi))^3}\int_c^{\varkappa}\int_c^{\widetilde{\varkappa}}\int_c^{\widehat{\varkappa}}\frac{m'(s)k'(r)l'(u)\left[\zeta_1 z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_2 z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_3 z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right]}{(m(\varkappa)-m(s))^{1-\varpi}(k(\widetilde{\varkappa})-k(r))^{1-\varpi}\left(l(\widehat{\varkappa})-l(u)\right)^{1-\varpi}}dudrds \\
= \ & \frac{\zeta_1}{(\Gamma(\varpi))^3}\int_c^{\varkappa}\int_c^{\widetilde{\varkappa}}\int_c^{\widehat{\varkappa}}\frac{m'(s)k'(r)l'(u)z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})}{(m(\varkappa)-m(s))^{1-\varpi}(k(\widetilde{\varkappa})-k(r))^{1-\varpi}\left(l(\widehat{\varkappa})-l(u)\right)^{1-\varpi}}dudrds \\
& + \frac{\zeta_2}{(\Gamma(\varpi))^3}\int_c^{\varkappa}\int_c^{\widetilde{\varkappa}}\int_c^{\widehat{\varkappa}}\frac{m'(s)k'(r)l'(u)z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})}{(m(\varkappa)-m(s))^{1-\varpi}(k(\widetilde{\varkappa})-k(r))^{1-\varpi}\left(l(\widehat{\varkappa})-l(u)\right)^{1-\varpi}}dudrds \\
& + \frac{\zeta_3}{(\Gamma(\varpi))^3}\int_c^{\varkappa}\int_c^{\widetilde{\varkappa}}\int_c^{\widehat{\varkappa}}\frac{m'(s)k'(r)l'(u)z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})}{(m(\varkappa)-m(s))^{1-\varpi}(k(\widetilde{\varkappa})-k(r))^{1-\varpi}\left(l(\widehat{\varkappa})-l(u)\right)^{1-\varpi}}dudrds \\
= \ & \zeta_1 I^{\varpi}_{c^+,m,k,l}z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_2 I^{\varpi}_{c^+,m,k,l}z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + \zeta_3 I^{\varpi}_{c^+,m,k,l}z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}).
\end{aligned}$$

This proves that the operator (4.2) is linear operator.

Further, for $z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}), z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}), z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) \geq 0$, one can write

$$\begin{aligned}
& I^{\varpi}_{c^+,m,k,l}\left[z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa}) + z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right] \\
\neq \ & \left[I^{\varpi}_{c^+,m,k,l}z_1(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right]\left[I^{\varpi}_{c^+,m,k,l}z_2(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right]\left[I^{\varpi}_{c^+,m,k,l}z_3(\varkappa,\widetilde{\varkappa},\widehat{\varkappa})\right],
\end{aligned}$$

and $I^{\varpi}_{c^+,m,k,l}(0) = 0 \neq I$. As a result, we draw the conclusion that the operator (4.2) is not a SCS on $\beth$.

Assume that $\Theta = C(J\times J\times J)$ is the space of all real continuous functions on $J=[0,1]$. Clearly, the pair $(\Theta, \|.\|)$ is a BS under the norm

$$\|\varkappa\| = \sup\left\{\left|\varkappa(s,t,u)\right| : s,t,u \in J,\ \varkappa \in \Theta\right\}.$$

Consider $\Im\ (\neq \emptyset)$ is a fixed bounded subset of Θ. Also, consider $\urcorner(\varkappa,\epsilon)$ refers to the modulus of continuity of $\varkappa$, that is,

$$\urcorner(\varkappa,\epsilon)=\sup\left\{\begin{array}{c}\left|\varkappa(\varphi_1,\varphi_2,\varphi_3)-\varkappa(\varrho_1,\varrho_2,\varrho_3)\right|:\varphi_1,\varphi_2,\varphi_3,\varrho_1,\varrho_2,\varrho_3\in I,\\ \left|\varphi_1-\varrho_1\right|\leq\epsilon,\ \left|\varphi_2-\varrho_2\right|\leq\epsilon,\ \left|\varphi_3-\varrho_3\right|\leq\epsilon,\end{array}\right\}$$

