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Perfect Quadratic Forms Connected With a Lattice and Cubature Formulas

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Abstract. In the present work, a new improved Voronoi algorithm is proposed for calculating the Voronoi neighborhood of a perfect form in many variables, and using this algorithm, all non-equivalent adjacent perfect forms in five variables are calculated.

Introduction

Let $A = \|a_{\mu\nu}\|$ be a real square matrix of order n, i.e. $\mu = 1, 2, ..., n$; $\nu = 1, 2, ..., n$ and $\det A > 0$. In the space R^n we consider a set of points

$$\Lambda\left(A|x^{(0)}\right) = \left\{x : \; x = A\beta + x^{(0)}\right\}, \tag{0.1}$$

where $x^{(0)}$ is any fixed vector, and β runs through the set B^n of integer valued n- dimensional column-vectors. A set $\Lambda\left(A|x^{(0)}\right)$ is called n- dimensional lattice in R^n.

Studying vectors $x = (x_1, x_2, ..., x_n)^* \in \Lambda$ of the lattice, it is usually more convenient to work not with the length, but with the square of the length or the norm of vectors.

The norm of vector is denoted as

$$N(x) = x \cdot x = (x, x) = \sum_{i=1}^{n} x_i^2$$

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Suppose Λ is a lattice in n-dimensional space R^n having a basis $a_1, ..., a_n$ (vectors $a_1, ..., a_n$ are columns $a_1 = (a_{11}, a_{21}, ..., a_{n1})^*$, $a_2 = (a_{12}, a_{22}, ..., a_{n2})^*$, ..., $a_n = (a_{1n}, a_{2n}, ..., a_{nn})^*$ generating the matrix A).

General lattice vector $x = (x_1, x_2, ..., x_n)^* \in \Lambda$ can be written as

$$x = A\beta = \begin{pmatrix} a_{11}\beta_1 + a_{12}\beta_2 + ... + a_{1n}\beta_n \\ a_{21}\beta_1 + a_{22}\beta_2 + ... + a_{2n}\beta_n \\ .... \\ a_{n1}\beta_1 + a_{n2}\beta_2 + ... + a_{nn}\beta_n \end{pmatrix}.$$

On the other side using denotations of the work [10] general vector x can be written in the form

$$x = a_1\beta_1 + a_2\beta_2 + ... + a_n\beta_n.$$

The norm of this vector is

$$N(x) = N(a_1\beta_1 + a_2\beta_2 + ... + a_n\beta_n)$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n} \beta_j\beta_i\left(a_j, a_i\right) = \beta^* A^* A\beta = \beta^* B\beta = r_A^2(\beta),$$

where $B = A^*A$ is the Gramm matrix of the lattice Λ.
The function $r_A^2(\beta)$ considered as a function of n integer valued vectors $\beta_1, \beta_2, ..., \beta_n$ is a quadratic form associated with the lattice. Thus, the study of lattices is equivalent to the study of quadratic forms in n integer-valued arguments.

The arithmetic properties of such forms have been the subject of numerous studies. The quantities that will be of interest to us include various lattice functionals that are invariant under orthogonal space transformations. Furthermore, functionals are obviously functions of the quadratic form $r_A^2(\beta)$.

The matrix A we present in the form $A = hH$, where $h^n = \det A$; $\det H = 1$. Then $r_A^2(\beta) = h^2 r_H^2(\beta)$. Therefore, it is sufficient to study the function $r_H(\beta)$ of a matrix H with unit determinant. The theory of cubature formulas is related to the problem of finding the minimum of the Epstein zeta function

$$\zeta(H/2m) = \sum_{\beta\neq 0} \frac{1}{r_H^{2m}(\beta)} \tag{0.2}$$

as a function of the matrix $H$for a given integer $m > \frac{n}{2}$. For large m in series (0.2) all terms, except for those, where $r_H(\beta)$ has smallest value, become negligible. In this case, the problem of finding the smallest value of $\zeta(H/2m)$ approximately reduces to finding such matrices H for which the quantity $\frac{1}{\min\limits_{\beta\neq 0} r_H(\beta)}$ is minimal, i.e. the quantity $\max\limits_{H}\min\limits_{\beta\neq 0} r_A(\beta)$ is attained, which is we denote as r_0. The Lattices H for which this maximum is reached are called lattices on the densest packing of balls in R^n. Densest packings, as we have already mentioned, can be obtained in a regular way, because of a finite number of actions. The algorithm for finding them was given by Voronoi. It is known [17] that for the matrix H, realizing the minimum of the function $\zeta(H/2m)$, the matrix H^{-1*} gives optimal lattice of cubature formulas.

The classical Voronoi problem of finding perfect forms, closely related to Hermite's well-known problem of finding the arithmetic minimum of positive quadratic forms, are interesting and non-trivial problems in geometric number theory that have been studied by many mathematicians. They also appeared in the works of S.L. Sobolev in connection with the construction of lattice optimal cubature formulas [17].

The present work is devoted to the development of an algorithm and finding all adjacent perfect forms in five variables.

The technique presented in the work allows us to study the classical Voronoi problems, its results and calculation methods can be used to further search for new perfect forms in many variables.

Perfect forms are calculated in the works of [1–9]

In recent years, several papers on the theory of perfect forms have been published. Let us note the works [10–16], in which problems of the theory of perfect forms were considered.

Many mathematicians have studied the construction of lattice optimal cubature and quadrature formulas [18–28].

Let a positive-definite quadratic form

$$f \equiv f(x) \equiv f(x_1, \ ..., \ x_n) = \sum_{1\le \ i,j \ \le n} a_{i\,j}\, x_i\, x_j \tag{0.3}$$

in n variables $x_1, ..., x_n$ with real coefficients $a_{ij} = a_{ij}$and with determinant $\det f = \det(a_{ij}) > 0$ be given and $m = m(f)$ which is the arithmetic minimum of the form f is attained at 2s integer points

$$\pm\, m_k = \pm\,(m_{1k}, ...,m_{nk}), \ \ \mathrm{k} = 1, ..., \mathrm{s}, , \tag{0.4}$$

called representations of the minimum m of the form f. We sometimes call points (0.4) the minimal vectors of the form f, and the matrix

$$M(f) = \begin{pmatrix} m_{11} \ m_{21} \dots m_{n1} \\ m_{12} \ m_{22} \dots m_{n2} \\ \dots \ \ \dots \ \ \dots \ \dots \\ m_{1s} \ m_{2s} \dots m_{ns} \end{pmatrix}$$

minimal matrix of the form f.

A positive-definite quadratic form is called a perfect form f (see [1]) if it is completely given by the value of its arithmetic minimum and its minimum representations (0.4), i.e. if the system of equations

$$\sum a_{ij} m_{ik} m_{jk} = m \qquad \mathrm{k} = 1, ..., \mathrm{s}$$

has a unique solution with respect to unknowns a_{ij}.

Two positive-definite quadratic forms $f_1(x)$ and $f_2(y)$ are called integrally equivalent (equivalent $f_1 \cong f_2$) if there exists an integer unimodular substitution $x_i = \sum_{i=1}^{n} u_{ij} y_i \quad (x = Uy)$, of variables that transforms the form $f_1(x)$ into $f_2(x)$, i.e. $f_1(Uy) = f_2(y)$, or the same that $f_1 U = f_2$.

In the case $f_1 = f_2 = f$ the substitution U is called automorphism of the form f, i.e. $fU = f$.

It is known [1] that the number of nonequivalent perfect forms in n variables for a given n is finite. This implies the problem: for a given n, find all non-equivalent perfect forms. Perfect forms in $n \leq 5$ variables are known from the classical work of Voronoi [1]. Perfect forms in six variables were found by Barnes [2].

1. Voronoi's algorithm

According to Voronoi's theory [1], each perfect form of the form (0.3) is placed in the corresponding area - a dimensional infinite pyramid with a finite number of $(N-1)-$ dimensional faces and with a vertex at the beginning coordinates (perfect gonohedron [4–6]) is the set of all non-negative quadratic forms represented as

$$\sum_{1\leq i,j\leq n} a_{ij}x_ix_j = \sum_{1\leq k\leq s} \rho_k \lambda_k^2 \left(x_1, ..., x_n\right), \tag{1.1}$$

where $\bar{K}^N-$ is the closure of the cone K^N, $\rho_k \geq 0$,

$$\lambda_k = \lambda_k(x) = \lambda_k\left(x_1, ..., x_n\right) = m_{1k}x_1 + ... + m_{nk}x_n, \qquad k = 1, ..., s.$$

In the space E^N the domain $V^N\left(f\right)$ is a set of solutions to some system of homogeneous inequalities with unknowns a_{ij}:

$$\Psi_k\left(a_{ij}\right) = \sum_{1\leq i,j\leq n} P_{ij}^{(k)} a_{ij} \geq 0, \quad k = 1, ..., \sigma. \tag{1.2}$$

Then, according to the Voronoi algorithm [5], the perfect forms f_k adjacent to the perfect form f are constructed as follows:

$$f_k(x) = f(x) + r_k\Psi_k(x), \qquad k = 1, ..., \sigma., \tag{1.3}$$

where

$$r_k = \min_{\left\{x\in Z^n/\{0\}:\Psi_k(x)<0\right\}} \left\{\frac{f(x)-m}{\left[-\Psi_k(x)\right]}\right\}, \tag{1.4}$$

$$\Psi_k(x) = \Psi_k\left(x_1, ..., x_n\right) = \sum_{1\leq i,j\leq n} P_{ij}^{(k)} x_ix_j. \tag{1.5}$$

Selecting from the set $\{f, f_1, ..., f_\sigma\}$ those that are not equivalent with respect to the group $G\left(n; Z\right)G\left(n; Z\right)$ (the group of integer unimodular permutations of the variables $x_1, ..., x_n$), we obtain a Voronoi neighborhood $\{f, f_1, ..., f_\tau\}$ (see [1,2]) of a perfect form f with respect to the group $G\left(n; Z\right)$ or simply a Voronoi neighborhood, which is denoted by $VN\left(f; G\left(n; Z\right)\right)$ or $VN\left(f\right)$.

