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On KU-Modules Over KU-Algebras

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Abstract. The paper introduces the concept of modules for KU-algebras, named as KU-modules. It presents basic isomorphism theorems for KU-modules and explores their applications, particularly concerning chains of KU-modules. Additionally, it defines and examines exact sequences of KU-modules. The paper discusses various properties of chains of KU-modules and establishes the butterfly lemma in the context of KU-modules.

1. Introduction

KU algebra is a type of logical algebras that was firstly introduced by Prabpayak and Leerawat [6]. Homomorphisms and results for them for KU-algebras are given in ([6], [7]). Next to the introduction of KU-algebras, numerous authors have extensively explored KU-algebras in various research directions, such as fuzzy, neutrosophic, and intuitionistic contexts, as well as in soft and rough senses. For instance, Naveed et al. [9] introduced concepts on cubic KU ideals, whereas fuzzy ideals within this context was given by Mostafa et al. [10]. Additionally, Mostafa et al. [11] delved into the study of Interval-valued fuzzy concepts for KU ideals. Moin and Ali studies and described roughness in a KU algebra [12], while Ali et al. [13] explored the notion of a pseudo-metric for a KU-algebras. Further, Atanassovs intuitionistic fuzzy binormed KU ideals within the framework of KU-algebras was given and studied by Senapati and K.P. Shum [14]. More recently, Ali et al. [3] constructed graphs based on the KU ideals.

The notion of BCK-algebras that is an interesting class of logical algebras and that gives generalization for set-theoretic differences and proportional calculi was introduced by Imai and Iseki [8] as an important tools in logical algebras. This concept shares similarities with the development of Boolean logic based on Boolean algebras. Furthermore, Iseki [5] introduced a superclass of

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BCK-algebras and said that as BCI-algebras that encompasses as a class of BCK-algebras. This article is mainly distributed in six sections.
Section 1 is introduction part for motivation and interest based on KU algebra. Section 2 focuses on general concepts associated with KU-algebras. It initiates by discussing the fundamental concepts of KU-algebras, elucidating their elementary properties. Subsequently, the discussion extends to the exploration of ideals within a KU algebra along with their associated properties. In Section 3, we delve into the exploration of KU-modules, presenting various examples and elucidating their properties. Section 4 is dedicated to the examination of chains of KU-modules. We establish the minimal and maximal conditions for submodules, introduce the Schreier Refinement Theorem, and expound upon the Jordan-Holder Theorem within the context of KU-modules. In Section 5, our focus centers on the study of exact sequences with some of their algebraic properties. Section 6, is investigation of projective and injective KU-modules, outlining their respective properties and implications.

2. Preliminaries

This section is based on related definitions and notations concerning KU-algebras, KU ideals and their posets are discussed.

Definition 2.1. [6] The structure $(X, \circ, 1)$ with binary operation $\circ$ of type $(2, 0)$ is called a KU algebra if for any $a, b, c \in X$ we have:
ku(1) $(a \circ b) \circ [(b \circ c) \circ (a \circ c)] = 1$,
ku(2) $a \circ 1 = 1$,
ku(3) $1 \circ a = a$,
ku(4) $a \circ b = b \circ a = 1 \Rightarrow a = b$.

If not specified then, we consider $(X, \circ, 1)$, a KU algebra. X will indicate a KU algebra throughout the text. 1 is a constant element and fixed for X. The condition $u \leq v$ indicates a partial order $'' \leq ''$ in X iff $v \circ u = 1$.

Lemma 2.1. [6] The structure $(X, \circ, 1)$ is a KU algebra iff:
ku(5) $a \circ b \leq (b \circ c) \circ (a \circ c)$,
ku(6) $a \leq 1$,
ku(7) $a \leq b, b \leq a \Rightarrow a = b$,

Lemma 2.2. *In the structure $(X, \circ, 1)$, the given identities are valid* [10]:
1. $c \circ c = 1$,
2. $c \circ (a \circ c) = 1$,
3. $a \leq b$ imply $b \circ c \leq a \circ c$,
4. $c \circ (b \circ a) = b \circ (c \circ a)$, for all $a, b, c \in X$,
5. $b \circ [(b \circ a) \circ a = 1$.

Example 2.1. [10] Considering $X = \{u, v, w, x, y\}$ we observe that X is a KU algebra with single binary operation $\circ$ that is given by the following table

$\circ$	u	v	w	x	y
u	u	v	w	x	y
v	u	u	w	x	y
w	u	v	u	x	x
x	u	u	w	u	w
y	u	u	u	u	u

Definition 2.2. *The following is defined for a KU algebra X :*
(i) *commutativity:* $(y \circ x) \circ x = (x \circ y) \circ y$, *for all* $x, y \in X$.
(ii) *Implicativity:* $(x \circ y) \circ x = x$, *for all* $x, y \in X$.
(iii) *Boundedness: if there exists an element* $1 \in X$ *so that for each* $x \in X, x \leq 1$.

Definition 2.3. [6] A KU ideal is a non-void subset A of X that satisfies:
1. $1 \in A$,
2. $u \circ (v \circ w) \in A, v \in A \Rightarrow u \circ w \in A$, for all $u, v, w \in X$.

