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The Convergence Solution of Mixed Variational-Hemivariational Inequality Problems

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Abstract. In this paper, we considered a mixed variational-hemivariational inequality problem in a reflexive Banach space with a set of constraints, a nonlinear operator, a sequence generated for data perturbation, and a parameter. We proved the existence and uniqueness of the solution for the problem and its perturbation. The research focuses on the strong convergence of the sequences suggested by the problem.

1. Introduction

Panagiotopoulos [1] was the first to apply variational-hemivariational inequalities to engineering challenges. The mathematical literature on variational-hemivariational inequalities has grown in recent decades, owing to significant applications in Physics, Mechanics, Biotechnology, Nanotechnology, and Engineering Sciences. In a recent reference, the book [2,3] reviewed the current state of the art and relevant applications in Contact Mechanics

Variational-hemivariational inequalities are a type of inequalities that arise during the investigation of nonsmooth boundary value problems. They are governed by convex functions and local Lipschitz functions, which may or may not be convex. As a result, their work necessitates an understanding of both convex and nonsmooth analysis. Let $\mathbb{X}$ be a real reflexive Banach space, with dual spaces $\mathbb{X}^*$. Assume $\langle \cdot, \cdot \rangle$ is a dual pairing of $\mathbb{X}$ and $\mathbb{X}^*$. Consider $\Omega \subset \mathbb{X}$, $\mathcal{D} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}^*$, $\varphi : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}$, $\jmath : \mathbb{X} \longrightarrow \mathbb{R}$, and $f \in \mathbb{X}^*$. The inequality problem is then considered when finding an element $u \in \Omega$ such that

$$\langle \mathcal{D}(u,u), v-u \rangle + \varphi(u,v) - \varphi(u,u) + \jmath^0(u, v-u) \geq \langle f, v-u \rangle, \ v \in \Omega, \tag{1.1}$$

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where $\jmath$ is a locally Lipschitz function and $\jmath^0(u,v)$ a generalized directional derivative of $\jmath$ at the point u in the direction v.
The perturbation of problem (1.1) for finding $u_n \in \Omega$ such that

$$\langle \mathcal{D}(u_n,u_n), v-u_n\rangle + \varphi(u_n,v) - \varphi(u_n,u_n) + \jmath^0(u_n, v-u_n) + \varepsilon_n \|v-u_n\|_{\mathbb{X}} \geq \langle f, v-u_n\rangle,\ v \in \Omega, \quad (1.2)$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers, for each $n \in \mathbb{N}$.

It should be noted that the perturbed problem (1.2) has a solution in the case of problem (1.1). However, it is possible that the solution to problem (1.2) is not unique. Assuming u_n, a sequence that converges to $u \in \mathbb{X}$ under suitable assumptions, was shown in [4] to be an approximating sequence for each $n \in \mathbb{N}$. According to Tykhonov's proposal in [5–7], this property is the main component of the well-posedness of problem (1.1).

Other convergence results for problem (1.1) are associated with the penalty method. Let $\mathcal{P} : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{X}^*$ be a penalty operator. The classical penalty method consists to replace (1.1) by a sequence of problems for finding $u_n \in \mathbb{X}$ such that

$$\langle \mathcal{D}(u_n,u_n), v-u_n\rangle + \frac{1}{\gamma_n}\langle \mathcal{P}(u_n,u_n), v-u_n\rangle + \varphi(u_n,v) - \varphi(u_n,u_n) + \jmath^0(u_n, v-u_n) \geq \langle f, v-u_n\rangle, v \in \mathbb{X}, \quad (1.3)$$

where γ_n is a sequence, for each $n \in \mathbb{N}$.

It should be noted that (1.3) is formally obtained from (1.1) by eliminating the constraint $u \in \Omega$ and inserting a penalty term regulated by a parameter $\gamma_n > 0$ and an operator $\mathcal{P} : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{X}^*$. Penalty methods have been used as an approximation tool to deal with constraints in variational inequalities and variational-hemivariational inequalities, *see* [8–10]. The numerical approximation of the variational-hemivariational inequality (1.1) affords another class of convergence result. A solution to problem (1.1) is given for a sequence $\{\Omega_n\}$, where the objective is to find $u_n \in \Omega_n$ such that

$$\langle \mathcal{D}(u_n,u_n), v-u_n\rangle + \varphi(u_n,v) - \varphi(u_n,u_n) + \jmath^0(u_n, v-u_n) \geq \langle f, v-u_n\rangle, v \in \Omega_n. \quad (1.4)$$

Take note that in a wide range of applications

$$\Omega_n = \mathbb{X}_n \bigcap \Omega$$

where the finite element method is used to construct a finite-dimensional space $\mathbb{X}$.
This paper aims to first propose a sequence of problem (1.1), which is denoted by problem $(1.1)_n$ and to show that problem $(1.1)_n$ has at least one solution, that converges to the solution of problem (1.1).

2. Preliminaries

Throughout this work, we use the symbols $\rightharpoonup$ and $\rightarrow$ to represent weak and strong convergence in different spaces, and all of the limits, upper and lower limits below are treated as $n \longrightarrow \infty$, even if we do not explicitly state it. Assume that $\|\cdot\|_{\mathbb{X}}$, $\|\cdot\|_{\mathbb{X}^*}$, and the zero element $0_{\mathbb{X}}, 0_{\mathbb{X}^*}$ are on the spaces $\mathbb{X}$ and $\mathbb{X}^*$. The space $\mathbb{X}^*$ equipped with the weak* topology is denoted by X^*_{ω}. Any reflexive

Banach space $\mathbb{X}$ is known to always be strictly convex, and its duality mapping $J : \mathbb{X} \longrightarrow 2^{\mathbb{X}^*}$, defined by

$$Jx = \{x^* \in \mathbb{X}^* : \langle x^*, x\rangle = ||x||^2_{\mathbb{X}} = ||x^*||^2_{\mathbb{X}^*}\} \ \forall x \in \mathbb{X}$$

an operator with a single value.