The main difficulties in implementing the Voronoi algorithm are as follows: finding equations for all $(N-1)$-dimensional faces of the domain $V^N\left(f\right)$; selection among all $(N-1)$-dimensional faces non-equivalent with respect to the group $G\left(f\right)$ of integral automorphisms of the perfect form f; finding a number r_k and calculating $VN\left(f\right)$.

2. Additional algorithms

In this subsection, the goal is to bring together all the available additional algorithms (see below) of Lemmas 2 and 3 and trace how they work together with the Voronoi algorithm - this is the improved Voronoi algorithm.

Let $\Psi = \Psi\left(a_{ij}\right) = \sum_{1\le i,j\le n} P_{ij}a_{ij} = 0$ be an equation of $(N-1)$-dimensional face $\Psi$of the domain $V^N(f)$, corresponding to the perfect form f of the form (0.1), and let $\lambda_1^2, ..., \lambda_t^2$ be form-points lying on the face Ψ; $\lambda_{t+1}^2, ..., \lambda_s^2$ be form-points lying outside the face Ψ.

Lemma 2.1. *The set of forms $\lambda_1^2, ..., \lambda_t^2$ is the set of all form-points lying on the face Ψ if and only if*
a) the system of equations

$$\Psi(m_k) = \sum_{1\le i,j\le n} P_{ij}m_{ik}m_{jk} = 0 \quad (k = 1, ..., t),$$

has a rank $N-1$;
b) the solution $\left\{P_{ij}\right\}$ satisfies the inequality up to a sign

$$\Psi(m_k) = \sum_{1\le i,j\le n} P_{ij}m_{ik}m_{jk} > 0 \quad (k = t+1, ..., s).$$

Then

$$\left.\begin{array}{l}\Psi\left(\lambda_k^2\right) = 0 \quad (k = 1, ..., t), \\ \Psi\left(\lambda_k^2\right) > 0 \quad (k = t+1, ..., s\end{array}\right\}. \tag{2.1}$$

From (2.1) we have

$$\Psi(m_k) = \sum_{1\le i,j\le n} P_{ij}m_{ik}m_{jk} = 0 \quad (k = 1, ..., t), \tag{2.2}$$

$$\Psi(m_k) = \sum_{1\le i,j\le n} P_{ij}m_{ik}m_{jk} > 0 \quad (k = t+1, ..., s). \tag{2.3}$$

From equality (1.1), equating the coefficients at the same powers of x_ix_j, we obtain the following system of equations with unknowns $\rho_1, ...,$
ρ_s:

$$\sum_{1\le k\le s} \rho_k m_{ik}m_{jk} = a_{ij}, \;\; i, j = 1, ..., n. \tag{2.4}$$

In the case $s = N$ the system has a unique solution

$$\rho_k = \Psi_k\left(a_{ij}\right) = \sum_{1\le i,j\le n} P_{ij}^{(k)}a_{ij} \ge 0 \;\; (k = 1, ..., N). \tag{2.5}$$

N equalities in (2.5) completely determine all $(N-1)$ -dimensional faces of the domain $V^N(f)$..

In the case $s > N$, an arbitrary solution $\rho_k \;\; (k = 1, ..., s)$ to system (2.4) depends on $\nu = s - N$ parameters $u_1, ..., u_\nu$ and has the form

$$\rho_k = L_k\left(a_{ij}\right) + M_k(u), \tag{2.6}$$

where $L_k,\; M_k$ are linear forms and $u = (u_1, ..., u_\nu)$.

Lemma 2.2. *Form-points $\lambda_{t+1}^2, ..., \lambda_s^2$ do not lie on a face if and only if*
a) there is a unique linear relation

$$\sum_{t+1 \le k \le s} \alpha_k M_k(u) = 0,$$

b) all coefficients α_k $(k = t+1, ..., s)$ of this relation are positive.

Lemma 2.3. *Let $\{\lambda\} = \left\{\lambda_{t+1}^2, ..., \lambda_s^2\right\}$ and $\{\lambda'\} = \left\{\lambda_{t'+1}^2, ..., \lambda_{s'}^2\right\}$ be sets of form points lying outside of $N-1$ dimensional faces Ψ and Ψ' of the domain $V^N(f)$, respectively, then $\{\lambda\}$ cannot be equivalent to its own subset $\{\lambda'\}$, in particular, $\{\lambda\} \not\subset \{\lambda'\}$.*

Equations of the face Ψ are

$$\Psi\left(a_{ij}\right) = \sum_{t+1 \le k \le s} \alpha_k L_k\left(a_{ij}\right) = 0.$$

It follows from lemma 2 that set of form-points $\left\{\lambda_k^2(x)\right\}$ $(k = t+1, ..., s)$, lying outside the face Ψ, or, the same, corresponding set of representations $\{m_k\}$ $(k = t+1, ..., s)$ of the minimum m of the form f completely defines the face Ψ of the domain $V^N(f)$. Such set we briefly call "face".

The goal of the present work is to get all nonequivalent perfect forms of five variables by the improved Voronoi algorithm developed in works [7–9]. Exactly this algorithm is optimal than known methods [1–4] in calculations of perfect forms. The application of the improved Voronoi algorithm leads to the goal faster than the Voronoi algorithm itself, since here the amount of calculations is sharply reduced.

3. Necessary lemmas for the proof of the theorem

Evidence of the perfect form φ_1^5. The arithmetic minimum of the perfect form φ_1^5 is equal to 1. The representations of the minimum are as follows:

$$\begin{aligned}
&1)\ (1,0,0,0,0),\ 2)\ (0,0,1,0,0),\ 3)\ (0,0,0,1,0),\ 4)\ (0,0,0,0,1),\\
&5)\ (0,1,-1,0,0),\ 6)\ (0,1,0,-1,0),\ 7)\ (0,1,0,0,-1),\\
&8)\ (1,1,-1,-1,0),\ 9)\ (1,1,-1,0,-1),\ 10)\ (1,1,0,-1,-1),\\
&11)\ (0,1,0,0,0),\ 12)\ (1,1,-1,0,0),\ 13)\ (1,1,0,-1,0),\\
&14)\ (1,1,0,0,-1),\ 15)\ (1,0,-1,0,0),\ 16)\ (1,0,0,-1,0),\\
&17)\ (1,0,0,0,-1),\ 18)\ (0,0,1,-1,0),\ 19)\ (0,0,1,0,-1),\\
&20)\ (0,0,0,1,-1)
\end{aligned} \tag{3.1}$$

The set of points (3.1) are representations of the arithmetic minimum of the perfect form φ_1^5.

Voronoi's domain $VN^{15}\left(\varphi_1^5\right)$ of the perfect form $\varphi_1^5(x)$ consists of a set of quadratic forms representable in the form:

$$\begin{aligned}&\sum_{1\leq i,j\leq 5} a_{ij}x_ix_j = \rho_1 x_1^2+\rho_2 x_3^2+\rho_3 x_4^2+\rho_4 x_5^2+\rho_5\left(x_2-x_3\right)^2\\&+\rho_6\left(x_2-x_4\right)^2+\rho_7\left(x_2-x_5\right)^2+\rho_8\left(x_1+x_2-x_3-x_4\right)^2\\&+\rho_9\left(x_1+x_2-x_3-x_5\right)^2+\rho_{10}\left(x_1+x_2-x_4-x_5\right)^2+\rho_{11}x_2^2\\&+\rho_{12}\left(x_1+x_2-x_3\right)^2+\rho_{13}\left(x_1+x_2-x_4\right)^2+\rho_{14}\left(x_1+x_2-x_5\right)^2\\&+\rho_{15}\left(x_1-x_3\right)^2+\rho_{16}\left(x_1-x_4\right)^2+\rho_{17}\left(x_1-x_5\right)^2\\&+\rho_{18}\left(x_3-x_4\right)^2+\rho_{19}\left(x_3-x_5\right)^2+\rho_{20}\left(x_4-x_5\right)^2.\end{aligned} \tag{3.2}$$

Hence the proof of the theorem is based on the following lemmas.