Definition 2.4. *Maximal ideal in X is a KU ideal I_M of X so that proper ideal I_M of X is not a proper subset of any proper ideal of X. Such maximal ideal in KU algebra is called maximal KU ideal.*

Definition 2.5. *For* $|X| \geq 2$, *in a bounded KU algebra X, there is at least one maximal ideal.*

Definition 2.6. *A proper commutative ideal of a bounded KU algebra is called maximal commutative ideal containing I_M.*

Definition 2.7. *Any implicative KU algebra (that is proper commutative) ideal of a bounded KU algebra is called maximal commutative ideal containing I_M.*

Example 2.2. [12] Let $X = \{1, b, c, u, v, w\}$ in which $\circ$ is defined by the following table:

$\circ$	1	b	c	u	v	w
1	1	b	c	u	v	w
b	1	1	c	c	v	w
c	1	1	1	b	v	w
u	1	1	1	1	v	w
v	1	1	1	b	1	w
w	1	1	b	1	1	1

We see that $(X, \circ, 1)$ is a KU algebra. Further we can show that $A = \{1, b\}$ is a KU ideal and $B = \{1, b, c, u, v\}$ is KU ideal of X.

The following conditions for a non-void set with a single binary operation construct a KU algebra.

Proposition 2.1. $(X, \circ, 1)$ *is a KU algebra iff:*
i. $(b \circ c) \circ ((c \circ a) \circ (b \circ a)) = 1$ *for all* $a, b, c \in X$;
ii. $(b \circ 1) \circ a = a$ *for all* $a, b \in X$;
iii. *For all* $a, b, c \in X$ *such that* $a \circ b = 1, b \circ a = 1 \Rightarrow a = b$.

Proof. Considering direct part and saying $(X, \circ, 1)$ is a KU algebra, we see that i. follows from ku(1). Further, ii. follows from fourth condition of a KU algebra. iii. is directly followed from ku(2) and ku(3) as $(b \circ 1) \circ a = 1 \circ a = a$.
Considering indirect part, if $(X, \circ, 1)$ satisfies all three conditions above, then ku(1) and ku(4) holds true from i. and ii.. Now, replace b by a, a by 1 and c by 1 in i. and use ii. we find that, $(a \circ 1) \circ [(1 \circ 1) \circ (a \circ 1)] = 1 \Rightarrow (1 \circ 1) \circ (a \circ 1) = 1 \Rightarrow a \circ 1 = 1$ that proves ku(2). Further, using $a \circ 1 = 1$ in ii. we have, $1 \circ a = a$ for every $a \in X$. Hence this proves that $(X, \circ, 1)$ is a KU algebra. □

Definition 2.8. *A prime ideal P of X where X is a KU algebra is defined as, if* $(b * a) * a \in P \Rightarrow a \in P$ *or* $b \in P$.

Definition 2.9. *A KU algebra X is called a bounded KUalgebra if for an* $e \in X$ *that exists in X so that* $x \leq e$ *for any* $x \in X$. *The element e is called unit of X. In a bounded KU algebra, we write* $x * e$ *for* $\mathfrak{N}(X)$.

Theorem 2.1. *If X is a bounded KU algebra having greatest element 1, then for any* $x, y \in X$:
1. $\mathfrak{N}1 = 0$ *and* $\mathfrak{N}0 = 1$.
2. $\mathfrak{N}y * \mathfrak{N}x \leq x * y$.
3. $y \leq x \Rightarrow \mathfrak{N}x \leq \mathfrak{N}y$.

Theorem 2.2. *For a bounded KU algebra X, it is commutative iff* $x \wedge y = (y * x) * x, xvy = \mathfrak{N}(\mathfrak{N}x \wedge \mathfrak{N}y)$.

We have the following theorem that we shall use later on:

Theorem 2.3. *We have given assertions that hold true:*
i. *If X is implicative, then it is commutative.*
ii. *If X is commutative, then it forms a lattice with* $x \vee y = \mathfrak{N}(\mathfrak{N}_x \vee \mathfrak{N}_y)$ *and* $x \wedge y = (x * y) * y$.
iii. *Bounded implicative KU algebra and Boolean algebra are equivalent.*

Lemma 2.3. *If X is a commutative KU algebra, then* $(X, \leq)$ *is a lower KU-semilattice. If X is bounded and commutative, then* $(X, \leq)$ *is a KU-lattice.*

Lemma 2.4. *If I is an ideal of a KU algebra X, we get* $I_1 \vee I_2 \in I$, *for any* $I_1, I_2 \in I$.

Theorem 2.4. *If* $(X, *, \wedge, \vee, 0, 1)$ *is a bounded implicative KU algebra, then we have, for all* $u, v \in X$:
i. $\mathfrak{N}\mathfrak{N}_u = u$.
ii. $\mathfrak{N}_u \vee \mathfrak{N}_v = \mathfrak{N}(u \wedge v)$ *and* $\mathfrak{N}_u \wedge \mathfrak{N}_v = \mathfrak{N}(u \vee v)$.
iii. $\mathfrak{N}_u * \mathfrak{N}_v = v \wedge u$.
iv. $u \wedge \mathfrak{N}_u = 0$.

v. $u \vee \mathfrak{N}_u = 1$.
vi. $(v * u) * u = \mathfrak{N}_v * u = \mathfrak{N}_v \wedge u = v * u$.
vii. $\mathfrak{N}\mathfrak{N}_u = u$.