We recall the following definitions for set-valued and single-valued operators defined on $\mathbb{X}$.

Definition 2.1. *A set-valued operator, $\mathcal{T} : \mathbb{X} \longrightarrow 2^{\mathbb{X}^*}$ is referred to as*

(a) *pseudo monotone if*

(i) *A nonempty set $\mathcal{T}(u) \subset \mathbb{X}^*$ is closed and convex for all $u \in \mathbb{X}$.*

(ii) *$\mathcal{T} : \mathbb{X} \longrightarrow \mathbb{X}^*_\omega$ is a upper semicontinuous.*

(iii) *For any sequences $\{u_n\} \subset \mathbb{X}$ and $\{u^*_n\} \subset \mathbb{X}^*$ such that $u_n \xrightarrow{weakly} u \in \mathbb{X}$ and*

$$\lim\sup\langle u^*_n, u_n - u\rangle \leq 0, \forall u^*_n \in \mathcal{T}(u), \forall n \in \mathbb{N},$$

then there exists $u^*(v) \in \mathcal{T}(u)$ such that

$$\langle u^*(v), u - v\rangle \leq \lim\inf\langle u^*_n, u_n - v\rangle \forall v \in \mathbb{X}.$$

(b) *generalized pseudo monotone if*

any sequences $\{u_n\} \subset \mathbb{X}$ and $\{u^*_n\} \subset \mathbb{X}^*$ such that $u_n \xrightarrow{weakly} u \in \mathbb{X}$, and for all $n \in \mathbb{N}$, $u^*_n \in \mathcal{T}(u_n)$ and

$$u^*_n \longrightarrow u^* \in \mathbb{X}^*_w$$

implies that

$$\lim\sup\langle u^*_n, u_n - u\rangle \leq 0,$$

and

$$\lim\langle u^*_n, u_n\rangle = \langle u^*, u\rangle, \forall u^* \in \mathcal{T}(u).$$

Definition 2.2. *[11] A single-valued operator $\mathcal{D} : \mathbb{X} \longrightarrow \mathbb{X}^\star$ is said to be*

(i) *monotone, if*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v\rangle \geq 0, \ \forall u, v \in \mathbb{X}.$$

(ii) *inverse strongly monotone, if there exists $\alpha_{\mathcal{D}} > 0$ such that*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v\rangle \geq \alpha_{\mathcal{D}}||\mathcal{D}(u) - \mathcal{D}(v)||^2, \ \forall \ u, v \in \mathbb{X}.$$

(iii) *Lipschitz continuous if there exists $\beta_{\mathcal{D}} \geq 0$ such that*

$$||\mathcal{D}(u) - \mathcal{D}(v)|| \leq \beta_{\mathcal{D}}||u - v||, \ \forall \ u, v \in \mathbb{X}.$$

(iv) *bounded, if $\mathcal{D}$ maps bounded sets of $\mathbb{X}$ to bounded sets of $\mathbb{X}^*$.*

(v) *pseudo monotone, if it is bounded and $u_n \xrightarrow{weakly} u \in \mathbb{X}$ with*

$$\lim\sup\langle \mathcal{D}(u_n), u_n - u\rangle \leq 0,$$

implies that

$$\lim\inf\langle \mathcal{D}(u_n), u_n - v\rangle \geq \langle \mathcal{D}(u), u - v\rangle, \ \forall \ v \in \mathbb{X}.$$

(vi) *stable pseudomonotone with respect to the set* $W \subset \mathbb{X}^*$, *if* $\mathcal{D}$ *and* $u \mapsto \mathcal{D}u - u$ *are pseudomonotone for all* $w \in W$.

(vii) *demi-continuous, if* $u_n \longrightarrow u \in \mathbb{X}$ *implies that*

$$\mathcal{D}(u_n) \xrightarrow{weakly} \mathcal{D}(u) \in \mathbb{X}^*.$$

Proposition 2.1. *[12,13]*

(a) *If a function* $\mathcal{D} : \mathbb{X} \longrightarrow \mathbb{X}^*$ *is bounded, demicontinuous and monotone, then it is pseudomonotone.*

(b) *If* $\mathcal{D}, \mathcal{P} : \mathbb{X} \longrightarrow \mathbb{X}^*$ *are pseudomonotone operators, then the sum* $\mathcal{D} + \mathcal{P} : \mathbb{X} \longrightarrow \mathbb{X}^*$ *is also pseudomonotone.*

Definition 2.3. *[14,15] A function* $\jmath : \mathbb{X} \longrightarrow \mathbb{R}$ *is considered locally Lipschitz if, for each* $u \in \mathbb{X}$, *there is a neighbourhood of* x *denoted by* $\mathcal{U}_x$ *and a constant* $\mathfrak{L}_x > 0$ *such that*

$$\mid \jmath(y) - \jmath(z) \mid \leq \mathfrak{L}_x \|y - z\|_{\mathbb{X}}, \forall\, y, z \in \mathcal{U}_x.$$

Given a direction $v \in \mathbb{X}$ *and a point* $x \in \mathbb{X}$, *the generalised directional derivative of* $\jmath$ *is defined as*

$$\jmath^0(x, v) = \limsup_{y \longrightarrow x, \gamma \downarrow 0} \frac{\jmath(y + \gamma v) - \jmath(y)}{\gamma}.$$

The Clarke subdifferential of $\jmath$ *at* x *is a subset of the dual space* $\mathbb{X}$ *defined by*

$$\partial \jmath(x) = \left\{ \zeta \in \mathbb{X}^* : \jmath^0(x, v) \geq \langle \zeta, v \rangle \; \forall \, v \in \mathbb{X} \right\}.$$

The function $\jmath$ *is regular at the point* $x \in \mathbb{X}$ *if the one-sided directional derivative* $\jmath'(x, v)$ *exists for* $v \in \mathbb{X}$ *and*

$$\jmath^0(x, v) = \jmath'(x, v).$$

Proposition 2.2. *Given a local Lipschitz function* $\jmath : \mathbb{X} \longrightarrow \mathbb{R}$. *Then the subsequent hold:*