Lemma 3.1. *Linear forms $M_k = M_k\left(u_1,...,u_5\right)$ from equality (2.6), corresponding perfect form φ_1^5, have the form:*

1) $M_k = u_1$ *for* $m_k \in \{10,12\}$; 2) $M_k = u_2$ *for* $m_k \in \{9,19\}$; 3) $M_k = u_3$ *for* $m_k \in \{8,18\}$; 4) $M_k = u_4$ *for* $m_k \in \{7,17\}$; 5) $M_k = u_5$ *for* $m_k \in \{6,16\}$; 6) $M_k = -u_1-u_2-u_3-u_4-u_5$ *for* $m_k \in \{5,15\}$; 7) $M_k = u_1+u_2+u_3$ *for* $m_k \in \{1,11\}$; 8) $M_k = u_1+u_4+u_5$ *for* $m_k \in \{2,12\}$; 9) $M_k = -u_1-u_2-u_4$ *for* $m_k \in \{4,14\}$; 10) $M_k = -u_1-u_3-u_5$ *for* $m_k \in \{3,13\}$.

Proof. From equality (3.2) equating coefficients of the same telms x_ix_j in both sides we get the following system with unknowns $\rho_1,...,\rho_{20}$:

$$\begin{aligned}&\{1\}: \ a_{11}=\rho_1+\rho_8+\rho_9+\rho_{10}+\rho_{12}+\rho_{13}+\rho_{14}+\rho_{15}+\rho_{16}+\rho_{17},\\&\{2\}: \ a_{22}=\rho_5+\rho_6+\rho_7+\rho_8+\rho_9+\rho_{10}+\rho_{11}+\rho_{12}+\rho_{13}+\rho_{14},\\&\{3\}: \ a_{33}=\rho_2+\rho_5+\rho_8+\rho_9+\rho_{12}+\rho_{15}+\rho_{18}+\rho_{19},\\&\{4\}: \ a_{44}=\rho_3+\rho_6+\rho_8+\rho_{10}+\rho_{13}+\rho_{16}+\rho_{18}+\rho_{20},\\&\{5\}: \ a_{55}=\rho_4+\rho_7+\rho_9+\rho_{10}+\rho_{14}+\rho_{17}+\rho_{19}+\rho_{20},\\&\{6\}: \ a_{12}=\rho_8+\rho_9+\rho_{10}+\rho_{12}+\rho_{13}+\rho_{14},\end{aligned}$$

$$\begin{aligned}&\{7\}: \ a_{13}=-\rho_8-\rho_9-\rho_{12}-\rho_{15},\\&\{8\}: \ a_{13}=-\rho_8-\rho_{10}-\rho_{13}-\rho_{16},\\&\{9\}: \ a_{15}=-\rho_9-\rho_{10}-\rho_{14}-\rho_{17},\\&\{10\}: \ a_{23}=-\rho_5-\rho_8-\rho_9-\rho_{12},\\&\{11\}: \ a_{24}=-\rho_6-\rho_8-\rho_{10}-\rho_{13},\\&\{12\}: \ a_{25}=-\rho_7-\rho_9-\rho_{10}-\rho_{14},\\&\{13\}: \ a_{34}=\rho_8-\rho_{18},\\&\{14\}: \ a_{35}=\rho_9-\rho_{19},\\&\{15\}: \ a_{45}=\rho_{10}-\rho_{20},\end{aligned} \tag{3.3}$$

System (3.3) is overdetermined: the number of equations is $N = 15$, the number of unknowns is $s = 20$. This belongs to the case $s > N$ and any solution ρ_k of system (3.1) depends on 5 parameters u_1, u_2, u_3, u_4, u_5. We calculate arbitrary solution of system (3.3).

Let $\rho_{10} = u_1$, then form $\{15\}$ $: \rho_{20} = -a_{45} + u_1$. Let $\rho_9 = u_2$, then from $\{14\}$ $: \rho_{19} = -a_{35} + u_2$. Assume $\rho_8 = u_3$, then from $\{13\}$ $: \rho_{18} = -a_{34} + u_3$. Let $\rho_7 = u_4$, then from $\{12\}$ $: \rho_{14} = -a_{25} - u_1 - u_2 - u_4$. Assume $\rho_6 = u_5$, then from $\{11\}$ $: \rho_{14} = -a_{25} - u_1 - u_2 - u_4$.

Let $\rho_{10} = u_1$, then from $\{15\}$: $\rho_{20} = -a_{45} + u_1$. Let $\rho_9 = u_2$, then from $\{14\}$: $\rho_{19} = -a_{35} + u_2$. Assume $\rho_8 = u_3$, the from $\{13\}$: $\rho_{18} = -a_{34} + u_3$. Let $\rho_7 = u_4$, then from $\{12\}$: $\rho_{14} = -a_{25} - u_1 - u_2 - u_4$. Let $\rho_6 = u_5$, then from $\{11\}$: $\rho_{13} = -a_{24} - u_1 - u_3 - u_5$. From $\{9\}$: $\rho_{17} = -a_{15} + a_{25} + u_4$; $\{8\}$: $\rho_{16} = -a_{14} + a_{24} + u_5$; $\{6\}$: $\rho_{12} = a_{12} + a_{25} + u_1 + u_4 + u_5$;$\{7\}$: $\rho_{15} = -a_{12} - a_{13} - a_{24} - a_{25} - u_1 - u_2 - u_3 - u_4 - u_5$;$\{10\}$: $\rho_5 = -a_{12} - a_{13} - a_{23} - a_{25} - u_1 - u_2 - u_3 - u_4 - u_5$;$\{7\},\{8\},\{9\}$: $\{16\} \cdot a_{13} + a_{14} + a_{15} = -2(\rho_8 + \rho_9 + \rho_{10}) - \rho_{12} - \rho_{13} - \rho_{14} - \rho_{15} - \rho_{16} - \rho_{17}$.Now from $\{2\}$ and $\{17\}$: $\rho_1 = a_{11} + a_{13} + a_{14} + a_{15} + u_1 + u_2 + u_3$. From $\{10\},\{11\},\{12\}$: $\{10\},\{11\},\{12\}$: $\{17\} \cdot a_{23} + a_{24} + a_{25} + =$ $-2(\rho_8 + \rho_9 + \rho_{10}) - \rho_5 - \rho_6 - \rho_7 - \rho_{12} - \rho_{13} - \rho_{14}$.hence $\rho_{11} = a_{22} + a_{23} + a_{24} + a_{25} + u_1 + u_2 + u_3$. From $\{3\},\{7\}$: $a_{33} + a_{13} = \rho_2 + \rho_5 - \rho_{18} - \rho_{19},$ hence $\rho_2 = a_{33} + a_{12} + a_{13} + a_{23} + a_{24} + a_{25} + a_{34} + a_{35} + u_1 + u_4 + u_5$. From $\{4\},\{8\}$: $a_{44} + a_{14} = \rho_3 + \rho_8 + \rho_{18} + \rho_{20},$ hence $\rho_3 = a_{44} + a_{14} + a_{34} + a_{45} - u_1 - u_3 - u_5$. From $\{5\},\{9\}$: $a_{55} + a_{15} = \rho_4 + \rho_7 + \rho_{19} + \rho_{20},$ hence $\rho_4 = a_{55} + a_{15} + a_{35} + a_{45} - u_1 - u_2 - u_4$.

Thus, in the case $n = 5$and $f = \varphi_1^5$ equality (2.6) takes the form:

$$\rho_1 = a_{11} + a_{13} + a_{14} + a_{15} + u_1 + u_2 + u_3$$
$$\rho_2 = a_{33} + a_{12} + a_{13} + a_{23} + a_{24} + a_{25} + a_{34} + a_{35} + u_1 + u_4 + u_5$$
$$\rho_3 = a_{44} + a_{14} + a_{34} + a_{45} - u_1 - u_3 - u_5$$
$$\rho_4 = a_{55} + a_{15} + a_{35} + a_{45} - u_1 - u_2 - u_4.$$
$$\rho_5 = -a_{12} - a_{23} - a_{24} - a_{25} - u_1 - u_2 - u_3 - u_4 - u_5$$
$$\rho_6 = u_5$$
$$\rho_7 = u_4$$
$$\rho_8 = u_3$$
$$\rho_9 = u_2$$
$$\rho_{10} = u_1 \tag{3.4}$$
$$\rho_{11} = a_{22} + a_{23} + a_{24} + a_{25} + u_1 + u_2 + u_3$$
$$\rho_{12} = a_{22} + a_{24} + a_{25} + u_1 + u_4 + u_5$$
$$\rho_{13} = -a_{24} - u_1 - u_3 - u_5$$
$$\rho_{14} = -a_{25} - u_1 - u_2 - u_4$$
$$\rho_{15} = -a_{12} - a_{13} - a_{24} - a_{25} - u_1 - u_2 - u_3 - u_4 - u_5$$
$$\rho_{16} = -a_{14} + a_{24} + u_5$$
$$\rho_{17} = -a_{15} + a_{25} + u_4$$
$$\rho_{18} = -a_{34} + u_3$$
$$\rho_{19} = -a_{35} + u_2$$

$$\rho_{20} = -a_{45} + u_1.$$

From (3.4) we have:

$$\begin{gathered} u_1 : \{10,\ 20\},\ \ -u_1 - u_2 - u_3 - u_4 - u_5 : \{5,\ 15\}, \\ u_2 : \{9, 19\},\ \ u_1 + u_2 + u_3 : \{1, 11\}, \\ \\ u_3 : \{8, 18\},\ \ u_1 + u_4 + u_5 : \{2, 12\}, \\ u_4 : \{7, 17\},\ \ -u_1 - u_2 - u_4 : \{4, 14\}, \\ u_5 : \{6, 16\},\ \ -u_1 - u_3 - u_5 : \{3, 13\}. \end{gathered} \tag{3.5}$$

