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GENERALIZATION OF INTEGRAL INEQUALITIES FOR PRODUCT OF CONVEX FUNCTIONS

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Abstract. In this paper, generalizations of some inequalities for product of convex functions are given.

1. Introduction

A function $f:[a,b]\to\mathbb{R}$, with $[a,b]\subset\mathbb{R}$, is said to be convex if whenever, x, $y\in[a,b]$ and $t\in[0,1]$ the following inequality holds:

$$f\left(tx+(1-t)y\right)\le tf(x)+(1-t)f(y).$$

This definition has its origin in Jensen's result from [1] and has opened up a very useful and multi-disciplinary domain of mathematics, namely, convex analysis. A largely applied inequality for convex functions, due to its geometrical significance, is the Hermite-Hadamard's inequality which has generated a wide range of directions for extensions and rich mathematical literature.

Hermite-Hadamrd's inequality is stated as follows:

A convex function satisfies:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

In a recent paper, Pachpatte [2] established the following inequalities for product of convex functions which can be derived from Hermite-Hadamard's inequality:

Theorem 1. [2] *Let f and g be real valued, nonnegative and convex functions on $[a,b]$. Then*

$$\frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left(tx+(1-t)y\right)g\left(tx+(1-t)y\right)dtdxdy \le\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{8}\left[\frac{M(a,b)+N(a,b)}{(b-a)^2}\right] \tag{1.2}$$

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and

$$\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)dtdx \le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{4}\cdot\frac{1+b-a}{b-a}\left[M(a,b)+N(a,b)\right], \tag{1.3}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b)$$
$$N(a,b)=f(a)g(b)+f(b)g(a).$$

The inequalities (1.2) and (1.3) are valid when the length of the interval $[a,b]$ does not exceed 1. The inequality (1.2) is sharp for linear functions defined on $[0,1]$, while the inequality (1.3) does not have the same property.

In [3], Cristescu improved these inequalities by eliminating the condition $b-a\le 1$ and derived the inequalities which are sharp for the whole class of linear functions defined on $[0,1]$. The main result from [3] is the following:

Theorem 2. [3] *Let f and g be real valued, nonnegative and convex functions on $[a,b]$. Then*

$$\frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdxdy \le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{8}\left[M(a,b)+N(a,b)\right] \tag{1.4}$$

and

$$\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)dtdx \le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{2}\left[M(a,b)+N(a,b)\right], \tag{1.5}$$

where $M(a,b)$ and $N(a,b)$ are defined in Theorem 1.

The main aim of this paper is to generalize the inequalities (1.4) and (1.5).

2. Main Results

Let I be an interval of $\mathbb{R}$ and let $f:I\to\mathbb{R}$ be a convex functions on I, $h:[a,b]\to\mathbb{R}$ be continuous function such that $h([a,b])\subset I$ and $p:[a,b]\to\mathbb{R}$ be a positive integrable function a, $b\in\mathbb{R}$ with $a<b$. Then the Jenesen's inequality

$$f\left(\frac{\int_a^b p(x)h(x)\,dx}{\int_a^b p(x)\,dx}\right)\le\frac{\int_a^b p(x)f(h(x))\,dx}{\int_a^b p(x)\,dx}$$

holds.

Assume that f and p are as above. Let us denote

$$P=\int_a^b p(x)dx, \qquad \bar{h}=\frac{1}{P}\int_a^b p(x)h(x)dx.$$

The following is the Hermite-Hdamard type inequality in this case:

$$f\left(\bar{h}\right) \leq \frac{1}{P}\int_a^b f\left(h(x)\right)p(x)dx \leq \frac{f\left(h(a)\right)+f\left(h(b)\right)}{2}. \tag{2.1}$$

We now state the following lemma which is very useful in this section:

Lemma 1. *Let $[a,b]\subset\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ be a function and $h:[a,b]\to\mathbb{R}$ be a continuous function such that $h\left([a,b]\right)\subset[a,b]$. Then the following statements are equivalent*

1. *function f is convex on $[a,b]$*
2. *For every $x,\ y\in[a,b]$, the function $\gamma:[0,1]\to\mathbb{R}$ defined by*

$$\gamma\left(t\right)=f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)$$

is convex on $[0,1]$ for any positive real number λ.