(a) *Positive homogeneity and subadditiveness characterise the function* $\mathbb{X} \ni v \rightarrowtail \jmath^0(x, v) \in \mathbb{R}$ *for each* $x \in \mathbb{X}$ *and*

$$\jmath^0(x, \gamma v) = \gamma \jmath^0(x, v), \forall \gamma \geq 0, \; v \in \mathbb{X}$$

and

$$\jmath^0(x, v_1 + v_2) \leq \jmath^0(x, v_1) + \jmath^0(x, v_2), \forall v_1, v_2 \in \mathbb{X}.$$

(b) *Considering each* $v \in \mathbb{X}$, *we get*

$$\jmath^0(x, v) = \max\{\langle \xi, v \rangle : \xi \in \partial \jmath(x)\}.$$

(c) *The gradient* $\partial \jmath(x)$ *is a nonempty, convex, and compact subset of* $\mathbb{X}^*_w$, *bounded by the Lipschitz constant* $\mathcal{L} > 0$ *of* $\jmath$ *near* x.

Definition 2.4. *Consider* $\{\Omega_n\}$ *a sequence of nonempty subsets of* V, *and* Ω *a nonempty subset of* $\mathbb{X}$. *If* $\{\Omega_n\} \xrightarrow{Mosco} \tilde{\Omega}$, *the conditions below are met.*

(a) *There is a sequence* $\{v_n\} \subset \mathbb{X}$ *such that* $v_n \in \Omega_n$ *for each* $n \in \mathbb{N}$.

$$v_n \longrightarrow v \in \tilde{\Omega} \subset \mathbb{X}.$$

(b) *For each sequence* $\{v_n\}$ *such that* $v_n \in \Omega_n$ *for* $n \in \mathbb{N}$ *and*

$$v_n \rightharpoonup v \in \tilde{\Omega} \subset \mathbb{X}.$$

Proposition 2.3. *Assume* C *is a nonempty closed convex subset of* $\mathbb{X}$*, and let* y *be an arbitrary element of* C*. Additional, let* C^* *be a bounded and nonempty closed convex subset of* $\mathbb{X}^*_\omega$*. Let* $\varphi\colon \mathbb{X} \longrightarrow \mathbb{R}$ *be a proper, convex lower semi-continuous function. There exists* $x^*(x) \in C$ *such that*

$$\langle x^*(x), x - y\rangle \geq \varphi(y) - \varphi(x), \forall x \in C.$$

Then, there exists $y^* \in C^*$ *such that*

$$\langle y^*, x - y\rangle \geq \varphi(y) - \varphi(x)\ \forall\ x \in C.$$

Definition 2.5. *An operator* $\mathcal{P} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}^*$ *is said to be a penalty operator of the set* $\Omega \subset \mathbb{X}$ *if* $\mathcal{P}$ *is bounded, demicontinuous, monotone and*

$$\Omega = \{x \in \mathbb{X} \mid \mathcal{P}(x, x) = 0_{\mathbb{X}^*}\}.$$

The mixed variational-hemivariational inequality (1.1) is unique and its existence is based on the following data assumptions.

(i)

$$\Omega \text{ is a nonempty, closed and convex subset of } \mathbb{X}. \tag{2.1}$$

(ii) $\mathcal{D} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}^*$ is a pseudo monotone and inverse strongly monotone for the constant $\alpha_{\mathcal{D}} > 0$, such that

$$\langle \mathcal{D}(u, u) - \mathcal{D}(v, v), u - v\rangle \geq \alpha_{\mathcal{D}}\|\mathcal{D}(u, u) - \mathcal{D}(v, v)\|^2, \forall u, v \in \mathbb{X}. \tag{2.2}$$

Lipschitz continuous for the first variable with constant $\beta_{\mathcal{D}} > 0$ and the second variable with constant $\rho_{\mathcal{D}} > 0$, such that

$$\|\mathcal{D}(u, u) - \mathcal{D}(v, v)\| \leq \beta_{\mathcal{D}}\|u - v\| + \rho_{\mathcal{D}}\|u - v\|, \forall u, v \in \mathbb{X}. \tag{2.3}$$

(iii) $\varphi : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}$ is such that

(a)

$$\varphi(\eta, \cdot) : \mathbb{X} \longrightarrow \mathbb{R} \text{ is convex and lower semicontinuous, for all } \eta \in \mathbb{X}. \tag{2.4}$$

(b) there exists $\alpha_\varphi \geq 0$ such that

$$\varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \leq \alpha_\varphi\|\eta_1 - \eta_2\|_{\mathbb{X}}\|v_1 - v_2\|_{\mathbb{X}}, \forall \eta_1, \eta_2, v_1, v_2 \in \mathbb{X}. \tag{2.5}$$

(iv) $\jmath : \mathbb{X} \longrightarrow \mathbb{R}$ ensures that

(a) $\jmath$ is locally Lipschitz and

$$\|\xi\|_{\mathbb{X}^*} \leq \bar{\varrho}_0 + \bar{\varrho}_1\|v\|_{\mathbb{X}}\ \forall\ v \in \mathbb{X}, \xi \in \partial\jmath(v), \text{ with } \bar{\varrho}_0, \bar{\varrho}_1 \geq 0. \tag{2.6}$$

(b) $\alpha_\jmath \geq 0$ exists such that

$$\jmath^0(v_1, v_2 - v_1) + \jmath^0(v_2, v_1 - v_2) \leq \alpha_\jmath\|v_1 - v_2\|^2_{\mathbb{X}},\ \forall v_1, v_2 \in \mathbb{X}. \tag{2.7}$$

(v)

$$\alpha_{\varphi} + \alpha_{J} < \alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2. \tag{2.8}$$

(vi)

$$f \in \mathbb{X}^*. \tag{2.9}$$

Theorem 2.1. *Assume that*(2.1)-(2.9) *are satisfied. Then inequality* (1.1) *has a unique solution* $u \in \Omega$.