Lemma 4 is completely proved. □

According to lemma 1 on 14-dimensional face Ψ of the domain $V^{15}\ (\varphi_1^5)$ lie at least 14 forms of the form λ_k^2. Since the number of representations of the minimum of the form φ_1^5 is equal to 20, then the number of forms λ_k^2,lying outer of the face Ψ at least 1 and no more than 6. Therefore, all possible linear nontrivial relations, satisfying the condition of lemma 2, in principle, can be obtained from elements of the set

$$\begin{aligned} A = \{&u_1, u_2, u_3, u_4, u_5, u_1 + u_2 + u_3, u_1 + u_4 + u_5, \\ &-u_1 - u_2 - u_4, -u_1 - u_3 - u_5, -u_1 - u_2 - u_3 - u_4 - u_5\} \end{aligned}$$

(see lemma 4) as a combination of $k\,(k = 2, 3, 4, 5, 6)$, i.e. $C_{10}^k(k = 2, 3, 4,\ 5, 6)$. Further, will be shown how this enumeration can be reduced.

Lemma 3.2. *14-dimensional "faces" of the domain $V^{15}\left(\varphi_1^5\right)$,outside of which lie one point from $M\left(\varphi_1^5\right)$ (see (15)) are absent. Such faces really no, since all ρ_i from (3.5) depend on parameters u_j.*

Lemma 3.3. *14-dimensional "faces" of the domain $V^{15}\left(\varphi_1^5\right)$, outside of which lie two points from $M\left(\varphi_1^5\right)$ are absent.*

Proof. The set A of linear forms $M_k\,(u)$ from lemma 4 we split into subsets

$$\begin{aligned} A &= \{u_1, u_2, u_3, u_4, u_5,\ \ u_1 + u_2 + u_3,\ \ u_1 + u_4 + u_5\} \\ A &= \{-u_1 - u_2 - u_4,\ \ -u_1 - u_3 - u_5,\ \ -u_1 - u_2 -\} \end{aligned}$$

It is easy to see that elements of $A_i\,(i = 1, 2)$ do not combine in pairs, i.e. $\alpha_1^{(i)} u_1^{(i)} + \alpha_2^{(i)} u_2^{(i)} \neq 0$ for $u_1^{(i)} u_2^{(i)} \in A_i,\ \ \alpha_1^{(i)} > 0.\ \ \alpha_2^{(i)} > 0$. Thus, there are not linear relations satisfying conditions of lemma 2. □

Lemma 3.4. *14-dimensional "faces" of the domain $V^{15}\left(\varphi_1^5\right)$, outside of which lie three points from $M\left(\varphi_1^5\right)$, are absent.*

Lemma 3.5. *14-dimensional "faces" of the domain $V^{15}\left(\varphi_1^5\right)$, outside of which lie five points from $M\left(\varphi_1^5\right)$, are absent.*

Lemmas 7 and 8 are proved similarly as Lemma 6.

Lemma 3.6. *14-dimensional "faces" of the domain* $V^{15}\left(\varphi_1^5\right)$*, which are nonequivalent with respect to* $Aut\left(\varphi_1^5\right)$*, outside of which lie four points from* $M\left(\varphi_1^5\right)$*, only three:*$(4,20,19,17)$, $(4,20,19,7)$, $(4,10,9,7)$.

Proof. All possible combinations of four elements from the set A satisfying the condition of Lemma 2 are the following

$$\{2\} \quad (-u_1-u_2-u_4,\ \ u_1,u_2,u_4)$$

$$\{3\} \quad (-u_1-u_3-u_5,\ \ u_1,u_3,u_5)$$

$$\{4\} \quad (u_1+u_2+u_3,\ \ u_1+u_4+u_5,\ \ -u_1-u_2-u_4,\ \ -u_1-u_3-u_5)$$

$$\{5\} \quad (-u_1-u_2-u_3-u_5,\ \ u_1+u_2+u_3,u_4,u_5)$$

$$\{6\} \quad (-u_1-u_2-u_3-u_4-u_5,\ \ u_1+u_4+u_5,u_2,u_3)$$

There is no other fours from A, satisfying the condition of Lemma 2. In the case {2}, in view of Lemma 4, taking the corresponding sets of points from (2.6), (3.5), firstly we get 16 faces:

{2}:	I	II	III	IV
	(4,20,19,17)	(4,10,19,17)	(14,20,19,17)	(14,10,19,17)
	(4,20,19,7)	(4,10,19,7)	(14,20,19,7)	(14,10,19,7)
	(4,20,9,17)	(4,10,9,17)	(14,20,9,17)	(14,10,9,17)
	(4,20,9,7)	(4,10,9,7)	(14,20,9,7)	(14,10,9,7)

Similarly, as in {2} we have

{3}:	(3,20,18,16)	(3,10,18,16)	(13,20,18,16)	(13,10,18,16)
	(3,20,18,6)	(3,10,18,6)	(13,20,18,6)	(13,10,18,6)
	(3,20,8,16)	(3,10,8,16)	(13,20,8,16)	(13,10,8,16)
	(3,20,8,6)	(3,10,8,6)	(13,20,8,6)	(13,10,8,6)

{4}:	V	VI	VII	VIII
	(1,2,3,4)	(1,12,3,4)	(11,2,3,4)	(11,12,3,4)
	(1,2,3,14)	(1,12,3,14)	(11,2,3,14)	(11,12,3,14)
	(1,2,13,4)	(1,12,3,4)	(11,2,13,4)	(11,12,13,4)
	(1,2,13,14)	(1,12,13,14)	(11,2,13,14)	(11,12,13,14)

{5}:	(5,1,7,6)	(5,11,7,6)	(15,1,7,6)	(15,11,7,6)
	(5,1,7,16)	(5,11,7,16)	(15,1,7,16)	(15,11,7,16)
	(5,1,17,6)	(5,11,17,6)	(15,1,17,6)	(15, 11,17,6)
	(5,1,17,16)	(5,11,17,16)	(15,1,17,16)	(15, 11,17,16)

{6}:	(5,2,9,8)	(5,12,9 8)	(15,2,9,8)	(15,12,9,8)
	(5,2,9,18)	(5,12,9,18)	(15,2,9,18)	(15,12, 9,18)
	(5,2,19,8)	(5,12,19,8)	(15,2,19 8)	(15,12,19,8)
	(5,2,19,18)	(5,12,19,18)	(15,2,19,18)	(15,12,19,18)

Now we apply $Aut\left(\varphi_1^5\right)$ to {{2},{3},{4},{5},{6}} with respect to $y_4 \to y_5, y_5 \to y_4 : 4 \to 3, 20 \to 20, 19 \to 18, 17 \to 16, 9 \to 8, 7 \to 6, 10 \to 10, 13 \to 14$. Therefore, in total {{2}} ∼ {{3}} . with respect to $y_1 \to y_2, y_2 \to y_1 :: 1, 2 \to 5, 3 \to 6, 4 \to 7, 11 \to 11, 12 \to 15, 13 \to 16, 14 \to 17$. Thus, in total {{4}} ∼ {{5}} . With respect to $y_2 \to y_3, y_3 \to y_2 : 5 \to 5, 1 \to 2, 6 \to \to 8, 7 \to 9, 16 \to 18, 17 \to 19, 15 \to 15, 11 \to 12$. Then, in total {{5}} ∼ {{6}} . Therefore, it suffices to consider {2} and {4} with respect to $y_i \to y_i\,(i = 1, 2, 3, 4)\,, y_5 \to -y_5 : 4 \to 14, 20 \to 10, 19 \to \to 9, 17 \to 7$. Therefore, in total $\{I\} \sim \{IV\}\,, \{II\} \sim \{III\}$. with respect to $y_i \to y_i\,(i = 1, 3, 4, 5)\,, y_2 \to -y_2 : 1 \to 11, 2 \to 2, 3 \to 3, 4 \to 4, 12 \to \to 12, 13 \to 13, 14 \to 14$. Thus, in total $\{V\} \sim \{VII\}\,, \{VI\} \sim \{VIII\}$. Therefore, it suffices to consider from {{2} , {4}} to consider $\{I, II, V, VI\}$. Further, with respect to $y_1 \to y_5, y_2 \to y_2, y_3 \to y_3,$ $y_4 \to y_4, y_5 \to y_1 : 1 \to 4, 2 \to 9, 3 \to 10, 4 \to 7, 13 \to 20, 14 \to 17$. Thus,

(1,2,13,4)∼(4,20,9,7)	(1,2,13,14)∼(4,20,9,17)
(1,2,3,4)∼(4,10,9,7)	(1,2,3,14)∼(4,10,9,17)
(1,12,3,9)∼(4,10,19,7)	(1,12,3,14)∼(4,10,19,17)
(1,12,13,4)∼(4,20,19,7)	(1,12,13,14)∼(4,20,19,17)

i.e. in total $\{I, II\} \sim \{V, VI\}$. Therefore, it suffices to consider $\{I, II\}$. with respect to $y_1 \to -y_1, y_2 \to y_2, y_3 \to -y_3, y_4 \to y_4, y_5 \to -y_5 : 4 \to 4, 20 \to \to 10, 9 \to 19, 7 \to 17, 19 \to 9, 17 \to 7$. Therefore,

(4,20,19,17)∼(4,10,9,7)	(4,20,19,7)∼(4,10,9,17)
(4,20,9,17)∼(4,10,19,7)	(4,20,9,7)∼(4,10,19,7)

i.e., in total $\{I\} \sim \{II\}$. with respect to $y_i \to y_i\,(i = 1, 4, 5)\,, y_2 \to y_3,\ y_3 \to y_2 : 4 \to 4, 20 \to 20, 7 \to 9, 9 \to 7, 17 \to 19, 19 \to 17$. Therefore, (4, 20, 19, 17)∼(4, 20, 9, 7). Thus, from 80 "faces" nonequivalent with respect to $Aut\left(\varphi_1^5\right)$, there are only three: (4, 20, 19, 17), (4, 20, 19, 7), (4, 20, 9, 7).