3. KU-Modules

In this section we connect KU algebra with module theory concepts defining KU-modules and studying properties for them.

Definition 3.1. *For a KU algebra $(X, *, 0)$, an abelian group A under usual addition and a mapping $X \times A \rightarrow A$ defined by $(x, m) \rightarrow x \times m$ such that*
A_1. $(x \wedge y)m = x(ym)$
A_2. $x(A_1 + A_2) = xA_1 + xA_2$
A_3. $0m = 0$ *for all $x, y \in X$ and $A_1, A_2 \in A$ where $x \wedge y = (x * y) * y$. is called a left KU-module. In a similar fashion right KU-module is defined. In case X is bounded then for some $m \in A$, $1m = m$, A is called a unitary KU-module.*

Definition 3.2. *Consider A_1 and A_2 are KU-modules. A homomorphism is a mapping $f : A_1 \rightarrow A_2$ for any $A_1, A_2 \in A_1$ satisfying the conditions:*
(i) $f(A_1 + A_2) = f(A_1) + f(A_2)$.
(ii) $f(xA_1) = xf(A_1) \forall x \in X$.
$Ker(f) = \{x \in a_1 : f(x) = 0\}$ is kernel for f and $Im(f) = \{f(A_1) \in a_2 : A_1 \in A_1\}$ is image for f.

Ker$\{f\}$ is a submodule of A_1 and Im$\{f\}$ is a submodule of A_2. Also f is monomorphism iff Ker$\{f\}$ is 0. Now we have the following isomorphism theorem:

Definition 3.3. *Let A_1 and A_2 are KU-modules and $f : A_1 \rightarrow A_2$ be an epimorphism. If B is a submodule of A_2 and $A' = f^{-1}(B)$ then A_1'/A' is isomorphic to A_2'/B. Moreover, if $B = \{0\}$, then $A_1'/$ Ker$\{f\} \geq A_2'$.*

Let A be a KU-module and A_1, A_2, A_3 are submodules of A then $A_1 + A_2/A_3$ is isomorphic to $A_1/A_2 \cap A_2$. Next if $A_3 \subset A_2 \subset A_1$, then A_2/A_3 is submodule of A_3/A_1, we have $(A_1/A_3)/(A_2/A_3)$ is isomorphic to A_1/A_2.

4. Chains of KU-Modules

Based on the concept of chains, connections and relationship between KU ideals and module theory will be established.

Theorem 4.1. *Ideals of a bounded implicative KU algebra X is a submodule.*

Proof. We define $x_1 + x_2 = \{(x_1 * x_2) \wedge (x_2 * x_1)\}$ in a bounded implicative KU algebra I. By ideal property of KU-algebras we have $x_2 * x_1 \in I$ and $x_1 * x_2 \in I$. Further in a bounded implicative KU algebra KU ideals are lattice ideals. Hence $x_1 + x_2 \in I$. Further, $x_1 + 0 = 0 + x_1 = x_1$ for all $x_1 \in I$. That means 0 is the identity element in addition. Furthermore, $x_1 \in I, 0 \in I \Rightarrow x_1 + x_1 = 0$. That is x_1 is inverse of x_1. That shows $(I, +)$ is a subgroup of bounded implicative KU algebra. Next $x_1 \in X, x_2 \in I$ then $x_1 \wedge x_2 = x_1 x_2 \in I$. This means I is a submodule of X. □

Definition 4.1. *Let A is a KU-module. A will satisfy minimal(maximal) conditions for submodules if each non-void collections of submodules of A contains a maximal(resp. minimal) member of A.*

Definition 4.2. *Let A be KU-module of a commutative KU algebra. Then A will satisfy the descending chain condition of submodules if for any chain of submodules, e.g. $A_1 \supseteq A_2 \supseteq \ldots\ldots \supseteq \ldots\ldots A_n \supseteq \ldots\ldots$ is terminated in the sense that $A_k = A_k + 1$ for some $k \geq 0$. Ascending chain condition is defined in a similar fashion.*

Definition 4.3. *Let A is a module of a commutative KU algebra. Then we say that A will satisfy maximal(minimal) conditions for submodules if every non-empty collections of submodules contains a maximal(minimal) member.*

Proposition 4.1. *If A is a KU-module then following are equivalent:*
(a) *A satisfies the minimal(maximal) condition for submodules.*
(b) *Ascending (descending) chain condition are satisfied by A for its submodules.*

Proof. $(a) \Rightarrow (b)$. Let A satisfies the maximal condition for submodules of a KU-module.
Further let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots\ldots$ be given chain of submodules of A. Next, $A = \{A_i | i \in I\}$ where A_i repeats in the sequence. According to our assumption A has maximal element, say A_μ then $A_m = A_\mu$ for all $m \geq \mu$. For a minimal condition similar fashion will work.
$(b) \Rightarrow (a)$. We consider A satisfies the ascending chain condition for submodules of KU-module. Let Ω be a non void set of submodules of M. We claim that Ω must contain a maximal member. Let us assume that Ω has no maximal element. Let $A_1 \in \Omega$ then since A_1 is not maximal in Ω, we can find $A_2 \in \Omega$ such that $A_1 \subset A_2$. But A_2 is not maximal in Ω. Consequently; there exists $A_3 \in \Omega$, for which $A_2 \subset A_3$ and so on. In this way a strictly increasing sequence of submodules $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots\ldots$ is generated, contrary to our hypothesis about (b). For a descending chain condition similar fashion will work. □

The following theorem can be established on the lines of the proof of Proposition 2.4. Butterfly Lemma that can be considered as another isomorphism theorem.