Proof. We refer to Theorem 18 in [10] and [3, 16]. □

3. Main Results

In this segment, we assert and demonstrate the results' existence and convergence. We investigate a family $\{\Omega_n\}$ of subsets $\mathbb{X}$, a family of operators $\{\mathcal{P}_n\}$ defined on $\mathbb{X}$ with values in $\mathbb{X}^*$, and two sequences $\{\gamma_n\}$, $\{\varepsilon_n\} \subset \mathbb{R}$. Then, for each $n \in \mathbb{R}$, we approach the problem of finding $u_n \in \Omega_n$ such that

$$\langle \mathcal{D}(u_n, u_n), v - u_n\rangle + \frac{1}{\gamma_n}\langle \mathcal{P}_n(u_n, u_n), v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + J^0(u_n, v - u_n) \\ + \varepsilon_n \|v - u_n\|_{\mathbb{X}} \geq \langle f, v - u_n\rangle, \ \forall v \in \Omega_n. \tag{3.1}$$

To analyse (3.1), we assume that the following are true for each $n \in \mathbb{N}$.

$$\Omega_n \text{ is a closed convex subset of } \mathbb{X} \text{ that is not empty.} \tag{3.2}$$

$$\mathcal{P} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}^* \text{ is a monotone operator that is bounded and demicontinuous .} \tag{3.3}$$

$$\gamma_n > 0. \tag{3.4}$$

$$\varepsilon_n \geq 0. \tag{3.5}$$

$$\Omega \subset \Omega_n. \tag{3.6}$$

$$\langle \mathcal{P}_n(u, u), v - u\rangle \leq 0, \forall u \in \Omega_n, v \in \Omega. \tag{3.7}$$

$\tilde{\Omega} \subset \mathbb{X}$ is a set such that

$$\Omega_n \xrightarrow{Mosco} \tilde{\Omega} \text{ as } n \longrightarrow \infty. \tag{3.8}$$

$\mathcal{P}:\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}^*$ is an operator and a sequence $\{\varrho_n\}\subset\mathbb{R}$ such that

$$\begin{cases}(a)\ \|\mathcal{P}_n(u,u)-\mathcal{P}(u,u)\|_{\mathbb{X}^*}\leq\varrho_n(1+2\|u\|_{\mathbb{X}}),\ \forall u\in\Omega_n, n\in\mathbb{N}.\\(b)\ \varrho_n\longrightarrow 0 \text{ as } n\longrightarrow\infty.\\(c)\ \text{The operator } \mathcal{P} \text{ is bounded, demicontinuous and monotone.}\\(d)\ \langle\mathcal{P}(u,u),v-u\rangle\leq 0,\ \forall u\in\tilde{\Omega}, v\in\Omega.\\(e)\ \text{Any one of the following applies:}\\(i)\ \tilde{\Omega}=\mathbb{X} \text{ and } u\in\mathbb{X},\ \mathcal{P}(u,u)=0_{\mathbb{X}^*}\Rightarrow u\in\Omega.\\(ii)\ u\in\tilde{\Omega},\ \langle\mathcal{P}(u,u),v-u\rangle=0,\ \forall v\in\Omega\Rightarrow u\in\Omega.\end{cases}\tag{3.9}$$

There exists $\varrho_\varphi(u)\geq 0$ such that, for every $u\in\Omega$,

$$\varphi(u,v_1)-\varphi(u,v_2)\leq\varrho_\varphi(u)\|v_1-v_2\|_{\mathbb{X}},\ \forall v_1,v_2\in\mathbb{X}.\tag{3.10}$$

$$\gamma_n\longrightarrow 0 \text{ as } n\longrightarrow\infty.\tag{3.11}$$

$$\varepsilon_n\longrightarrow 0 \text{ as } n\longrightarrow\infty.\tag{3.12}$$

Theorem 3.1. *Assume that* (2.1)(2.9) *and* (3.2)-(3.5) *are satisfied. Then the following assertions are true.*

(a) *Each $n\in\mathbb{N}$ has at least one solution $u_n\in\Omega_n$ to* $(1.1)_n$. *Furthermore, the solution is unique when*

$$\varepsilon_n=0.$$

(b) *If* (2.1) *and* (3.6)-(3.12) *are satisfied, u is the solution of* (1.1), *and $\{u_n\}\subset\mathbb{X}$ is a sequence such that u_n is a solution of* $(1.1)_n$, *for each $n\in\mathbb{N}$, then*

$$u_n\longrightarrow u\in\mathbb{X}.$$

Proof. (a) Assume $n\in\mathbb{N}$. Proposition 2.1(a) and (3.3)-(3.4) show that the operator

$$\frac{1}{\gamma_n}\mathcal{P}_n:\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}^*$$

is pseudomonotone. Therefore, Proposition 2.1(b) and (2.2)-(2.3) demonstrate that the operator $\mathcal{D}_n:\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}^*$ defined by

$$\mathcal{D}_n=\mathcal{D}+\frac{1}{\gamma_n}\mathcal{P}_n$$

is also considered pseudomonotone. Thus, we can conclude that the operator $\mathcal{D}_n$ satisfies the conditions (2.2)-(2.3). Since the set Ω_n satisfies condition (3.2), and from Theorem 2.1, we can deduce the existence of a unique element $u_n\in\Omega_n$ such that

$$\langle\mathcal{D}(u_n,u_n),v-u_n\rangle+\frac{1}{\gamma_n}\langle\mathcal{P}_n(u_n,u_n),v-u_n\rangle+\varphi(u_n,v)-\varphi(u_n,u_n)+J^0(u_n,v-u_n)$$
$$\geq\langle f,v-u_n\rangle,\forall v\in\Omega_n.\tag{3.13}$$

This establishes $(1.1)_n$'s unique solvability for $\varepsilon_n = 0$.

Thus, for $\varepsilon_n > 0$, the solution u_n of (3.13) satisfies (3.1). This shows that there is at least one solution to the problem $(1.1)_n$.