Proof. It is a direct consequence of the convexity of the function f. □

Now we state and prove the main result of this section which will generalize the Theorem 2.

Theorem 3. *Let f and g be real valued, nonnegative and convex functions on $[a,b]$. Let $h:[a,b]\to\mathbb{R}$ be continuous function such that $h\left([a,b]\right)\subset[a,b]$ and $p:[a,b]\to\mathbb{R}$ be a positive integrable function. Then*

$$\begin{aligned}&\frac{3}{2P^2}\int_a^b\int_a^b\int_0^1 p\left(x\right)p\left(y\right)f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)g\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)dtdxdy\\&\qquad\leq\frac{1}{P}\int_a^b f\left(h\left(x\right)\right)g\left(h\left(x\right)\right)p\left(x\right)dx+\frac{1}{8}\left[M^{'}\left(a,b\right)+N^{'}\left(a,b\right)\right],\end{aligned} \tag{2.2}$$

where

$$M^{'}\left(a,b\right)=f(h(a))g(h(a))+f(h(b))g(h(b))$$

and

$$N^{'}\left(a,b\right)=f(h(a))g(h(b))+f(h(b))g(h(a)).$$

Proof. Since both functions f and g are convex, for every two points $x,y\in[a,b]$ and $t\in[0,1]$, the following inequalities are valid

$$f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)\leq tf\left(h\left(x\right)\right)+\left(1-t\right)f\left(h\left(y\right)\right)$$

and

$$g\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)\leq tg\left(h\left(x\right)\right)+\left(1-t\right)g\left(h\left(y\right)\right)$$

Multiplying these inequalities side by side, we obtain

$$\begin{aligned}&f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)g\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)\\&\quad\leq t^2f\left(h\left(x\right)\right)g\left(h\left(x\right)\right)+\left(1-t\right)^2f\left(h\left(y\right)\right)g\left(h\left(y\right)\right)\\&\qquad+t\left(1-t\right)\left[f\left(h\left(x\right)\right)g\left(h\left(y\right)\right)+f\left(h\left(y\right)\right)g\left(h\left(x\right)\right)\right].\end{aligned} \tag{2.3}$$

Due to Lemma 1 and known properties of convex functions, both sides of the inequality (2.3) are integrable. Multiplying both sides of (2.3) by $p\left(x\right)p\left(y\right)$ and integrating both sides of the inequality (2.3) with respect to t over $[0,1]$, with respect to x and y over $[a,b]$, we get

(2.4)
$$\int_a^b\int_a^b\int_0^1 p\left(x\right)p\left(y\right)f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)g\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)dtdxdy$$
$$\leq \frac{2}{3}P\int_a^b f\left(h\left(x\right)\right)g\left(h\left(x\right)\right)p(x)dx$$
$$+\frac{1}{3}\left(\int_a^b f\left(h\left(x\right)\right)p(x)dx\right)\left(\int_a^b g\left(h\left(x\right)\right)p(x)dx\right).$$

The convexity property of f and g allow us to use right side of the inequality (2.1) and thus the above inequality (2.4) takes the form:

(2.5)
$$\int_a^b\int_a^b\int_0^1 p\left(x\right)p\left(y\right)f\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)g\left(th\left(x\right)+\left(1-t\right)h\left(y\right)\right)dtdxdy$$
$$\leq \frac{2}{3}P\int_a^b f\left(h\left(x\right)\right)g\left(h\left(x\right)\right)p(x)dx$$
$$+\frac{P^2}{12}\left[f\left(h(a)\right)+f\left(h(b)\right)\right]\left[g\left(h(a)\right)+g\left(h(b)\right)\right]$$
$$=\frac{2}{3}P\int_a^b f\left(h\left(x\right)\right)g\left(h\left(x\right)\right)p(x)dx$$
$$+\frac{P^2}{12}\left[M^{'}\left(a,b\right)+N^{'}\left(a,b\right)\right].$$