Theorem 4.2. *Let N, N', K and K' are submodules of a KU-module so that $N' \subset N$ and $K' \subset K$. Then $[N' + (N \cap K)]/[N' + N \cap K'] \cong [K' + (K \cap N)]/[(K' + (K \cap N'))]$.*

Proof. Let $A_1 = N' + (N \cap K'), A_2 = N \cap K$. Then $A_1 + A_2 = N' + N \cap K$. Now, we have $A_1 \cap A_2 = [N' + (N \cap K')] \cap (N \cap K) = N' \cap K + N \cap K'$. By Isomorphism theorem, we have $A_1 + A_2/A_1 \cong A_2/A_1 \cap A_2[N' + N \cap K]/N' + (N \cap K') \cong [N \cap K]/[N \cap K' + (N' \cap K)]$. By the symmetry of the above expression, it shows that $[K' + (K \cap N)]/[K' + (K \cap N')] \cong [K \cap N]/[N \cap K' + (N' \cap K)]$. □

For a given module M, the chain of submodules. length of a chain and its refinement is defined in a a similar way. By simple module, we mean a module, which has no non-trivial submodules. A simple module is the module that has no non-trivial submodules.

Schreir Refinement Theorem in terms of KU-module

Theorem 4.3. *Any two chains of KU-modules say*
$N = A_0 \subset A_1 \subset \cdot\cdot \subset A_t = M$. *and* $N = A'_0 \subset A'_1 \subset \cdot\cdot \subset A'_s = M$ *can be easily refined so that the output chains have the same length and a factor module* A_{t+1}/A_t *of the first chain is isomorphic to some factor module* M'_{i+1}/M'_j *of the second chain.*

Definition 4.4. *Let A be a non-zero KU-module, then a finite descending chain of submodules of A starting with A and ending with* (0) *is called a KU-composition series. i.e.* $M = A_0 \supset A_1 \supset \cdot\cdot \supset A_m = (0)$ *in such a way that their successive quotient* A_i/A_{i+1} *are simple for all* $i, 0 < i < m-1$. *Here A is called length of the series.*

Jordan-Holder Theorem in case of KU-module

In a non-zero KU-module any two composition series are equivalent if they have the same length and same simple quotients up to the order and isomorphisms. Then we have

(i.) $m = n$ and

(ii.) for each $i, 0 < i < m-1$, there exists $j = j(i)$ for $0 \leq j \leq n-1$ such that $A_i/A_{i+1} \cong N_j/N_{j+1}$.

5. Exact Sequences in KU-Module

Homology of KU-modules through exact sequences is mentioned in this section. Some homological properties are studied.

Definition 5.1. *Let* $f_1 : A_1 \to A_2$ *and* $f_2 : A_2 \to A_3$ *be homomorphism of KU-modules. Then* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ *is called exact sequence if* $Im(f_1) = Ker f_2$. *It is called semi exact sequence if* $Im(f_1) \supset Ker(f_2)$. *We can extend this concept up to* A_n *exact KU-modules.*

Remark 5.1. (i.) *If* f_1 *is one-one at* A_1 *then* $0 \to A_1 \xrightarrow{f_1} A_2$ *is exact sequence.*

(ii.) *If* f_1 *is onto at* A_2 *then* $A_1 \xrightarrow{f_1} A_2 \to 0$ *is exact sequence.*

Theorem 5.1. *In the homology of a KU-module with an exact sequence* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$, *then the below conditions are equivalent:*

(i.) f_1 *is an epimorphism.*

(ii.) f_2 *is the trivial homomorphism.*

(iii.) f_3 *is monomorphism.*

Theorem 5.2. *Let* $f_1 : A_1 \to A_2$ *and* $g : A_2 \to A_3$ *be two homomorphism of KU-module. Then it is trivial if and only if* $Im(f_1) \subseteq Ker(f_2)$.

Proof. We first prove direct part.

Let f_3 is trivial homomorphism and let A_2 be any element in $\text{Im}(f_1)$. We have that there exists

an element $A_1 \in A_1$ such that $f_1(A_1) = A_2$ and hence $f_2(A_2) = f_2(f_1(A_1)) = f_3(A_1) = 0$. Thus $A_2 \in Ker(h_2)$. This proves that $Im(f_1) \supset Ker(f_2)$.
Now we consider indirect part.
Consider $\text{lm}(f_1) \subset Ker(f_2)$ and let A_1 be any element of A_l. Then we have $f_3(A_1) = f_2(f_1(A_1))$. Since $f_1(A_1) \in Im(f_1) \supset Ker(f_2)$. Then we have $f_2(f_1(A_1)) = 0 \Rightarrow$ that $f_3(A_1) = 0$ for all $A_1 \in A_1$. This proves that f_3 is a trivial homomorphism. □