where $\varkappa\in\Theta$ and $\epsilon>0$. In addition, let

$$\urcorner(\Im,\epsilon)=\sup\left\{\urcorner(\varkappa,\epsilon):\varkappa\in\Im\right\},$$

and

$$\urcorner_0(\Im)=\lim_{\epsilon\to 0}\urcorner(\Im,\epsilon).$$

It can be demonstrated that the function $\urcorner_0$ is a MNC in the space Θ, similar to [34]. This section examines the solvability of the following generalized fractional order IE:

$$\begin{aligned}&\sqsupset(\varphi,\varrho,\rho)\\ =\ & R\left(\varphi,\varrho,\rho,\sqsupset(\varphi,\varrho,\rho),\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right),\end{aligned}\tag{4.3}$$

where $\varpi\in(0,1)$, $\varphi,\varrho,\rho\in[0,S]$, $S>0$.

To reach our desired goal here we need the hypotheses below:

(H_1) The function $R:J^3\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there are $B_1,B_2\geq 0$ with $B_1\in[0,1)$ so that
$$\left|R(\varphi,\varrho,\rho,P_1,P_2)-R\left(\varphi,\varrho,\rho,\widetilde{P}_1,\widetilde{P}_2\right)\right|\leq B_1\left|P_1-\widetilde{P}_1\right|+B_2\left|P_2-\widetilde{P}_2\right|,$$
where $\varphi,\varrho,\rho\in J$, $P_1,P_2,\widetilde{P}_1,\widetilde{P}_2\in\mathbb{R}$ and $J^3=J\times J\times J$.

(H_2) The functions $m,k,l:J\to\mathbb{R}_+$ are C^1 nondecreasing. Further, $m',k',l'\geq 0$.

(H_3) The function $q:J^6\times\mathbb{R}\to\mathbb{R}$ is continuous, where $J^6=J^3\times J^3$.

(H_4) We assume that
$$Q=\sup\left\{\left|q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))\right|:\varphi,\varrho,\rho,\varsigma,\vartheta,\theta\in J,\ \sqsupset\in C(J\times J\times J)\right\},$$
and
$$\widehat{R}=\sup\left\{\left|R(\varphi,\varrho,\rho,0,0)\right|:\varphi,\varrho,\rho\in J\right\}.$$
In addition, assume that there is z_0 so that
$$B_1z_0+\frac{B_2Q}{\varpi^3}(l(S)-l(0))^{\varpi}(k(S)-k(0))^{\varpi}(m(S)-m(0))^{\varpi}+\widehat{R}\leq z_0.$$
Also, we consider $\Xi_{z_0}=\{\varkappa\in\Theta:\|\varkappa\|\leq z_0\}$

Theorem 4.1. *In the light of the hypotheses (H_1)-(H_4), Equation (4.3) has at least one solution in Θ.*

Proof. For $\sqsupset\in\Theta$, describe the operator Q on Θ as

$$\begin{aligned}&Q(\sqsupset)(\varphi,\varrho,\rho)\\ =\ & R\left(\varphi,\varrho,\rho,\sqsupset(\varphi,\varrho,\rho),\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right),\end{aligned}$$

for all $\varphi, \varrho, \rho \in J$. We split the proof into the following steps:

(1) Show that Q is well defined. Let $\varphi, \varrho, \rho \in J$ be a fixed and $\{\varphi_n\}$, $\{\varrho_n\}$ and $\{\rho_n\}$ be sequences in J so that, $\varphi_n \to \varphi$, $\varrho_n \to \varrho$ and $\rho_n \to \rho$ as $n \to \infty$. Choose $\varphi_n \geq \varphi$, $\varrho_n \geq \varrho$ and $\rho_n \geq \rho$, (without loss of generality). Then, we get

$$\begin{aligned}
&\left|(Q\beth)\left(\varphi_n, \varrho_n, \rho_n\right)-(Q\beth)\left(\varphi, \varrho, \rho\right)\right| \\
\leq\ & B_1\left|\beth\left(\varphi_n, \varrho_n, \rho_n\right)-\beth\left(\varphi, \varrho, \rho\right)\right| \\
&+B_2\left|\int_0^{\varphi_n}\int_0^{\varrho_n}\int_0^{\rho_n} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|.
\end{aligned}$$