(b) Suppose $n \in \mathbb{N}$. We start by assuming the auxiliary mixed variational-hemivariational inequality problem for finding $\tilde{u}_n \in \Omega_n$ such

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n), v - \tilde{u}_n \rangle + \frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n), v - \tilde{u}_n \rangle + \varphi(u, v) - \varphi(u, \tilde{u}_n) + \jmath^0(\tilde{u}_n, v - \tilde{u}_n) \geq \langle f, v - \tilde{u}_n \rangle, \ \forall v \in \Omega_n. \tag{3.14}$$

The inequality (3.14) is the same as (3.13), except that the first argument of φ in (3.14) is the solution u of (1.1). Using the same arguments as (3.13), Theorem 2.1 guarantees the existence of a unique solution to inequality (3.14). The remaining steps in the proof are then divided into four.

(i) We claim that there exists $\tilde{u} \in \tilde{\Omega}$ and a subsequence of $\{\tilde{u}_n\}$, represented by $\{\tilde{u}_n\}$, such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \in \mathbb{X}, \text{ as } n \longrightarrow \infty.$$

To prove the assertion, we first show that the sequence $\{\tilde{u}_n\}$ in $\mathbb{X}$ is bounded. Let $n \in \mathbb{N}$ and $u_0 \in \Omega$. Using the assumption (3.6), it follows that

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n), \tilde{u}_n - u_0 \rangle \leq \frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n), u_0 - \tilde{u}_n \rangle + \varphi(u, u_0) - \varphi(u, \tilde{u}_n) + \jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) + \langle f, \tilde{u}_n - u_0 \rangle. \tag{3.15}$$

Using the inverse strong monotonicity and Lipschitz continuity of $\mathcal{D}$ for the first and the second variables, we get

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2)\|\tilde{u}_n - u_0\|_{\mathbb{X}}^2 \leq \langle \mathcal{D}(u_0, u_0), u_0 - \tilde{u}_n \rangle + \frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n), u_0 - \tilde{u}_n \rangle + \varphi(u, u_0) - \varphi(u, \tilde{u}_n) + \jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) + \langle f, \tilde{u}_n - u_0 \rangle. \tag{3.16}$$

Using (3.4) and (3.7), we have

$$\frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n), u_0 - \tilde{u}_n \rangle \leq 0. \tag{3.17}$$

Based on (3.10), we have

$$\varphi(u, u_0) - \varphi(u, \tilde{u}_n) \leq \varrho_\varphi(u)\|\tilde{u}_n - u_0\|_{\mathbb{X}}. \tag{3.18}$$

We, however, write

$$\begin{aligned} \jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) &= \jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) + \jmath^0(u_0, \tilde{u}_n - u_0) - \jmath^0(u_0, \tilde{u}_n - u_0) \\ &\leq \jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) + \jmath^0(u_0, \tilde{u}_n - u_0) + | \jmath^0(u_0, \tilde{u}_n - u_0) | . \end{aligned}$$

Using (2.6) and Proposition 2.2(b), we obtain

$$\jmath^0(\tilde{u}_n, u_0 - \tilde{u}_n) \leq \alpha_\jmath\|\tilde{u}_n - u_0\|_{\mathbb{X}}^2 - (\bar{\varrho}_0 + \bar{\varrho}_1\|u_0\|_{\mathbb{X}})\|\tilde{u}_n - u_0\|_{\mathbb{X}}. \tag{3.19}$$

Now

$$\langle \mathcal{D}(u_0,u_0),u_0-\tilde{u}_n\rangle+\langle f,\tilde{u}_n-u_0\rangle \le \|f-\mathcal{D}(u_0,u_0)\|_{\mathbb{X}^*}\,\|\tilde{u}_n-u_0\|_{\mathbb{X}^*}. \tag{3.20}$$

Subsequently, we employ the combination of (3.16)-(3.20) to determine that

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}}+\rho_{\mathcal{D}})^2-\alpha_J)\|\tilde{u}_n-u_0\|_{\mathbb{X}} \le \varrho_\varphi(u)+\bar{\varrho}_0+\bar{\varrho}_1\|u_0\|_{\mathbb{X}}+\|f-\mathcal{D}(u_0,u_0)\|_{\mathbb{X}^*}. \tag{3.21}$$

We use (2.8) and (3.21) to show that $\{\tilde{u}_n\}$ is a bounded sequence in $\mathbb{X}$. Since $\mathbb{X}$ is reflexive, there exists $\tilde{u}\in\mathbb{X}$ and a subsequence of $\{\tilde{u}_n\}$, which is still denoted by $\{\tilde{u}_n\}$, such that

$$\tilde{u}_n \rightharpoonup \tilde{u}\in\mathbb{X}.$$

Since $\tilde{u}_n\in\Omega_n$ for all $n\in\mathbb{N}$. Using (3.8) and Definition 2.4, we conclude that

$$\tilde{u}\in\tilde{\Omega}.$$

(ii) We then assert that $\tilde{u}$ is the solution of (1.1), *i.e.*

$$\tilde{u}=u.$$

To demonstrate this claim, we utilize assumption (3.8) and consider an element $v\in\tilde{\Omega}$ in conjunction with a sequence $\{v_n\}\subset\mathbb{X}$ such that $v_n\in\Omega_n$ for every $n\in\mathbb{N}$ and $v_n\longrightarrow v\in\mathbb{X}$ as $n\longrightarrow\infty$. We now apply the inequality (3.14) with $v=v_n$ and the assumptions (2.2)-(2.3), (3.10), (2.6) to show that

$$\begin{aligned}\frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle &\le \langle \mathcal{D}(\tilde{u}_n,\tilde{u}_n)-\mathcal{D}(v_n,v_n),v_n-\tilde{u}_n\rangle+\varphi(u,v_n)-\varphi(u,\tilde{u}_n)\\ &\quad+J^0(\tilde{u}_n,v_n-\tilde{u}_n)+\langle f,\tilde{u}_n-v_n\rangle+\langle \mathcal{D}(v_n,v_n),v_n-\tilde{u}_n\rangle\\ &\le(\varrho_\varphi(u)+\bar{\varrho}_0+\bar{\varrho}_1\|\tilde{u}_n\|_{\mathbb{X}}+\|f-\mathcal{D}(v_n,v_n)\|_{\mathbb{X}^*})\|\tilde{u}_n-v_n\|_{\mathbb{X}}.\end{aligned}$$