Lemma 5.1. *Considering A_1, A_2, A_3 be KU-modules and let $f_3 : A_1 \to A_2$ be an epimorphism, and let $f_2 : A_1 \to A_3$ be a homomorphism. If $Ker(f_3) \subseteq Ker(f_2)$, then there is a unique homomorphism $f_1 : A_2 \to A_3$ satisfying $f_1 * f_2 = f_3$.*

Proof. Let $A_2 \in A_2$. Since f_3 is an epimorphism hence $A_1 \in A_1$ and $A_2 \in f_3(A_1)$. We have the following mapping equations:

$$A_1 \xrightarrow{f_3} A_2 \xrightarrow{f_1} A_3 \ \ and \ \ A_1 \xrightarrow{f_2} A_3 \qquad \ldots(5.1)$$

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_3} A_3 \ \ and \ \ A_2 \xrightarrow{f_2} A_3 \qquad \ldots(5.2)$$

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0 \ \ and \ \ A_2 \xrightarrow{f_3} A_4 \ \ and \ \ A_3 \xrightarrow{f_4} A_4 \qquad \ldots(5.3)$$

$$A_4 \xrightarrow{f_4} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \ \ and \ \ 0 \to A_1 \ \ and \ \ A_4 \xrightarrow{f_2} A_1 \qquad \ldots(5.4)$$

$$A_1 \xrightarrow{\alpha} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \ \ and \ \ A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{\gamma} A_3 \ \ and \ \ A_2 \xrightarrow{\beta} A_2 \qquad \ldots(5.5)$$

$$A_1 \xrightarrow{\alpha} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \ \ and \ \ A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{\gamma} A_3 \ \ and \ \ A_2 \xrightarrow{\beta} A_2' \qquad \ldots(5.6)$$

Define $f_1 : A_2 \to A_3$ by $f_1(A_2) = f_2(A_1)$, where $A_2 = f_3(A_1)$.
Next, $f_1 o f_3 = f_2$. Let $f_1 : A_2 \to A_3$ so that $f_1 f_3 = f_1' f_3$. Then, $f_1 = f_1'$. Hence f_1 is unique. □

Proposition 5.1. *Let $A_1, A_2.A_3$ be KU-modules and let $f_2 : A_1 \to A_3$ is a homomorphism and $f_3 : A_2 \to A_3$ is a monomorphism with $lm(f_2) \subset lm(f_3)$. Then there is a homomorphism $f_1 : A_1 \to A_2$ that is unique and satisfy $f_2 = f_3 \circ f_1$ i.e. equation (2) commutes.*

Proof. For every $A_1 \in A_1, f_2(A_1) \in A_3$ and hence $f_2(A_1) \in \text{Im} f_2$. Since $\text{Im}(f_2) \subseteq Im(f_3)$ and f_3 is a monomorphism, therefore there exists a unique $A_2 \in A_2$ so that $f_3(A_2) = f_2(A_1)$. Therefore, there is a function $f_1 : A_1 \to A_2, A_1 \to A_2$ so that $f_2 = f_1 \circ f_3$. Hence f_3 is a unique homomorphism so that $f_2 = f_1 \circ f_3$. □

Theorem 5.3. *In a KU-module with f_1 and f_2 as exact homomorphism with Equation (5.6), and $f_1 o f_3 = 0$, we have a unique homomorphism $f_4 : A_3 \to A_4$ so that $f_2 \circ f_4 = f_3$.*

Proof. Since $f_1 \circ f_3 = 0$. we get that Ker (f_2) =Im$(f_1) \subset Ker(f_3)$. By using Proposition 5.5, there is a unique homomorphism $f_4 : A_4 \to A_1$ satisfying $k \circ f_1 = f_3$. □

Theorem 5.4. *Considering the Equation 4 of homomorphism of KU-module, f_1 and f_2 are exact homomorphisms and $f_3 \circ f_2 = 0$, then there is a unique homomorphism $f_4 : A_4 \to A_1$ that satisfy $f_4 \circ f_1 = f_3$.*

Proof. $f_3 \circ f_2 = 0$, so we have lm$(f_3) \subset Ker(f_2)$ =Imf. By Proposition 5.5, there is a unique homomorphism $f_4 : A_4 \to A_1$ satisfying $k \circ f_1 = f_3$. □

Theorem 5.5. *Consider A_1, A_2, A_3 and A'_1, A'_2, A'_3 be KU-modules over X, in Equation (5.6) given homomorphism commutes and are exact. Next if α, γ and f' are monomorphism, then β is also monomorphism.*

Proof. We see that all homomorphisms commutes, hence $f'\alpha = \beta f$ and $g'\beta = \gamma g$ and it is exact. Thus lm(f) =Ker(g) and lm(f') =Ker(g').
Further let A_2 ∈Ker(β). Since they are commutative, we have $\gamma g(A_2) = g'\beta(A_2) = 0, g(A_2) = 0$, because γ is one to one. Therefore, A_2 ⊆Ker$g$ =Imf. □

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{\alpha_3} A'_3 \xrightarrow{\beta_3} A_3 \ \ and \ \ A_1 \xrightarrow{\alpha_1} A'_1 \xrightarrow{\beta_1} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \qquad \ldots (5.7)$$

$$A_2 \xrightarrow{\alpha_2} A'_2 \xrightarrow{\beta_2} A_2 \ and \ \ A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} A'_3. \qquad \ldots (5.8)$$

The exactness of the Equation (5.7) and by the fact that the Equation (5.7) and the Equation (5.8) commutes, f_1 is a monomorphism, all these imply that $A_2 = 0$. This proves that $ker\beta = \{0\}$.