Now,

$$\begin{aligned}
&\left|\int_0^{\varphi_n}\int_0^{\varrho_n}\int_0^{\rho_n} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right| \\
\leq\ &\left|\int_0^{\varphi_n}\int_0^{\varrho_n}\int_0^{\rho_n} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right| \\
&+\left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right| \\
&+\left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right| \\
=\ & T_1^n+T_2^n+T_3^n,
\end{aligned}$$

where

$$\begin{aligned}
T_1^n =\ &\left|\int_0^{\varphi_n}\int_0^{\varrho_n}\int_0^{\rho_n} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right. \\
&\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\int_{\varphi}^{\varphi_n}\int_0^{\varrho_n}\int_0^{\rho_n}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.+\int_0^{\varphi}\int_0^{\varrho}\int_{\rho_n}^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq \frac{Q}{\varpi^3}(l(\varphi_n)-l(\varphi))^{\varpi}(k(\varrho_n)-k(0))^{\varpi}(m(\rho_n)-m(0))^{\varpi}\\
&\quad+\frac{Q}{\varpi^3}\left[(l(\varphi_n)-l(\varphi))^{\varpi}-(l(\varphi_n)-l(0))^{\varpi}\right](k(\varrho_n)-k(0))^{\varpi}(m(\rho_n)-m(\rho))^{\varpi}.
\end{aligned}
$$

Since m, l, k are continuous, $\varphi_n \to \varphi$ and $\rho_n \to \rho$ as $n \to \infty$, then $T_1^n \to 0$ as $n \to \infty$. Also,

$$
\begin{aligned}
T_2^n &= \left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq Q\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)}{(m(\varphi_n)-m(\theta))^{1-\varpi}(k(\varrho_n)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\
&\quad+Q\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\
&= \frac{Q}{\varpi^3}\left[\begin{array}{c}
(m(\varphi_n)-m(\varphi))^{\varpi}(k(\varrho_n)-k(0))^{\varpi}(l(\rho_n)-l(0))^{\varpi}\\
+(m(\varphi_n)-m(0))^{\varpi}(k(\varrho_n)-k(\varrho))^{\varpi}(l(\rho_n)-l(0))^{\varpi}\\
+(m(\varphi_n)-m(0))^{\varpi}(k(\varrho_n)-k(0))^{\varpi}(l(\rho_n)-l(\rho))^{\varpi}\\
-3(m(\varphi_n)-m(\varphi))^{\varpi}(k(\varrho_n)-k(\varrho))^{\varpi}(l(\rho_n)-l(\rho))^{\varpi}
\end{array}\right].
\end{aligned}
$$

Since m, l, k are continuous, $\varphi_n \to \varphi$, $\varrho_n \to \varrho$ and $\rho_n \to \rho$ as $n \to \infty$, then $T_2^n \to 0$ as $n \to \infty$. Again

$$
\begin{aligned}
T_3^n &= \left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)\left|q(\varphi_n,\varrho_n,\rho_n,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))-q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))\right|}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho_n)-l(\varsigma))^{1-\varpi}},
\end{aligned}
$$

Since $\varphi_n \to \varphi$, $\varrho_n \to \varrho$, $\rho_n \to \rho$ as $n \to \infty$ and q is continuous, we have $T_3^n \to 0$ as $n \to \infty$. Thus, $\beth(\varphi, \varrho, \rho) \in \Theta$ implies $Q\beth \in \Theta$. Hence, Q is well defined.

(2) Prove that $Q(\Xi_{z_0}) \subseteq \Xi_{z_0}$ and $Q : \Xi_{z_0} \to \Xi_{z_0}$ is well defined. Let $\Xi_{z_0} = \{\beth \in \Theta : \|\beth\| \leq z_0\}$. Then for all $\varphi, \varrho, \rho \in J$ and for $\beth \in \Xi_{z_0}$, we get

$$\begin{aligned}
&\left|(Q\beth)(\varphi,\varrho,\rho)\right| \\
\leq\ & \left|R\left(\varphi,\varrho,\rho,\beth(\varphi,\varrho,\rho), \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right)\right. \\
& \left.-R(\varphi,\varrho,\rho,0,0)\right| + \left|R(\varphi,\varrho,\rho,0,0)\right| \\
\leq\ & B_1\left|\beth(\varphi,\varrho,\rho)\right| + B_2 Q\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma \\
& +\widehat{R} \\
\leq\ & B_1 z_0 + \frac{B_2 Q}{\varpi^3}(l(S)-l(0))^{\varpi}(k(S)-k(0))^{\varpi}(m(S)-m(0))^{\varpi} + \widehat{R} \\
\leq\ & z_0.
\end{aligned}$$

Hence, $Q(\Xi_{z_0}) \subseteq \Xi_{z_0}$, that is $Q : \Xi_{z_0} \to \Xi_{z_0}$ is well defined.