Lemma 9 is completely proved. □

Lemma 3.7. *14-dimensional "faces" of the domain $V^{15}\left(\varphi_1^5\right)$ nonequivalnet with respect to $Aut\left(\varphi_1^5\right)$, outside of which lie six points from $M\left(\varphi_1^5\right)$, are only six: (5, 20, 19, 18, 17, 16), (5, 20, 19, 18, 17, 6), (5, 20, 19, 8, 17, 16), (5, 20, 19, 8, 17, 6), (5, 20, 19).*

Proof. All possible combinations of six elements of the set A satisfying the condition of Lemma 2 are the following:

{1} $(-u_1 - u_2 - u_3 - u_4 - u_5, u_1, u_2, u_3, u_4, u_5)$
{7} $(-u_1 - u_2 - u_4, -u_1 - u_3 - u_5, u_1 + u_2 + u_3, u_1, u_4, u_5)$
{8} $(-u_1 - u_2 - u_4, -u_1 - u_3 - u_5, u_1 + u_4 + u_5, u_1, u_2, u_3)$
{9} $(-u_1 - u_2 - u_3 - u_4 - u_5, -u_1 - u_3 - u_5, u_1 + u_2 + u_3, u_1 + u_4$
$+u_5, u_3, u_5)$
{10} $(-u_1 - u_2 - u_3 - u_4 - u_5, -u_1 - u_2 - u_4, u_1 + u_2 + u_3, u_1 + u_4$
$+u_5, u_2, u_4)$.

There are no other sixes from A that satisfy the condition of Lemma 2.
In the case {1}, in view of Lemma 4, taking the corresponding sets of points from (3.4), we initially

obtain 64 faces:

{1}:		I_1	II_1
	1.1	$I_1^1\,(5,20,19,18,17,16)$	$I_1^{17}\,(5,20,9,18,17,16)$
	1.2	$I_1^2\,(5,20,19,18,17,6)$	$I_1^{18}\,(5,20,9,18,17,6)$
	1.3	$I_1^3\,(5,20,19,18,7,16)$	$I_1^{19}\,(5,20,9,18,7,16)$
	1.4	$I_1^4\,(5,20,19,18,7,6)$	$I_1^{20}\,(5,20,9,18,7,6)$
		III_1	IV_1
	1.5	$I_1^5\,(5,20,19,8,17,16)$	$I_1^{21}\,(5,20,9,8,17,16)$
	1.6	$I_1^6\,(5,20,19,8,17,6)$	$I_1^{22}\,(5,20,9,8,17,6)$
	1.7	$I_1^7\,(5,20,19,8,7,16)$	$I_1^{23}\,(5,20,9,8,7,16)$
	1.8	$I_1^8\,(5,20,19,8,7,6)$	$I_1^{24}\,(5,20,9,8,7,6)$
		V_1	VI_1
	1.9	$I_1^9\,(5,10,19,18,17,16)$	$I_1^{25}\,(5,10,9,18,17,16)$
	1.10	$I_1^{10}\,(5,10,19,18,17,6)$	$I_1^{26}\,(5,10,9,18,17,6)$
	1.11	$I_1^{11}\,(5,10,19,18,7,16)$	$I_1^{27}\,(5,10,9,18,7,16)$
	1.12	$I_1^{12}\,(5,10,19,18,7,6)$	$I_1^{28}\,(5,10,9,18,7,6)$
		VII_1	VIII_1
	1.13	$I_1^{13}\,(5,10,19,8,17,16)$	$I_1^{29}\,(5,10,9,8,17,16)$
	1.14	$I_1^{14}\,(5,10,19,8,17,6)$	$I_1^{30}\,(5,10,9,8,17,6)$
	1.15	$I_1^{15}\,(5,10,19,8,7,16)$	$I_1^{31}\,(5,10,9,8,7,16)$
	1.16	$I_1^{16}\,(5,10,19,8,7,6)$	$I_1^{32}\,(5,10,9,8,7,6)$

{1*}: here replacing 5 to 15 we get more 32 sets of points. We denote them $\left\{I_1^*, II_1^*, III_1^*, IV_1^*, V_1^*, VI_1^*, VII_1^*, VIII_1^*\right\}$. There are 64 "faces" in total. Similarly as in{1} we have

{7}:	I$_7$	V$_7$
	$I_7^1\,(4,3,1,20,17,16)$	$I_7^{17}\,(4,13,1,20,17,16)$
	$I_7^2\,(4,3,1,20,17,6)$	$I_7^{18}\,(4,13,1,20,17,6)$
	$I_7^3\,(4,3,1,20,7,16)$	$I_7^{19}\,(4,13,1,20,7,16)$
	$I_7^4\,(4,3,1,20,7,6)$	$I_7^{20}\,(4,13,1,20,7,6)$
	II$_7$	VI$_7$
	$I_7^5\,(4,3,1,10,17,16)$	$I_7^{17}\,(4,13,1,10,17,16)$
	$I_7^6\,(4,3,1,10,17,6)$	$I_7^{22}\,(4,13,1,10,17,6)$
	$I_7^7\,(4,3,1,10,7,16)$	$I_7^{23}\,(4,13,1,10,7,16)$
	$I_7^8\,(4,3,1,10,7,6)$	$I_7^{24}\,(4,13,1,10,7,6)$
	III$_7$	VII$_7$
	$I_7^9\,(4,3,11,20,17,16)$	$I_7^{25}\,(4,13,11,20,17,16)$
	$I_7^{10}\,(4,3,11,20,17,6)$	$I_7^{26}\,(4,13,11,20,17,6)$
	$I_7^{11}\,(4,3,11,20,7,16)$	$I_7^{27}\,(4,13,11,20,7,16)$
	$I_7^{12}\,(4,3,11,20,7,6)$	$I_7^{28}\,(4,13,11,20,7,6)$
	IV$_7$	VIII$_7$
	$I_7^{13}\,(4,3,11,10,17,16)$	$I_7^{29}\,(4,13,11,10,17,16)$
	$I_7^{14}\,(4,3,11,10,17,6)$	$I_7^{30}\,(4,13,11,10,17,6)$
	$I_7^{15}\,(4,3,11,10,7,16)$	$I_7^{31}\,(4,13,11,10,7,16)$
	$I_7^{15}\,(4,3,11,10,7,16)$	$I_7^{32}\,(4,13,11,10,7,6)$
	IV$_7$	VIII$_7$
	$I_7^{13}\,(4,3,11,10,17,16)$	$I_7^{29}\,(4,13,11,10,17,16)$
	$I_7^{14}\,(4,3,11,10,17,6)$	$I_7^{30}\,(4,13,11,10,17,6)$
	$I_7^{15}\,(4,3,11,10,7,16)$	$I_7^{31}\,(4,13,11,10,7,16)$
	$I_7^{15}\,(4,3,11,10,7,16)$	$I_7^{32}\,(4,13,11,10,7,6)$

{7*}: here replacing 4 by 14 we obtain more 32 sets of "faces". We denote them as $\left\{I_7^*, II_7^*, ..., VIII_7^*\right\}$. There are 64 "faces" in total.