Theorem 5.6. *Let A_1, A_2, A_3 and A'_1, A'_2, A'_3 are KU-modules over X, as shown in the Equation (5.6) where each part commutes and α, β, γ are isomorphism. If the second part of the Equation (5.6) is exact then the third part, is also exact.*

Proof. Given that each part of the Equation (5.6) are commuting so $f'_1\alpha = \beta f_1$ and $f'_2\beta = \gamma f_2$. As first two parts are exact, therefore lmf'_1 =ker f'_2. By using these information and the fact that α, β, γ are isomorphisms, we can deduce that $lmf_1 = \ker f_2$. This shows that the last part is also exact. □

Theorem 5.7. *Let the Equation (5.7) and (5.8) be equations related to KU-modules and homomorphism with $\beta_i\alpha_i = I_i$, where I'_is denote associated identity correspondence, from $i = 1, 2, 3$. If the first part of the Equation (5.8) is exact, then $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is also exact.*

Proof. Since the Equations (5.7) and (5.8) commutes, hence

$$\begin{aligned} f'_1\alpha_1 = \alpha_2 f_1 \ \ &and \ \ f'_2\alpha_2 = \alpha_3 f_2 \\ f_1\beta_1 = \beta_2 f'_1 \ \ &and \ \ f_2\beta_2 = \beta_3 f'_2 \end{aligned} \tag{5.9}$$

As the middle row $A_1 \xrightarrow{f'_1} A_2 \xrightarrow{f'_2} A_3$ is exact, therefore $lmf'_1 = \ker\ f'_2$. In order to prove first part of the Equation (5.7) and last part of the Equation (5.8) are also exact, we have to show that

$lm(f_1) = ker(f_2)$. Let $A_i \in A_i$, and consider $\alpha_3 f_2 f_1(A_1) = f_2'(\alpha_2) f_1(A_1) = f_2' f_1' \alpha_1(A_1)$ because $\alpha_2 f_1 = f_1' \alpha_1$ (by Eq. 5.9). Since Ker $f_2' = lmf_1, f_2' f_1'(\alpha_1(A_1)) = 0$, and hence $\alpha_3 f_2 f_1(A_1) = 0$. This $\Rightarrow$ that $\alpha_3 f_2 f_1 = 0$ and hence $\beta_3 \alpha_3 f_2 f_1 = 0$. which shows that $f_2 f_1 = 0(\beta_3 \alpha_3 = I_3)$. This shows that $lmf_1 \subseteq Kerf_2$. For the converse part, consider $x \in Kerf_2$, we get then $f_2(x) = 0$ and hence $\alpha_3 f_2(x) = 0$. As $f_2' \alpha_2 = \alpha_3 f_2$, therefore, $\alpha_2(x) \in kerf_2'$ and hence $\alpha_2(x) \in kerf_2' = Imf_1'$, therefore $\alpha_2(x) \in Imf_1'$ and hence $\alpha_2(x) = f_1'(m_1')$ for some $m_1' \in A_1'$.

$$0 \to A_1' \xrightarrow{\alpha'} A_1 \xrightarrow{\beta'} A_1'' \to 0 \ \ and \ \ 0 \to A_2' \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_2'' \to 0 \ \ and \ \ 0 \to A_2' \xrightarrow{\alpha'} A_3 \xrightarrow{\beta''} A_3'' \to 0 \qquad \ldots (5.10)$$

and

$$0 \to A_1' \xrightarrow{f'} A_2' \xrightarrow{g'} A_3' \to 0 \ \ and \ \ 0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0 \ \ and \ \ 0 \to A_1'' \xrightarrow{f''} A_2'' \xrightarrow{g''} A_3'' \to 0 \qquad \ldots (5.11)$$

$\beta_2 \alpha_2(x) = \beta_2 f'(m_1')$. As $\beta_2 \alpha_2 = I_2$. therefore $x = \beta_2 f'(m_1') = f(\beta_1(m_1'))$ by Eq. 5.9.
This shows that $x \in lmf$ Therefore $Kerg \subseteq lmf$ that $\Rightarrow Kerg = lmf$ and hence required sequence is exact. ◻

Theorem 5.8. *Let $A_1, A_1', A_1'', A_2, A_2', A_2'', A_3, A_3', A_3''$ be KU-modules over X as in the Equation (5.10) and (5.11). If all three parts of the Equation (5.11) are exact, and if all parts of the Equation (5.10) are exact, then there exists unique homomorphism $\alpha'' : A_3' \to A_3$ and $\beta'' : A_3 \to A_3''$ such that the sequence $0 \to A_3' \xrightarrow{\alpha''} A_3 \xrightarrow{\beta''} A_3'' \to 0$ is semi-exact and the complete diagram commutes.*