(3) Claim that Q is continuous on Ξ_{z_0}. Assume that $\beth, L \in \Xi_{z_0}$ with $\|\beth - L\| \leq \epsilon$, where $\epsilon > 0$. For each $\varphi, \varrho, \rho \in J$, we obtain that

$$\begin{aligned}
&\left|(Q\beth)(\varphi,\varrho,\rho) - (QL)(\varphi,\varrho,\rho)\right| \\
=\ & \left|R\left(\varphi,\varrho,\rho,\beth(\varphi,\varrho,\rho), \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right)\right. \\
& \left. - R\left(\varphi,\varrho,\rho,L(\varphi,\varrho,\rho), \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,L(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right)\right| \\
\leq\ & B_1\left|\beth(\varphi,\varrho,\rho) - L(\varphi,\varrho,\rho)\right| \\
& + B_1\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)\left|q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta)) - q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,L(\varsigma,\vartheta,\theta))\right|}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma \\
\leq\ & B_1\|\beth - L\| + \frac{B_2 q_\epsilon}{\varpi^3}(l(S)-l(0))^{\varpi}(k(S)-k(0))^{\varpi}(m(S)-m(0))^{\varpi},
\end{aligned}$$

where

$$q_\epsilon = \sup\left\{\begin{array}{c}\left|q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta)) - q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,L(\varsigma,\vartheta,\theta))\right| : \varphi,\varrho,\rho,\varsigma,\vartheta,\theta \in J, \\ |\beth - L| \leq \epsilon,\ \ |\beth| \leq z_0,\ \ |L| \leq z_0.\end{array}\right\}$$

As q is uniformly continuous on $J^6 \times [-z_0, z_0]$, then $q_\epsilon \to 0$ as $\epsilon \to 0$. Hence, $\|Q\beth - QL\| \to 0$ as $\epsilon \to 0$.This proves that Q is continuous on Ξ_{z_0}.

(4) Prove that the contractive condition of Corollary 3.3 holds. Consider $V(\neq \emptyset) \subseteq \Xi_{z_0}$. For an arbitrary $\epsilon > 0$, set $\beth(\varphi,\varrho,\rho) \in V$ and $\varphi,\varrho,\rho,\varphi^*,\varrho^*,\rho^* \in J$ such that $\left|\varphi - \varphi^*\right| \leq \epsilon$, $\left|\varrho - \varrho^*\right| \leq \epsilon$ and $\left|\rho - \rho^*\right| \leq \epsilon$. Without loss of generality, take $\varphi^* \geq \varphi$, $\varrho^* \geq \varrho$ and $\rho^* \geq \rho$. Now,

$$\begin{aligned}
&\left|(Q\beth)(\varphi^*,\varrho^*,\rho^*) - (Q\beth)(\varphi,\varrho,\rho)\right| \\
=\ & \left|R\left(\begin{array}{c}\varphi^*,\varrho^*,\rho^*,\beth(\varphi^*,\varrho^*,\rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\end{array}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& - R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi, \varrho, \rho), \\ \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right)\Bigg| \\
\leq \; & \Bigg| R\left(\begin{array}{c} \varphi^*, \varrho^*, \rho^*, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right) \\
& - R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right)\Bigg| \\
& + \Bigg| R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right) \\
& - R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi, \varrho, \rho), \\ \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right)\Bigg| \\
= \; & T_1 + T_2. \qquad (4.4)
\end{aligned}
$$