{8}:	I_8	V_8
	$I_8^1\,(4,3,2,20,19,18)$	$I_8^{17}\,(4,13,2,20,19,18)$
	$I_8^2\,(4,3,2,20,19,8)$	$I_8^{18}\,(4,13,2,20,19,8)$
	$I_8^3\,(4,3,2,20,9,18)$	$I_8^{19}\,(4,13,2,20,9,18)$
	$I_8^4\,(4,3,2,20,9,8)$	$I_8^{20}\,(4,13,2,20,9,8)$
	II_8	VI_8
	$I_8^5\,(4,3,2,10,19,18)$	$I_7^{17}\,(4,13,1,10,17,16)$
	$I_8^6\,(4,3,2,10,19,8)$	$I_7^{22}\,(4,13,1,10,17,6)$
	$I_8^7\,(4,3,2,10,9,18)$	$I_7^{23}\,(4,13,1,10,7,16)$
	$I_8^8\,(4,3,2,10,9,8)$	$I_7^{24}\,(4,13,1,10,7,6)$
	III_8	VII_8
	$I_8^9\,(4,3,12,20,19,18)$	$I_8^{25}\,(4,13,12,20,19,18)$
	$I_8^{10}\,(4,3,12,20,19,8)$	$I_8^{26}\,(4,13,12,20,19,8)$
	$I_8^{11}\,(4,3,12,20,9,18)$	$I_8^{27}\,(4,13,12,20,9,18)$
	$I_8^{12}\,(4,3,12,20,9,8)$	$I_8^{28}\,(4,13,12,20,9,8)$
	IV_8	VIII_8
	$I_8^{13}\,(4,3,12,10,19,18)$	$I_8^{29}\,(4,13,12,10,19,18)$
	$I_8^{14}\,(4,3,12,10,19,8)$	$I_8^{30}\,(4,13,12,10,19,8)$
	$I_8^{15}\,(4,3,12,10,9,18)$	$I_8^{31}\,(4,13,12,10,9,18)$
	$I_8^{16}\,(4,3,12,10,9,8)$	$I_8^{32}\,(4,13,12,10,9,8)$

{8*}: here replacing 4 by 14 we get more 32 faces. We denote them by $\left\{I_8^*, II_8^*, ..., VIII_8^*\right\}$.

{9}:	I$_9$	V$_9$
	$I_9^1\,(5,3,1,2,18,16)$	$I_9^{17}\,(5,13,1,2,18,16)$
	$I_9^2\,(5,3,1,2,18,6)$	$I_9^{18}\,(5,13,1,2,18,6)$
	$I_9^3\,(5,3,1,2,8,16)$	$I_9^{19}\,(5,13,1,2,8,16)$
	$I_9^4\,(5,3,1,2,8,6)$	$I_9^{20}\,(5,13,1,2,8,16)$
	II$_9$	VI$_9$
	$I_9^5\,(5,3,1,12,18,16)$	$I_9^{21}\,(5,13,1,12,18,16)$
	$I_9^6\,(5,3,1,12,18,6)$	$I_9^{22}\,(5,13,1,12,18,6)$
	$I_9^7\,(5,3,1,12,8,16)$	$I_9^{23}\,(5,13,1,12,8,16)$
	$I_9^8\,(5,3,1,12,8,6)$	$I_9^{24}\,(5,13,1,12,8,16)$
	III$_9$	VII$_9$
	$I_9^9\,(5,3,11,2,18,16)$	$I_9^{25}\,(5,13,11,2,18,16)$
	$I_9^{10}\,(5,3,11,2,18,6)$	$I_9^{26}\,(5,13,11,2,18,6)$
	$I_9^{11}\,(5,3,11,2,8,16)$	$I_9^{27}\,(5,13,11,2,8,16)$
	$I_9^{12}\,(5,3,11,2,8,6)$	$I_9^{28}\,(5,13,11,2,8,16)$
	IV$_9$	VIII$_9$
	$I_9^{13}\,(5,3,11,12,18,16)$	$I_9^{29}\,(5,13,11,12,18,16)$
	$I_9^{14}\,(5,3,11,12,18,6)$	$I_9^{30}\,(5,13,11,12,18,6)$
	$I_9^{15}\,(5,3,11,12,8,16)$	$I_9^{31}\,(5,13,11,12,8,16)$
	$I_9^{16}\,(5,3,11,12,8,6)$	$I_9^{32}\,(5,13,11,12,8,16)$

{9*}: here replacing 5 by 15 we obtain more 32 sets of faces. We denote them as $\left\{I_9^*, II_9^*, ..., VIII_9^*\right\}$.

{10}:	I$_{10}$	V$_{10}$
	$I_{10}^1\,(5,4,1,2,19,17)$	$I_{10}^{17}\,(5,14,1,2,19,17)$
	$I_{10}^2\,(5,4,1,2,19,7)$	$I_{10}^{18}\,(5,14,1,2,19,7)$
	$I_{10}^3\,(5,4,1,2,9,17)$	$I_{10}^{19}\,(54,1,2,9,17)$
	$I_{10}^4\,(5,4,1,2,9,7)$	$I_{10}^{20}\,(5,14,1,2,9,17)$

	II_{10}	VI_{10}
	$I_{10}^{5}\,(5,4,1,12,19,17)$	$I_{10}^{21}\,(5,14,1,12,19,17)$
	$I_{10}^{6}\,(5,4,1,12,19,7)$	$I_{10}^{22}\,(5,14,1,12,19,7)$
	$I_{10}^{7}\,(5,4,1,12,9,17)$	$I_{10}^{23}\,(5,14,1,12,9,17)$
	$I_{10}^{8}\,(5,4,1,12,9,7)$	$I_{10}^{24}\,(5,14,1,12,9,17)$
	III_{10}	VII_{10}
	$I_{10}^{9}\,(5,4,11,12,19,17)$	$I_{10}^{25}\,(5,14,11,12,19,17)$
	$I_{10}^{10}\,(5,4,11,12,19,7)$	$I_{10}^{26}\,(5,14,11,12,19,7)$
	$I_{10}^{11}\,(5,4,11,12,9,17)$	$I_{10}^{27}\,(5,14,11,12,9,17)$
	$I_{10}^{12}\,(5,4,11,2,9,7)$	$I_{10}^{28}\,(5,14,11,2,9,17)$
	IV_{10}	$VIII_{10}$
	$I_{10}^{13}\,(5,4,11,12,19,17)$	$I_{10}^{29}\,(5,14,11,12,19,17)$
	$I_{10}^{14}\,(5,4,11,12,19,7)$	$I_{10}^{30}\,(5,14,11,12,19,7)$
	$I_{10}^{15}\,(5,4,11,12,9,17)$	$I_{10}^{31}\,(5,14,11,12,9,17)$
	$I_{10}^{16}\,(5,4,11,12,9,7)$	$I_{10}^{32}\,(5,14,11,12,9,17)$

{10*}: here we replace 5 by 15 we get more 32 sets of faces. We denote them as $\left\{I_{10}^{*}, II_{10}^{*}, ..., VIII_{10}^{*}\right\}$. Now we apply $Aut\varphi_1^5 k\,\{1,7,-10,1*,7*\}$.

With respect to $y_4 \to y_5, y_5 \to y_4$ we have $5 \to 5,\ 1 \to 1,\ 2 \to 2,\ 4 \to 3,\ 19 \to 18, 17 \to 16,\ 7 \to 6,\ 9 \to 8,\ 12 \to 12,\ 11 \to 11,\ 14 \to 13$. Therefore $I_{10} \to I_9,\ II_{10} \to II_9, III_{10} \to III_9,\ IV_{10} \to IV_9,\ V_{10} \to V_9,\ VI_{10} \to VI_9,\ VII_{10} \to VII_9,\ VIII_{10} \to VIII_9$. Hence it follows $\{10\} \sim \{9\}$.

With respect to $y_i \to y_i\,(i = 1,2,4,5)$, $y_3 \to -y_3$: $5 \to 15,\ 3 \to 3,\ 1 \to 1,\ 2 \to 12,\ 18 \to 8,\ 16 \to 6,\ 11 \to 11,\ 13 \to 13$. Therefore $IV_9 \to III_9^*,\ V_9 \to VI_9^*,\ VI_9 \to V_9^*,\ VII_9 \to VIII_9^*,\ VIII_9 \to VII_9^*$. Thus, $\{9\} \sim \{9*\}$.

Further, with respect to $y_i \to y_i\,(i = 1,2,4,5)$, $y_3 \to -y_3$: $4 \to 4,\ 19 \to 9, 17 \to 7,\ 14 \to 14$. Therefore $I_{10} \to II_{10}^*,\ II_9 \to I_{10}^*$,
$III_{10} \to IV_{10}^*,\ IV_{10} \to III_{10}^*, V_{10} \to VI_{10}^*, VI_{10} \to V_{10}^*, VII_{10} \to VIII_{10}^*, VIII_{10} \to VII_{10}^*$. Hence $\{10\} \sim \{10*\}$. From $\{9, 9^*, 10, 10^*\}$ only remain 9.

With respect to $y_i \to y_i\,(i = 1,2,3,4)$, $y_5 \to -y_5$ we have: $4 \to 4,\ 3 \to 3,\ 1 \to 1, 10 \to 10,\ 17 \to 7,\ 16 \to 16,\ 6 \to 6,\ 11 \to 11,\ 13 \to 13,\ 2 \to 2,\ 19 \to 9,\ 18 \to 18,\ 8 \to 8,12 \to 12$. Therefore $I_7 \to II_7^*,\ II_7 \to I_7^*,\ III_7 \to IV_7^*,\ IV_7 \to III_7^*, V_7 \to VI_7^*, VI_7 \to V_7^*, VII_7 \to VIII_7^*, VIII_7 \to VII_7^*$. Hence $\{7\} \sim \{7*\}$.

$$I_8 \to II_8^*,\ II_8 \to I_8^*,\ III_8 \to IV_8^*,\ IV_8 \to III_8^*,\ V_8 \to VI_8^*,\ VI_8 \to V_8^*, VII_8 \to VIII_8^*,$$

$VIII_8 \to VII_8^*$.Whence $\{8\} \sim \{8*\}$.