Multiplying both sides of the inequality (2.5) by $\frac{3}{2P^2}$, we get the desired result. This completes the proof of the theorem. □

Theorem 4. *Let f and g be real valued, nonnegative and convex functions on $[a,b]$. Let $h:[a,b]\to\mathbb{R}$ be continuous function such that $h\left([a,b]\right)\subset[a,b]$ and $p:[a,b]\to\mathbb{R}$ be a positive integrable function. Then*

$$\frac{3}{P}\int_a^b\int_0^1 p(x)f\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)g\left(th\left(x\right)+\left(1-t\right)\bar{h}\right) \tag{2.6}$$
$$\leq \frac{1}{P}\int_a^b f(h(x))g(h(x))p(x)dx+\frac{1}{2}\left[M^{'}\left(a,b\right)+N^{'}\left(a,b\right)\right],$$

where
$$M^{'}\left(a,b\right)=f(h(a))g(h(a))+f(h(b))g(h(b))$$
and
$$N^{'}\left(a,b\right)=f(h(a))g(h(b))+f(h(b))g(h(a)).$$

Proof. Again by the convexity of the functions f and g, we have
$$f\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)\leq tf(h(x))+\left(1-t\right)f\left(\bar{h}\right)$$
and
$$g\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)\leq tg(h(x))+\left(1-t\right)g\left(\bar{h}\right)$$

Multiplying the above two inequalities side by side, we get

$$\begin{aligned}(2.7)\quad & f\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)g\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)\\ &\leq t^{2}f(h(x))g(h(x))+t\left(1-t\right)\left[f(h(x))g(\bar{h})+g(h(x))f(\bar{h})\right]\\ &\qquad+\left(1-t\right)^{2}f(\bar{h})g(\bar{h}).\end{aligned}$$

Multiplying both sides of (2.7), by similar arguments as in obtaining (2.4) and using the Jensen's inequality, we have

$$\begin{aligned}(2.8)\quad & \int_{a}^{b}\int_{0}^{1}p(x)f\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)g\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)\\ &\leq\frac{1}{3}\int_{a}^{b}f(h(x))g(h(x))p(x)dx+\frac{1}{6}\int_{a}^{b}\left[f(h(x))g(\bar{h})+g(h(x))f(\bar{h})\right]p(x)dx\\ &\qquad+\frac{1}{3}\int_{a}^{b}f(\bar{h})g(\bar{h})p(x)dx\\ &\leq\frac{1}{3}\int_{a}^{b}f(h(x))g(h(x))p(x)dx+\frac{2}{3P}\left(\int_{a}^{b}f(h(x))p(x)dx\right)\left(\int_{a}^{b}g(h(x))p(x)dx\right).\end{aligned}$$

An application of the inequality (2.1), gives us

$$\begin{aligned}(2.9)\quad & \int_{a}^{b}\int_{0}^{1}p(x)f\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)g\left(th\left(x\right)+\left(1-t\right)\bar{h}\right)\\ &\leq\frac{1}{3}\int_{a}^{b}f(h(x))g(h(x))p(x)dx+\frac{P}{6}\left[f(h(a))+f(h(b))\right]\left[g(h(a))+g(h(b))\right]\\ &=\frac{1}{3}\int_{a}^{b}f(h(x))g(h(x))p(x)dx+\frac{P}{6}\left[M^{'}\left(a,b\right)+N^{'}\left(a,b\right)\right].\end{aligned}$$

Multiplying both sides of (2.9) by $\frac{3}{P}$, we get the desired inequality and hence the proof of the theorem is complete. □

Remark 1. *If in Theorem 3 and Theorem 4, $p(x) = 1$ and $h(x) = x$, $x \in [a, b]$, then $P = b - a$, $\bar{h} = \frac{a+b}{2}$, $M^{'}(a, b) = M(a, b)$ and $N^{'}(a, b) = N(a, b)$. Then the inequalities (2.2) and (2.6) reduce to the inequalities (1.4) and (1.5) respectively. This also shows that our results generalize those results proved in Theorem 2.*

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