Also,

$$
\begin{aligned}
& \left| \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \right| \\
\leq \; & \frac{Q}{\varpi^3}(l(S)-l(0))^{\varpi}(k(S)-k(0))^{\varpi}(m(S)-m(0))^{\varpi} = F(say).
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 = \; & \Bigg| R\left(\begin{array}{c} \varphi^*, \varrho^*, \rho^*, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right) \\
& - R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right)\Bigg| \\
\leq \; & C(R, \epsilon), \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 = \; & \Bigg| R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi^*, \varrho^*, \rho^*), \\ \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right) \\
& - R\left(\begin{array}{c} \varphi, \varrho, \rho, \sqsupset(\varphi, \varrho, \rho), \\ \int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma \end{array} \right)\Bigg| \\
\leq \; & B_1 \left| \sqsupset(\varphi^*, \varrho^*, \rho^*) - \sqsupset(\varphi, \varrho, \rho) \right| \\
& + B_2 \Bigg| \int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*} \frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\sqsupset(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}} d\theta d\vartheta d\varsigma
\end{aligned}
$$

$$-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\Bigg|. \tag{4.6}$$

Let

$$C(R,\epsilon)=\sup\left\{\begin{array}{c}\left|R(\varphi,\varrho,\rho,\beth,O)-R(\varphi^*,\varrho^*,\rho^*,\beth,O)\right|:\varphi,\varrho,\rho,\varphi^*,\varrho^*,\rho^*\in J,\\ \left|\varphi-\varphi^*\right|\le\epsilon,\ \left|\varrho-\varrho^*\right|\le\epsilon,\ \left|\rho-\rho^*\right|\le\epsilon,\ |\beth|\le z_0,\ |O|\le z_0.\end{array}\right\}$$

From the uniform continuity of R in $J^3\times[-z_0,z_0]\times[-F\times F]$, we conclude that $\lim_{\epsilon\to 0}C(R,\epsilon)=0$. Again, let

$$\begin{aligned}&C(q,\epsilon)\\ =\ &\sup\left\{\begin{array}{c}\left|q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth)-q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth)\right|:\varphi^*,\varrho^*,\rho^*,\varsigma^*,\vartheta^*,\theta^*,\varphi,\varrho,\rho,\varsigma,\vartheta,\theta\in J,\\ \left|\varphi^*-\varphi\right|\le\epsilon,\ \left|\varrho^*-\varrho\right|\le\epsilon,\ \left|\rho^*-\rho\right|\le\epsilon,\ |\beth|\le z_0,\end{array}\right\}\end{aligned}$$

and

$$\begin{aligned}C(m,\epsilon)&=\sup\left\{\left|m(\varphi^*)-m(\varphi)\right|:\varphi^*,\varphi\in J,\ \left|\varphi^*-\varphi\right|\le\epsilon\right\},\\ C(k,\epsilon)&=\sup\left\{\left|k(\varrho^*)-k(\varrho)\right|:\varrho^*,\varrho\in J,\ \left|\varrho^*-\varrho\right|\le\epsilon\right\},\\ C(l,\epsilon)&=\sup\left\{\left|l(\rho^*)-l(\rho)\right|:\rho^*,\rho\in J,\ \left|\rho^*-\rho\right|\le\epsilon\right\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\Bigg|\int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\ &-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\Bigg|\\ \le\ &\Bigg|\int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\ &-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\Bigg|\\ &+\Bigg|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\ &-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\Bigg|\\ &+\Bigg|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\\ &-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\beth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\Bigg|\\ =\ &W_1+W_2+W_3,\end{aligned}$$

where

$$\begin{aligned}
W_1 &= \left|\int_0^{\varphi^*}\int_0^{\varrho^*}\int_0^{\rho^*}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq \frac{Q}{\varpi^3}\left[\begin{array}{c}(m(\varphi^*)-m(0))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ -3(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(0))^{\varpi}\end{array}\right]\\
&\leq \frac{Q}{\varpi^3}\left[\begin{array}{c}(m(\varphi^*)-m(0))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(0))^{\varpi}\end{array}\right]\\
&\leq \frac{3Q\{C(m,\epsilon)C(k,\epsilon)C(l,\epsilon)\}^{\varpi}}{\varpi^3},
\end{aligned}$$