With regards $y_2 \to y_3,\ y_3 \to y_2$: $4 \to 4,\ 3 \to 3,\ 1 \to 2,\ 20 \to 20,\ 17 \to 19, 16 \to 18,\ 11 \to 12,\ 12 \to 11,\ 10 \to 10,\ 6 \to 8,\ 8 \to 6,\ 7 \to 9,\ 13 \to 13$.

Therefore $I_7 \to I_8,\ II_2 \to II_8,\ III_7 \to III_8,\ IV_7 \to IV_8,\ VI_7 \to VI_8,\ VII_7 \to VII_8,\ VIII_7 \to VIII_8$. Hence $\{7\} \sim \{8\}$. Thus from $\{7, 7^*,\ 8, 8^*\}$ remain only 7.

With respect $y_1 \to y_2),\ y_2 \to y_1,\ y_3 \to y_5),\ y_4 \to y_4,\ y_5 \to y_3)$, we get: $5 \to 4,\ 3 \to 6,\ 1 \to 1,\ 2 \to 7,\ 18 \to 20,\ 16 \to 13,\ 6 \to 3,\ 8 \to 10,\ 12 \to 17,\ 11 \to 11, 13 \to 16$. Therefore $I_9^1 \to I_7^{20},\ I_9^2 \to I_7^4,\ I_9^3 \to I_7^{24},\ I_9^4 \to I_7^8,\ I_9^5 \to I_7^{18},\ I_9^6 \to I_7^2, I_9^7 \to I_7^{22},\ I_9^8 \to I_7^6,\ I_9^9 \to I_7^{28},\ I_9^{10} \to I_7^{12},\ I_9^{11} \to I_7^{32},\ I_9^{12} \to I_7^{16},\ I_9^{13} \to I_7^{26},\ I_9^{14} \to I_7^{10}, I_9^{15} \to I_7^{30},\ I_9^{16} \to I_7^{14},\ I_9^{17} \to I_7^{19},\ I_9^{18} \to I_7^{13},\ I_9^{19} \to I_7^{23},\ I_9^{20} \to I_7^7,\ I_9^{21} \to I_7^{17},$

$$I_9^{22} \to I_7^1,\ I_9^{23} \to I_7^{21},\ I_9^{24} \to I_7^5,\ I_9^{25} \to I_7^{27},\ I_9^{26} \to I_7^{11},\ I_9^{27} \to I_7^{31},$$
$$I_9^{28} \to I_7^{15},\ I_9^{29} \to I_7^{25}, I_9^{30} \to I_7^9,\ I_9^{31} \to I_7^{29},\ I_9^{32} \to I_7^{13}.$$

Hence, $\{9\} \sim \{7\}$. Thus from $\{7, 7^*,\ 8, 8^*,\ 9, 9^*,\ 10, 10^*\}$ remain only 9.

With respect to $y_1 \to y_5),\ y_5 \to y_1$ we get: $5 \to 5,\ 3 \to 10,\ 1 \to 7,\ 2 \to 9,\ 18 \to 18,\ 16 \to 16,\ 6 \to 6,\ 8 \to 8,\ 12 \to 19,\ 11 \to 17,\ 13 \to 20$. Therefore $I_9^1 \to I_1^{27}, I_9^2 \to I_1^{28},\ I_9^3 \to I_1^{31},\ I_9^4 \to I_1^{32},\ I_9^5 \to I_1^{14},\ I_9^6 \to I_1^{12},\ I_9^7 \to I_1^{15},\ I_9^8 \to I_1^{16},\ I_9^9 \to I_1^{25}, I_9^{10} \to I_1^{26},\ I_9^{11} \to I_1^{29}, I_9^{12} \to I_1^{30},\ I_9^{13} \to I_1^9,\ I_9^{14} \to I_1^{10},\ I_9^{15} \to I_1^{13},\ I_9^{16} \to I_1^{14},$
$I_9^{17} \to I_1^{19}, I_9^{18} \to I_1^{20}, I_9^{19} \to I_1^{23}, I_9^{20} \to I_1^{24},\ I_9^{21} \to I_1^3,\ I_9^{22} \to I_1^4,\ I_9^{23} \to I_1^7,\ I_9^{24} \to I_1^8, I_9^{25} \to I_1^{17},\ I_9^{26} \to I_1^{18}, I_9^{27} \to I_1^{21}, I_9^{28} \to I_1^{22}, I_9^{29} \to I_1^1,$
$I_9^{30} \to I_1^{30}, I_9^{31} \to I_1^5, I_9^{32} \to I_1^6$. Hence, $\{9\} \sim \{1\}$.

Further, with respect to $y_i \to y_i\,(i = 1, 2, 4, 5)$, $y_3 \to -y_3$: $5 \to 15,\ 20 \to 20, 10 \to 10,\ 19 \to 9,\ 18 \to 8,\ 17 \to 17,\ 7 \to 7,\ 16 \to 16,\ 6 \to 6$. Therefore $I_1 \to IV_1^*, II_1 \to III_7^*,\ III_1 \to II_1^*,\ IV_1 \to I_1^*,\ V_1 \to VIII_1^*,\ VI_1 \to VII_1^*,\ VII_1 \to VI_1^*,\ VIII_1 \to V_1^*$.Hence, $\{1\} \sim \{1*\}$.

Thus, from $\{1, 1^*,\ 7, 7^*,\ 8, 8^*,\ 9, 9^*,\ 10, 10^*\}$only remain $\{1\}$.

Now we apply $Aut\varphi_1^5\ k\{1\}$. With respect to $y_4 \to -y_4, y_5 \to -y_5$ we have: $5 \to 15,\ 20 \to 20,\ 19 \to 9,\ 18 \to 8,\ 17 \to 17,\ 10 \to 10,\ 16 \to 6,\ 6 \to 6$. Therefore $\{I_1\} \sim \{IV_1\}$, $\{II_1\} \sim \{III_1\}$, $\{V_1\} \sim \{VIII_1\}$, $\{VI_1\} \sim \{VII_1\}$. From $\{1\}$ remain $\{I_1,\ III_1, V_1, VII_1\}$.

With respect$y_2 \to -y_5,\ y_5 \to -y_2$ we have: $5 \to 19, 20 \to 6,\ 18 \to 18,\ 10 \to 16, 17 \to 17,\ 8 \to 8$. Therefore $1.6 \sim 1.8,\ \ 1.3 \sim 1.9,\ \ 1.1 \sim 1.12,\ \ 1.9 \sim 1.11, 1.13 \sim 1.15, 1.7 \sim 1.14,\ \ 1.7 \sim 1.14,\ \ 1.5 \sim 1.16$.

Hence, from $\{I_1, III_1, V_1, VII_1\}$
remain {1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.9, 1.13}
$= \left\{I_1^1,\ I_1^2,\ I_1^3,\ I_1^4,\ I_1^5,\ I_1^6,\ I_1^7, I_1^9, I_1^13\right\}$.

With respect to $y_4 \to y_5, y_5 \to y_4$ we have: $5 \to 5,\ 20 \to 20,\ 19 \to 18,\ 17 \to 16,\ 6 \to 7$. Therefore $1.2 \sim 1.3$.

With respect to $y_4 \to -y_4,\ y_i \to y_i\,(i = 1,\ 2,\ 3,\ 5)$ we have: $5 \to 5,\ 20 \to 100, 19 \to 19,\ 18 \to 8,\ 17 \to 17,\ 16 \to 6$. Therefore $1.9 \sim 1.5,\ 1.2 \sim 1.13$.

Hence, definitively remain $\{1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7\}$.

Thus, from $5 \cdot 2^6 = 320$ "faces" nonequivalent with respect to $Aut\varphi_1^5$, are 6 : (1.1) : (5, 20, 19, 18, 17, 16) , (1.2) : (5, 20, 19, 18, 17, 6) , $I_1^4 =$ (1.4) : (5, 20, 19, 18,7, 6) , (1.5) :

$(5,\ 20,\ 19,\ 8,\ 17,\ 16)$,

$(1.6):\ (5, 20, 19, 8, 17, 6), (1.7)\ :\ (5, 20, 19, 8, 7, 16)$.

We note that with respect to $y_1 \to y_2,\ y_2 \to y_1,\ y_3 \to -y_5,\ y_4 \to -y_4,\ y_5 \to -y_3$ we have: $5 \to 14,\ 20 \to 18,\ 19 \to 19,\ 18 \to 20,\ 7 \to 12,\ 6 \to 13$. Therefore

$$I_1^4 =: (5,\ 20,\ 19,\ 18, 7,\ 6) \sim \left(I_8^*\right)^{26} : (14,\ 13,\ 12,\ 20,\ 19,\ 18).$$

Lemma 10 is completely proved. □

The main result of the paper is a detailed proof of the following proposition.