Given that $v_n\longrightarrow v\in\mathbb{X}$, the boundedness of the sequence $\{\tilde{u}_n\}$ and the operator $\mathcal{D}$, there exists a constant $\curlywedge>0$ that does not depend on n such that

$$\langle \mathcal{P}_n(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle\le\gamma_n\curlywedge.$$

Using the upper limit of the aforementioned inequality and (3.11), we get

$$\limsup\langle \mathcal{P}_n(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle\le 0, \tag{3.22}$$

On the other hand, we write

$$\begin{aligned}\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v\rangle&=\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle+\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),v_n-v\rangle\\ &\le\|\mathcal{P}(\tilde{u}_n,\tilde{u}_n)-\mathcal{P}_n(\tilde{u}_n,\tilde{u}_n)\|_{\mathbb{X}^*}\|\tilde{u}_n-v_n\|_{\mathbb{X}}\\ &\quad+\langle \mathcal{P}_n(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle+\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),v_n-v\rangle.\end{aligned}$$

From the assumption (3.9)(a) we have

$$\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v\rangle\le\varrho_n(1+2\|\tilde{u}_n\|_{\mathbb{X}})\|\tilde{u}_n-v_n\|_{\mathbb{X}}+\langle \mathcal{P}_n(\tilde{u}_n,\tilde{u}_n),\tilde{u}_n-v_n\rangle+\langle \mathcal{P}(\tilde{u}_n,\tilde{u}_n),v_n-v\rangle. \tag{3.23}$$

We now employ assumptions (3.9)(b),(c), the boundedness of sequence $\{\tilde{u}_n\}$ and the convergence $v_n \longrightarrow v \in \mathbb{X}$ to argue that

$$\lim\left[\varrho_n(1+2\|\tilde{u}_n\|_{\mathbb{X}})\|\tilde{u}_n - v_n\|_{\mathbb{X}}\right] = 0, \tag{3.24}$$

$$\lim\langle \mathcal{P}(\tilde{u}_n, \tilde{u}_n), v_n - v\rangle = 0. \tag{3.25}$$

Next, we go to the upper limit of (3.23) and utilise (3.22), (3.24) and (3.25) to find

$$\limsup\langle \mathcal{P}(\tilde{u}_n, \tilde{u}_n), \tilde{u}_n - v\rangle \leq 0. \tag{3.26}$$

Putting $v = \tilde{u} \in \tilde{\Omega}$ in (3.26), we get

$$\limsup\langle \mathcal{P}(\tilde{u}_n, \tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle \leq 0.$$

The assumption (3.9)(c) ensures that $\mathcal{P} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}^*$ is pseudomonotone. Thus, based on the pseudomonotonicity of $\mathcal{P}$, we can conclude that

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), \tilde{u} - v\rangle \leq \liminf\langle \mathcal{P}(\tilde{u}_n, \tilde{u}_n), \tilde{u}_n - v\rangle. \tag{3.27}$$

We may show by combining the inequalities (3.26) and (3.27) that

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), \tilde{u} - v\rangle \leq 0. \tag{3.28}$$

Remember that for every $v \in \tilde{\Omega}$, this inequality holds true.

Let us assume that (3.9)(e)(i) is met. Inequality (3.28) then suggests that

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), \tilde{u} - v\rangle \leq 0, \ \forall v \in \mathbb{X},$$

which yields

$$\mathcal{P}(\tilde{u}, \tilde{u}) = 0_{\mathbb{X}^*}.$$

Hence, $\tilde{u} \in \Omega$. Assume that (3.9)(e)(ii) is met. Using (3.6) and (3.8), we can see that $\Omega \subset \tilde{\Omega}$, and thus from (3.9), we get

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), \tilde{u} - v\rangle \leq 0, \ \forall v \in \Omega.$$

However, employing (3.9)(d) to obtain

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), v - \tilde{u}\rangle \leq 0, \ \forall v \in \Omega.$$

It can be seen from the last two inequality

$$\langle \mathcal{P}(\tilde{u}, \tilde{u}), v - \tilde{u}\rangle \leq 0, \ \forall v \in \Omega.$$

According to (3.9)(e)(ii),

$$\tilde{u} \in \Omega.$$

Based on the above, we can conclude that either (3.9)(e)(i) or (3.9)(e)(ii), and we have

$$\tilde{u} \in \Omega. \tag{3.29}$$

Then, we use (3.6) and (3.9) to obtain

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n), v - \tilde{u}_n \rangle + \frac{1}{\gamma_n} \langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n), v - \tilde{u}_n \rangle + J^0(\tilde{u}_n, v - \tilde{u}_n) - \langle f, v - \tilde{u}_n \rangle$$
$$\geq \varphi(u, \tilde{u}_n) - \varphi(u, v), \ \forall v \in \Omega, n \in \mathbb{N}. \tag{3.30}$$

According to Proposition 2.2(b), there exists a $\zeta_n(\tilde{u}_n, v) \in \partial J(\tilde{u}_n)$ such that

$$J^0(\tilde{u}_n, v - \tilde{u}_n) = \langle \zeta_n(\tilde{u}_n, v), v - \tilde{u}_n \rangle.$$

Hence, (3.30) gives

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \frac{1}{\gamma_n} \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n, v) - f, v - \tilde{u}_n \rangle \geq \varphi(u, \tilde{u}_n) - \varphi(u, v), \forall v \in \Omega. \tag{3.31}$$