$$\begin{aligned}
W_2 &= \left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi^*)-m(\theta))^{1-\varpi}(k(\varrho^*)-k(\vartheta))^{1-\varpi}(l(\rho^*)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq \frac{Q}{\varpi^3}\left[\begin{array}{c}(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(0))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(0))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ -3(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\end{array}\right]\\
&\leq \frac{Q}{\varpi^3}\left[\begin{array}{c}(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(0))^{\varpi}\\ +(m(\varphi^*)-m(\varphi))^{\varpi}(k(\varrho^*)-k(0))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\\ +(m(\varphi^*)-m(0))^{\varpi}(k(\varrho^*)-k(\varrho))^{\varpi}(l(\rho^*)-l(\rho))^{\varpi}\end{array}\right]\\
&\leq \frac{3Q\{C(m,\epsilon)C(k,\epsilon)C(l,\epsilon)\}^{\varpi}}{\varpi^3},
\end{aligned}$$

and

$$\begin{aligned}
W_3 &= \left|\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi^*,\varrho^*,\rho^*,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right.\\
&\quad\left.-\int_0^{\varphi}\int_0^{\varrho}\int_0^{\rho}\frac{m'(\theta)k'(\vartheta)l'(\varsigma)q(\varphi,\varrho,\rho,\varsigma,\vartheta,\theta,\daleth(\varsigma,\vartheta,\theta))}{(m(\varphi)-m(\theta))^{1-\varpi}(k(\varrho)-k(\vartheta))^{1-\varpi}(l(\rho)-l(\varsigma))^{1-\varpi}}d\theta d\vartheta d\varsigma\right|\\
&\leq \frac{C(q,\epsilon)}{\varpi^3}\left((m(\varphi)-m(0))^{1-\varpi}(k(\varrho)-k(0))^{1-\varpi}(l(\rho)-l(0))^{1-\varpi}\right)\\
&\leq \frac{C(q,\epsilon)\{C(m,\epsilon)C(k,\epsilon)C(l,\epsilon)\}^{\varpi}}{\varpi^3}.
\end{aligned}$$

Applying the above results in (4.6), we have

$$\begin{aligned} T_2 &\leq B_1 \left| \beth(\varphi^*, \varrho^*, \rho^*) - \beth(\varphi, \varrho, \rho) \right| + B_2(W_1 + W_2 + W_3) \\ &= B_1 \daleth(V, \epsilon) \\ &\quad + B_2 \left(\frac{6Q\{C(m,\epsilon)\, C(k,\epsilon)\, C(l,\epsilon)\}^{\varpi} + C(q,\epsilon)\{C(m,\epsilon)C(k,\epsilon)C(l,\epsilon)\}^{\varpi}}{\varpi^3} \right). \end{aligned} \tag{4.7}$$

From (4.5) and (4.7) in (4.4), we have

$$\begin{aligned} \daleth(QV, \epsilon) &\leq C(R, \epsilon) + B_1 \daleth(V, \epsilon) \\ &\quad + B_2 \left(\frac{6Q\{C(m,\epsilon)\, C(k,\epsilon)\, C(l,\epsilon)\}^{\varpi} + C(q,\epsilon)\{C(m,\epsilon)C(k,\epsilon)C(l,\epsilon)\}^{\varpi}}{\varpi^3} \right). \end{aligned}$$

Since R, m, k and l are continuous, $\daleth(QV, \epsilon) \to 0$ as $\epsilon \to 0$. Therefore, all requirements of Corollary 3.3 are satisfied. Thus, the mappings Q owns at least one FP in $V \subseteq \Xi_{z_0} \subseteq \Theta$, which is a solution to Problem (4.3).

□

To support our problem, we introduce the following example:

Example 4.1. *Consider the following FIE:*

$$\begin{aligned} \beth(\varphi, \varrho, \rho) &= \frac{\varphi\varrho\rho\,(1 + \beth(\varphi, \varrho, \rho))}{1 + \varphi\varrho\rho} \\ &\quad + \int_0^{\varphi} \int_0^{\varrho} \int_0^{\rho} \frac{\beth^3(\varsigma, \vartheta, \theta)}{(\varphi - \varsigma)^{\frac{1}{3}} (\varrho - \vartheta)^{\frac{1}{3}} (\rho - \theta)^{\frac{1}{3}} (1 + \beth^3(\varsigma, \vartheta, \theta))} d\theta d\vartheta d\varsigma, \end{aligned} \tag{4.8}$$