Proof. Since all parts of the Equation (5.11) are exact, we have $lm(f_1') = ker(f_2')$, $lm(f_1) = kerf_2$, $lm(f_1'') = ker(f_2'')$ (2) Also all parts of the Equation (5.10) are exact, we have $lm\alpha' = ker\beta'$ and $lm\alpha = ker\beta$ (3) Since both Equations commutes, we have $\alpha f_1' = f_1 \alpha'$, $f_1 \beta = f_1'' \beta'$ (4) We intend to show that $lm(\alpha'') \subset ker(\beta'')$. Observe that in this diagram α', α and f_1', f_1, f_1'' are one to one. Similarly f_2', f_2, f_2'' and β', β are onto. Now we have $f_2 \alpha f_1' = f_2 f_1 \alpha' = 0\ \alpha' = 0(lmf = kerf_2)$. First we prove the existence of α'' and β''. Define a mapping $\alpha'' : A_3' \to A_3$ by $\alpha''(m_3') = A_3$ where $f_2 \alpha(m_2') = A_3$ and $f_2'(m_2') = (m_3')$ (f_2' is an epimorphism). Then α'' is well-defined and indeed homomorphism. Moreover, $f_2 \alpha(m_2') = \alpha'' f_2'(m_2')$ and hence $f_2 \alpha = \alpha'' f_2'$. Now we prove the uniqueness of α''.
Since $\alpha'' f_2' = f_2 \alpha$, suppose there exists another homomorphism such that $\alpha_0 : A_3' \to A_3$ such that $f_2 \alpha = \alpha_0 f_2'$. We will show that $\alpha_0 = \alpha''$.
Let $m_3' \in A_3$. Then as f_2' is onto, we have $m_3' = f_2'(m_2')$ for some $m_2' \in A_2$. Thus, $\alpha_o(m_3') = \alpha_o f_2'(m_2')$. Similarly $\alpha''(m'3) = \alpha'' f_2'(m_2')$. Since $f_2 \alpha = \alpha_o f_2'$, $f_2 \alpha = \alpha'' f_2'$ and $f_2 \alpha(m'2) = \alpha'' f_2'(m_2')$ $f_2 \alpha(m_2') = \alpha_0 f_2'(m_2')$ which $\Rightarrow$ that $\alpha_0(m_3') = \alpha''(m_3')$ for all $m_3' \in A_3$. Thus $\alpha_0 = \alpha''$. This proves the uniqueness of α''.
Define $\beta'' : A_3 \to A_3''$ by $\beta''(A_3) = m_3''$, where $f_2'' \beta(A_2) = m_3''$ and $f_2(A_2) = A_3$. Then β'' is well-defined and indeed homomorphism. Moreover, $f_2'' \beta(A_2) = \beta'' f_2(A_2)$ (by(3)) therefore, $f_2'' \beta = \beta'' f_2$. The uniqueness of β'' can be established on the lines similar to α''.
Now for semi-exactness, let $m_3' \in ker(\alpha'')$, since f_2' is surjective, we see that there exists $m_2' \in M_2'$ such

that $f_2'(m_2') = m_3'$. Thus, by definition of α', $0 = \alpha''(m_3') = \alpha'' f_2'(m_2') = f_2\alpha(m_2')$. This shows that $\alpha(m_2') \in ker(f_2) = lm(f_1)$ (by the sequence). Therefore $\alpha(m_2') = f_1(A_1)$ for some $A_1 \in A_1$. By the exactness of middle row $\beta\alpha(m_2') = 0$. Since $0 = \beta\alpha(m_2') = \beta f(A_1) = f_1''\beta'(A_1)$ (by (3)). Therefore, $\beta'(A_1) \in ker(f_1'') = \{0\}$ and hence $A_1 \in ker(\beta') = lm(\alpha')$ (by the exactness of the sequence) which $\Rightarrow$ that $A_1 = \alpha'(m_1')$ for some $m_1' \in A_1'$. Therefore, we have $\alpha(m_2') = f_1(A_1) = f_1\alpha'(m_1') = \alpha f_1'(m_1')$. Since α is a monomorphism, $m_2' = f_1'(m_1')$, it follows that $m_3' = f_2'(m_2') = f_2' f_1'(m_1') = 0$ ($lmf' = kerf_2'$) and so $ker(\alpha'') = \{0\}$ i.e. α'' is monomorphism.
Next, since $\beta'' f_2 = f_2''\beta$ and since f_2'', β are each surjective, so $\beta'' f_2$ is surjective, which indeed $\Rightarrow$ that β'' is surjective. Finally, we have $\beta''\alpha'' f_2' = f_2''\beta\alpha = f_2''0 = 0(ker\beta = lma)$. Since f_2' is surjective, then indeed $f_2'(A_2') = A_3'$ and $\beta''\alpha'' f_2'(A_2') = \beta''\alpha''(A_3') = 0$. This mean that $\beta''\alpha''$ is zero mapping, therefore $\beta''\alpha'' = 0$. This shows that $lm(\alpha'') \subseteq ker(\beta'')$. □

6. Projective and Injective KU-Modules

In this section, we discuss Projective KU-modules and Injective KU-modules. We find that these modules are dual to each other.