Proposition 2.2(c) ensures a nonempty set

$$C^* = \{\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \frac{1}{\gamma_n} \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n) + \xi_n - f : \ \xi_n \in \partial J(\tilde{u}_n)\} \tag{3.32}$$

is closed, convex, and bounded in $\mathbb{X}^*_w$. Assumption (2.4) allows us to apply Proposition 2.3 to $C = \Omega$ and C^*, defined by (3.32), $x = v$ and $y = \tilde{u}_n$. This way, we discover that there exists $\zeta_n(\tilde{u}_n) \in \partial J(\tilde{u}_n)$ which does not depend on v so that

$$\langle \mathcal{P}(\tilde{u}_n, \tilde{u}_n) + \frac{1}{\gamma_n} \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n) - f, v - \tilde{u}_n \rangle \geq \varphi(u, \tilde{u}_n) - \varphi(u, v), \forall v \in \Omega. \tag{3.33}$$

Hence, (3.4) and (3.7) offer

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - v \rangle \leq \varphi(u, v) - \varphi(u, \tilde{u}_n) - \langle f, v - \tilde{u}_n \rangle, \forall v \in \Omega. \tag{3.34}$$

Using (3.29), replace $v = \tilde{u}$ in (3.34). In the resulting inequality, we pass to the upper limit, $\tilde{u}_n \rightharpoonup \tilde{u} \in \mathbb{X}$, and the lower semicontinuity of φ for its second argument to deduce that

$$\limsup \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - \tilde{u} \rangle \leq 0. \tag{3.35}$$

The boundedness of $\{\tilde{u}_n\}$ and $\mathcal{D}$ ensured that the sequence $\{\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n)\}$ is bounded in $\mathbb{X}^*$, as a result of the assumptions (2.2)-(2.3), (2.6). Thus, $\{\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n)\}$ represents a subsequence of the sequence, which is still denoted by $\{\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n)\}$, and an element $\theta \in \mathbb{X}^*$ such that

$$\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n) \rightharpoonup \theta \in \mathbb{X}^*_w. \tag{3.36}$$

According to [[10], Lemma 20], the set-valued operator $\mathcal{D} + \partial J : \mathbb{X} \longrightarrow 2^{\mathbb{X}^*}$ is a generalised pseudomonotone. Using the Definition 2.1 and the components $\{\tilde{u}_n\} \subset \mathbb{X}$, $\{\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n)\} \subset \mathbb{X}^*$, $\tilde{u}_n \rightharpoonup \tilde{u} \in \mathbb{X}$,

$$\mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n) \in \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \partial J(\tilde{u}_n).$$

and (3.35), (3.36), we have

$$\theta \in \mathcal{D}(\tilde{u}, \tilde{u}) + \partial J(\tilde{u})$$

and

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n \rangle \to \langle \theta, \tilde{u} \rangle. \tag{3.37}$$

On the other hand, (3.36) implies that

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}\rangle \to \langle \theta, \tilde{u}\rangle. \tag{3.38}$$

Now, we combine the convergence of (3.37) and (3.38) to find that

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle \to 0. \tag{3.39}$$

From the inclusion

$$\theta \in \mathcal{D}(\tilde{u}, \tilde{u}) + \partial \jmath(\tilde{u})$$

suggests that $\zeta(\tilde{u}) \in \partial \jmath(\tilde{u})$ exists such that

$$\theta = \mathcal{D}(\tilde{u}, \tilde{u}) + \zeta(\tilde{u}). \tag{3.40}$$

Now, consider the element $v \in \Omega$. We write

$$\begin{aligned}\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - v\rangle &= \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle \\ &\quad + \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u} - v\rangle.\end{aligned} \tag{3.41}$$

From (3.37), (3.39), and (3.40), it can be observed that

$$\lim \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - v\rangle = \langle \mathcal{D}(\tilde{u}, \tilde{u}) + \zeta(\tilde{u}), \tilde{u} - v\rangle. \tag{3.42}$$

Then, by applying (2.4) and going to the upper limit in (3.34), we obtain

$$\langle \mathcal{D}(\tilde{u}, \tilde{u}) + \zeta(\tilde{u}), \tilde{u} - v\rangle \leq \varphi(u, v) - \varphi(u, \tilde{u}) - \langle f, v - \tilde{u}\rangle$$

or, equivalently,

$$\langle f, v - \tilde{u}\rangle \leq \langle \mathcal{D}(\tilde{u}, \tilde{u}), v - \tilde{u}\rangle + \varphi(u, v) - \varphi(u, \tilde{u}) + \langle \zeta(\tilde{u}), v - \tilde{u}\rangle. \tag{3.43}$$

However, by applying the Clarke subdifferential definition, we have

$$\langle \zeta(\tilde{u}), v - \tilde{u}\rangle \leq \jmath^0(\tilde{u}, v - \tilde{u}). \tag{3.44}$$

Adding (3.43) and (3.44) we have

$$\langle f, v - \tilde{u}\rangle \leq \langle \mathcal{D}(\tilde{u}, \tilde{u}), v - \tilde{u}\rangle + \varphi(u, v) - \varphi(u, \tilde{u}) + \jmath^0(\tilde{u}, v - \tilde{u}). \tag{3.45}$$

Finally, we can see from (3.29) and (3.45) that $\tilde{u}$ is a solution to (1.1). Given the solution's uniqueness, we have

$$\bar{u} = u$$

as claimed.

(iii) We now demonstrate that the whole sequence $\{\tilde{u}_n\}$ converges to u.