Theorem 3.1. *The Voronoi neighborhood of the perfect form*

$$\varphi_1^5 = \varphi_1^5(x) = \varphi_1^5(x_1, ..., x_5)$$

$$= x_1^2 + ... + x_5^2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + ... + x_4x_5$$

consists of only three perfect forms:

$$\varphi_0^5 = \varphi_0^5(x) = \varphi_0^5(x_1, ..., x_5)$$

$$= x_1^2 + ... + x_5^2 + x_1x_2 + ... + x_1x_5 + x_2x_3 + ... + x_4x_5,$$

$$\varphi_1^5 = \varphi_1^5(x) = \varphi_1^5(x_1, ..., x_5) = \varphi_0^5 - x_1x_2,$$

$$\varphi_2^5 = \varphi_2^5(x) = \varphi_2^5(x_1, ..., x_5) = \varphi_0^5 - \frac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5).$$

Proof. According Lemmas 9, 10 and equality (3.5) by direct calculations we get equations 14-dimentional faces of the domain $V^{15}\left(\varphi_1^5\right)$ in the space R^{15} of coefficients of the quadratic forms.

We write them successively:

1) $(4,\ 20,\ 19,\ 7)$:

$$a_{55} + a_{15} + a_{35} + a_{45} - u_1 - u_2 - u_4 - a_{45} + u_1 - a_{35} + u_2 - a_{15} + a_{25} + u_4$$

$$= a_{55} + a_{25} = 0$$

The corresponding conjugate perfect form has the form:

$$2\left\{\varphi_1^5\right\}_1 = 2\left|\varphi_1^5 + x_5^2 + x_2x_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + x_1x_4\right.$$
$$+ x_1x_5 + x_2x_3 + x_2x_4 + 2x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5.$$

We calculate

$$\det\left\{2\left\{\varphi_1^5\right\}_1\right\} := \begin{vmatrix} 2 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 4 \end{vmatrix} = 4$$

Therefore, $\det\left\{\varphi_1^5\right\}_1 = \frac{1}{2^3}$.Thus $\det\left\{\varphi_1^5\right\}_1$coincides with $\det \varphi_1^5$.

2) (4, 20, 19, 7) :

$$a_{55}+a_{15}+a_{35}+a_{45}-u_1-u_2-u_4-a_{45}+u_1-a_{35}+u_2-a_{15}+a_{15}$$
$$+u_4=a_{55}+a_{15}=0.$$

The corresponding conjugate perfect form has the view:

$$2\left\{\varphi_1^5\right\}_1^*=2\left|\varphi_1^5+x_5^2+x_1x_5=x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+\right.$$
$$+x_1x_4+x_1x_5+x_2x_3+x_2x_4+2x_2x_5+x_3x_4+x_3x_5+x_4x_5.$$

This form is equivalent the form $\left\{\varphi_1^5\right\}_1$. Indeed, the substitution of the variables $x_1\to x_2, x_2\to x_1$ transforms $\left\{\varphi_1^5\right\}_1$ into $\left\{\varphi_1^5\right\}_1^*$.

3) (4, 20, 9, 7) :

$$a_{55}+a_{15}+a_{35}+a_{45}-u_1-u_2-u_4-a_{45}+u_1+u_2+u_4$$
$$=a_{55}+a_{15}+a_{35}=0.$$

The corresponding conjugate form has the view:

$$2\left\{\varphi_1^5\right\}_2=2\left|\varphi_1^5+x_5^2+x_1x_5+x_3x_5=x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+\right.$$
$$+x_1x_3+x_1x_4+2x_1x_5+x_2x_3+x_2x_4+x_2x_5+x_3x_4+2x_3x_5+x_4x_5.$$

Here $\det\left\{2\left\{\varphi_1^5\right\}_2\right\}:=4$

Hence, $\det\left\{\varphi_1^5\right\}_2=\frac{1}{2^3}$. Thus $\det\left\{\varphi_1^5\right\}_2$ coincides with $\det\varphi_1^5$.

4) (5, 20, 19, 18, 17, 16) :

$$-a_{12}-a_{14}-a_{15}-a_{23}-a_{34}-a_{35}-a_{45}=0.$$

The corresponding conjugate form has the view:

$$\left\{\varphi_1^5\right\}_3=2\left|\varphi_1^5-x_1x_2-x_1x_5-x_2x_3-x_3x_4-x_3x_5-x_4x_5\right.$$
$$=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-x_1x_2+x_1x_3+x_2x_4+x_2x_5.$$

With respect to transformation $x_1\to x_2, x_2\to x_1, x_3\to x_5, x_4\to x_4, x_5\to x_3$ we have: $\left\{\varphi_1^5\right\}_3\sim\left\{\varphi_1^5\right\}_3^*$.

6) (5, 20, 19, 8, 17, 16) :

$$-a_{12}-a_{15}-a_{23}-a_{35}-a_{45}=0.$$

The corresponding conjugate perfect form has the form:

$$\left\{\varphi_1^5\right\}_4=\varphi_1^5-x_1x_2-x_1x_4-x_1x_5-x_2x_3-x_3x_5-x_4x_5=$$
$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-x_1x_2+x_1x_3+x_2x_4+x_2x_5+x_3x_4.$$

Here $\det\left\{2\left\{\varphi_1^5\right\}_4\right\}:=4$

Hence, $\det\left\{\varphi_1^5\right\}_4=4$, $\det\left\{\varphi_1^5\right\}_4=\frac{2^2}{2^5}=\frac{1}{2^3}$. Thus $\det\left\{\varphi_1^5\right\}_4$ coincides with $\det\varphi_1^5$.

7) (5, 20, 19, 8, 17, 6) :

$$-a_{12}-a_{15}-a_{23}-a_{24}-a_{35}-a_{45}=0.$$

The corresponding conjugate perfect form has the view:

$$\left\{\varphi_1^5\right\}_1 = \varphi_1^5 - x_1x_2 - x_1x_5 - x_2x_3 - x_3x_5 - x_4x_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$+ x_5^2 - x_1x_2 + x_1x_3 + x_1x_4 + x_2x_5 + x_3x_4.$$

Here det $\left\{2\left\{\varphi_0^5\right\}_1\right\} := 6$

Hence, det $\left\{\varphi_0^5\right\}_1 = \frac{3}{2^4}$.Thus det $\left\{\varphi_0^5\right\}_1 = \det\left\{\varphi_0^5\right\}$.

8) (5, 20, 19, 8, 7, 16) :

$$-a_{12} - a_{14} - a_{23} - a_{25} - a_{35} - a_{45} = 0.$$

The corresponding conjugate perfect form has the form:

$$\left\{\varphi_1^5\right\}_5 = \varphi_1^5 - x_1x_2 - x_1x_4 - x_2x_3 - x_2x_5 - x_3x_5 - x_4x_5 = x_1^2 + x_2^2$$
$$+ x_3^2 + x_4^2 + x_5^2 - x_1x_2 + x_1x_3 + x_1x_5 + x_2x_4 + x_3x_4.$$

Here det $\left\{2\left\{\varphi_1^5\right\}_5\right\} := 4$

Hence, det $\left\{\varphi_1^5\right\}_5 = \frac{1}{2^3}$.Thus det $\left\{\varphi_1^5\right\}_5 = \det\left\{\varphi_1^5\right\}$.

9) (14, 13, 12, 20, 19, 18) :

$$-a_{12} - a_{34} - a_{35} - a_{45} = 0.$$

The corresponding conjugate perfect form has the form:

$$\left\{\varphi_2^5\right\} = \varphi_1^5 + \frac{1}{2}\left\{x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5\right\}.$$

Here det $\left\{4\left\{\varphi_2^5\right\}\right\} := 162.$

Hence, det $\left\{\varphi_2^5\right\}_5 = \frac{3^4}{2^9}$.

To calculate completely $VN\left(\varphi_1^5\right)$ it remains for us to show that

$$\left\{\varphi_1^5\right\}_1 \sim \varphi_1^5,\ \left\{\varphi_1^5\right\}_2 \sim \varphi_1^5,\ \left\{\varphi_1^5\right\}_3 \sim \varphi_1^5,\ \left\{\varphi_1^5\right\}_4 \sim \varphi_1^5,\ \left\{\varphi_0^5\right\}_1 \sim \varphi_0^5$$
$$\left\{\varphi_1^5\right\}_5 \sim \varphi_1^5.$$

Indeed, the substitution E_0^* of variables $x_1 \to x_1 \to x_2,\ x_2 \to x_2 + x_5,\ x_i \to x_i\,(i = 3,4,5)$ transforms the form φ_0^5 into $\left\{\varphi_0^5\right\}_1$. Hence $\varphi_0^5 \sim \left\{\varphi_0^5\right\}_1^*$.

Similarly, $\varphi_1^5 E_i^* = \left\{\varphi_0^5\right\}_i \quad (i = 1,2,3,4,5)$. Here

$$E_5^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},\ E_4^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$E_1^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, E_2^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$E_3^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The transformation E_i^* $(i = 0, ..., 5)$ were calculated with the help Lemma 3.

The theorem is completely proved. □

Conclusion

The classical Voronoi problem of finding perfect forms, closely related to Hermite's well-known problem of finding the arithmetic minimum of positive quadratic forms, are interesting and non-trivial problems in geometric number theory that have been studied by many mathematicians. They also appeared in the works of S.L. Sobolev in connection with the construction of lattice optimal cubature formulas.

The present work is devoted to the development of an algorithm and finding all adjacent perfect forms in five variables.

The technique presented in the work allows us to study the classical Voronoi problems in a complex, its results and calculation methods can be used to further search for new perfect forms in many variables.

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