Definition 6.1. *A KU-module Q is called projective if for given any Equations (6.1) to (6.8) in which the first part of the Equation is exact, there exists an X-homomorphism $f_3 : Q \to M_1$ such that $f_3 g_1 = f_1$ i.e., the Equations commutes.*

The following proposition follows from the above definition.

Proposition 6.1. *Consider the Equation (6.2) has M_1, M_2, M_3 KU-modules and Q is projective, $M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3$ is exact and $g_2 f_1 = 0$. Then there exists a KU-homomorphism $f_3 : Q \to M_1$ such that $f_3 g_1 = f_1$.*

Corollary 6.1. *Let the Equation (6.3) shows commutativity in which M_1, M_2, M_3 and Q, R, S are KU-modules and Q is projective, $f_6 f_5 = 0$ and $M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3$ is exact. Consequently, there exists an KU-homomorphism $f_3 : Q \to M_1$ for which the Equations commutes.*

Proof. Since the Equation commutes, $g_2 h_1 = h_2 f_2$. Now consider the Equation (6.4). From the Equation, we have $g_2 h_1 f_5 = (h_2 f_6) f_5 = h_2(f_6 f_5) = 0$ ($f_6 f_5 = 0$).

$$Q \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} 0 \ \ and \ \ Q \xrightarrow{f_1} M_2. \qquad \ldots(6.1)$$

$$Q \xrightarrow{f_3} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \ \ and \ \ Q \xrightarrow{f_1} M_2. \qquad \ldots(6.2)$$

$$Q \xrightarrow{f_3} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \ \ and \ \ Q \xrightarrow{f_1} R \xrightarrow{f_2} S \xrightarrow{h_2} M_3 and \ \ T \xrightarrow{h_1} M_2 \qquad \ldots(6.3)$$

$$Q \xrightarrow{f_3} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \ \ and \ \ Q \xrightarrow{h_1 f_1} M_2. \qquad \ldots(6.4)$$

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_3} Q \ \ and \ \ M_1 \xrightarrow{f_2} Q. \xrightarrow{\gamma} A_3 \ \ and \ \ A_2 \xrightarrow{\beta} A_2. \qquad \ldots(6.5)$$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{g_1} M_3 \xrightarrow{f_3} Q \ \ and \ \ M_2 \xrightarrow{g_2} Q. \qquad \ldots(6.6)$$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} Q \ \ and \ \ M_1 \xrightarrow{h_1} L \xrightarrow{g_1} M \xrightarrow{g_2} Qand \ \ M_2 \xrightarrow{h_2} M. \qquad \ldots(6.7)$$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} R \ \ and \ \ M_2 \xrightarrow{g_2h_2} R. \qquad \ldots(6.8)$$

Therefore $g_2h_1f_1 = 0$. Since Q is projective and $g_2h_1f_1 = 0$, therefore by proposition 4.2, there exists a homomorphism $f_3 : Q \to M_1$ such that $f_3g_1 = h_1f_1$. Consequently, all the squares are commutative. □

Definition 6.2. *A KU-module R is said to be injective if for an Equation say (6.5) with* $0 \xrightarrow{f_1} M_1 \to M_3$ *is an exact sequence there exists homomorphism* $f_3 : M_2 \to R$ *such that* $f_1f_3 = f_2$.
By using the injectivity and certain natural homomorphism, we establish the following proposition.

Proposition 6.2. *Consider Equation (6.6) in which* M_1, M_2, M_3 *are KU-modules and R is injective module,* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{g_1} M_3$ *is exact and* $g_2f_1 = 0$. *Then there exists a homomorphism* $f_3 : M_3 \to R$ *such that* $g_1f_3 = g_2$.

Corollary 6.2. *Consider the Equation (6.7) is commutative where R is an injective module,* $g_2g_1 = 0$ *and* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ *is an exact sequence. Then there exists a KU-homomorphism* $f_3 : M_3 \to R$ *for which the diagram, commutes.*

Proof. Since the Equation commutes, hence $g_1h_1 = h_2f_1$. Now consider the Equation (6.8). By this equation, we get that $g_2h_2f_1 = g_2(h_2f_1) = g_2(g_1h_1) = (g_2g_1)h_1$. Since $g_2g_1 = 0$ therefore $g_2h_2f_1 = 0$. Also R is injective and $g_2h_2f_1 = 0$, therefore by proposition 4.5, therefore there exists a homomorphism $f_3 : M_3 \to R$ such that $f_3f_2 = g_2h_2$. Consequently, all parts of the Equations are commutative. □

As an open *Problem* with this study we can say Theorem 4.1 provides a minor relationship of ideal theory of KU algebra and its Modules. It can be consider as ideal-theory of KU-algebras and its module theory. In special case we can say bounded implicative KU-algebras. It would be interesting to check and provide some general relationship between ideal theory and module theory of KU-algebras or other newly defined logical algebras.

7. Conclusion

In this article a discussion based on the Chains of KU-modules of KU-algebras are given. Basic isomorphism theorems for KU-modules, projective and injective modules are provided and explored with some of theirs applications. Schreir Refinement Theorem and Jordan-Holder Theorem in terms of KU-module is also studied. Additionally, it defines and examines exact sequences of KU-modules. Various properties of chains of KU-modules are established and the butterfly lemma in the context of KU-modules are given. As a future scope on this article, some further classical properties can be extended for a KU-module with chain conditions.

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