A closed analysis of the evidence in step (ii) reveals that any subsequence of $\{\tilde{u}_n\}$, that weakly converges in $\mathbb{X}$ has the same weak limit u. Furthermore, we recall that the sequence $\{\tilde{u}_n\}$ is bounded in $\mathbb{X}$. Therefore, we may derive a standard argument that the whole sequence $\{\tilde{u}_n\}$ converges weakly $u \in \mathbb{X}$ as $n \longrightarrow \infty$. This shows that all of the claims in step

(ii) are true for the whole sequence $\{\tilde{u}_n\}$. In particular, (3.39) paired with equality $\tilde{u} = u$ proves that

$$\langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - u\rangle \to 0. \tag{3.46}$$

Given $n \in \mathbb{N}$, let $\zeta(u) \in \partial \jmath(u)$, we have

$$\langle \zeta(u), \tilde{u}_n - u\rangle \leq \jmath^0(u, \tilde{u}_n - u),$$
$$\langle \zeta_n(\tilde{u}_n), u - \tilde{u}_n\rangle \leq \jmath^0(\tilde{u}_n, u - \tilde{u}_n),$$

implies that

$$\langle \zeta(u), \tilde{u}_n - u\rangle + \langle \zeta_n(\tilde{u}_n), u - \tilde{u}_n\rangle \leq \jmath^0(u, \tilde{u}_n - u) + \jmath^0(\tilde{u}_n, u - \tilde{u}_n).$$

Using (2.7), we can see

$$-\alpha_\jmath \|\tilde{u}_n - u\|_{\mathbb{X}}^2 \leq \langle \zeta(u), u - \tilde{u}_n\rangle + \langle \zeta_n(\tilde{u}_n), \tilde{u}_n - u\rangle. \tag{3.47}$$

On the other hand, (2.2)-(2.3) produces

$$\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 \|\tilde{u}_n - u\|_{\mathbb{X}}^2 \leq \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) - \mathcal{D}(u, u), \tilde{u}_n - u\rangle. \tag{3.48}$$

Adding the inequalities (3.47) and (3.48) to deduce that

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 - \alpha_\jmath)\|\tilde{u}_n - u\|_{\mathbb{X}}^2 \leq \langle \mathcal{D}(\tilde{u}_n, \tilde{u}_n) + \zeta_n(\tilde{u}_n), \tilde{u}_n - u\rangle + \langle \mathcal{D}(u, u) + \zeta(u), u - \tilde{u}_n\rangle. \tag{3.49}$$

Next, from the convergence of (3.46), $\tilde{u}_n \rightharpoonup u \in \mathbb{X}$ and (2.8) to find that

$$\|\tilde{u}_n - u\|_{\mathbb{X}}^2 \to 0, \tag{3.50}$$

which support the claim that $\tilde{u}_n \longrightarrow u \in \mathbb{X}$ as $n \longrightarrow \infty$.

(iv) In the final stage of the proof, we demonstrate that $u_n \longrightarrow u \in \mathbb{X}$, as $n \longrightarrow \infty$.

Assume that $n \in \mathbb{N}$. We test using $v = \tilde{u}_n$ in (3.1) and $v = u_n$ in (3.14). We then sum up the resulting inequalities to observe that

$$\begin{aligned}\langle \mathcal{D}(u_n, u_n) - \mathcal{D}(\tilde{u}_n, \tilde{u}_n), u_n - \tilde{u}_n\rangle \leq{}& \frac{1}{\gamma_n}\langle \mathcal{P}_n(\tilde{u}_n, \tilde{u}_n) - \mathcal{P}_n(u_n, u_n), u_n - \tilde{u}_n\rangle + \varphi(u_n, \tilde{u}_n)\\ &- \varphi(u_n, u_n) + \varphi(u, u_n) - \varphi(u, \tilde{u}_n)\\ &+ \jmath^0(u_n, \tilde{u}_n - u_n) + \jmath^0(\tilde{u}_n, u_n - \tilde{u}_n) + \varepsilon_n\|\tilde{u}_n - u_n\|_{\mathbb{X}}.\end{aligned}$$

By applying (3.3), (2.5), and (2.7), we may obtain

$$\begin{aligned}\langle \mathcal{D}(u_n, u_n) - \mathcal{D}(\tilde{u}_n, \tilde{u}_n), u_n - \tilde{u}_n\rangle \leq{}& \alpha_\varphi\|u_n - u\|_{\mathbb{X}}\|\tilde{u}_n - u_n\|_{\mathbb{X}} + \alpha_\jmath\|\tilde{u}_n - u_n\|_{\mathbb{X}}^2\\ &+ \varepsilon_n\|\tilde{u}_n - u_n\|_{\mathbb{X}}.\end{aligned}$$

By employing (2.2)-(2.3), we attain

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 - \alpha_\jmath)\|\tilde{u}_n - u_n\|_{\mathbb{X}} \leq \alpha_\varphi\|u_n - u\|_{\mathbb{X}} + \varepsilon_n. \tag{3.51}$$

Now, we compose

$$\alpha_\varphi\|u_n - u\|_{\mathbb{X}} \leq \alpha_\varphi\|u_n - \tilde{u}_n\|_{\mathbb{X}} + \alpha_\varphi\|\tilde{u}_n - u\|_{\mathbb{X}}. \tag{3.52}$$

Therefore from the inequalities (3.51) and (3.52), we obtain

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 - \alpha_{\varphi} - \alpha_{J})\|\tilde{u}_n - u_n\|_{\mathbb{X}} \leq \alpha_{\varphi}\|\tilde{u}_n - u\|_{\mathbb{X}} + \varepsilon_n. \tag{3.53}$$

Using (2.8), we find that

$$\|\tilde{u}_n - u_n\|_{\mathbb{X}} \leq \frac{\alpha_{\varphi}}{\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 - \alpha_{\varphi} - \alpha_{J}}\|\tilde{u}_n - u\|_{\mathbb{X}} + \frac{\varepsilon_n}{\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \rho_{\mathcal{D}})^2 - \alpha_{\varphi} - \alpha_{J}}. \tag{3.54}$$

The inequality (3.54), the convergence (3.50) and assumption (3.12) implies that

$$\|\tilde{u}_n - u_n\|_{\mathbb{X}} \to 0. \tag{3.55}$$

Finally, we can rewrite

$$\|u_n - u\|_{\mathbb{X}} \leq \|u_n - \tilde{u}_n\|_{\mathbb{X}} + \|\tilde{u}_n - u\|_{\mathbb{X}}.$$

Therefore, from (3.50) and (3.55) we have

$$u_n \to u \in \mathbb{X},$$

and the proof is completed.

□

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