for all $\varphi, \varrho, \rho, \varsigma, \vartheta, \theta \in [0,1]$. Problem (4.8) is another form of (4.3) with

$$\begin{aligned} m(\theta) &= \theta,\ k(\vartheta) = \vartheta,\ l(\varsigma) = \varsigma,\ q(\varphi, \varrho, \rho, \varsigma, \vartheta, \theta, \beth(\varsigma, \vartheta, \theta)) = \frac{\beth^3}{1 + \beth^3}, \\ \varpi &= \frac{2}{3},\ \text{and } R(\varphi, \varrho, \rho, \beth, P) = \frac{\varphi\varrho\rho\,(1 + \beth)}{1 + \varphi\varrho\rho} + P, \end{aligned}$$

where

$$P = \int_0^{\varphi} \int_0^{\varrho} \int_0^{\rho} \frac{\beth^3(\varsigma, \vartheta, \theta)}{(\varphi - \varsigma)^{\frac{1}{3}} (\varrho - \vartheta)^{\frac{1}{3}} (\rho - \theta)^{\frac{1}{3}} (1 + \beth^3(\varsigma, \vartheta, \theta))} d\theta d\vartheta d\varsigma.$$

Now, for $\varphi, \varrho, \rho \in [0,1]$ and $P_1, P_2, \widetilde{P}_1, \widetilde{P}_2 \in \mathbb{R}$

$$\left| R(\varphi, \varrho, \rho, P_1, P_2) - R\left(\varphi, \varrho, \rho, \widetilde{P}_1, \widetilde{P}_2\right) \right| \leq \frac{\varphi\varrho\rho}{1 + \varphi\varrho\rho} \left| P_1 - \widetilde{P}_1 \right| + \left| P_2 - \widetilde{P}_2 \right|.$$

Hence, $B_1 = \frac{1}{2}$ and $B_2 = 1$. The functions $m, k, l : J \to \mathbb{R}_+$ are C^1 nondecreasing. Further, $m', k', l' \geq 0$. The functions R and q are continuous and $Q = 1$, $\widehat{R} = \frac{1}{2}$. In addition, for $z_0 = 9$, the inequality

$$\begin{aligned} & B_1 z_0 + \frac{B_2 Q}{\varpi^3} (l(S) - l(0))^{\varpi} (k(S) - k(0))^{\varpi} (m(S) - m(0))^{\varpi} + \widehat{R} \\ &= \frac{z_0}{2} + \frac{27}{8} (1 - 0)^{\frac{2}{3}} (1 - 0)\, (1 - 0)^{\frac{2}{3}} + \frac{1}{2} \leq z_0 \end{aligned}$$

is true. Hence, the hypotheses (H_1)-(H_4) are fulfilled. According to Theorem 4.1, the problem (4.8) has at least on solution in $C\left([0,1]\times[0,1]\times[0,1]\right)$ *.*

As a special case, for all $\varphi, \varrho, \rho \in [0,1]$ *and* $\sqsupset$ *is a constant function, the exact solutions to Problem (4.8) are given by*

$$\sqsupset(\varphi,\varrho,\rho) = \frac{\varphi\varrho\rho\left(1+\sqsupset\right)}{1+\varphi\varrho\rho} + \frac{3\sqsupset^3\varphi^{\frac{1}{3}}\varrho^{\frac{1}{3}}\rho^{\frac{1}{3}}}{1+\sqsupset^3}.$$

5. Conclusion

It is known that we used contractive type conditions and their generalizations to establish several FP results and we applied them to develop some results of theoretical and word problems involving mathematical models describing integral and differential equations arising in fractional analysis. Working on the existence and uniqueness of variant forms of solutions to those equations becomes an intersecting and attractive field of research. MNC appeared in different applications in FP theory and in particular are useful in differential and integral equations. In our paper, we extended Darbo's FP theorem and we applied our findings to guarantee the existence of solutions of FIEs involving three variables.

6. Abbreviations

- MNC measure of noncompactness.
- FIE fractional integral equation
- IE integral equation.
- FP fixed point.
- FPT fixed point theorem.
- FC fractional calculus.
- ID integration and differentiation.
- BS Banach space.
- TFP tripled fixed point.
- NBCC nonempty, bounded, closed, and convex.
- NP natural projection.
- SCS strongly continuous semi-group.

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