Int. J. Anal. Appl. (2024), 22:49

International Journal of Analysis and Applications

On Cesaro-Hypercyclic Operators

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Abstract. In this paper we characterize some properties of the Cesàro-Hypercyclic and mixing operators. At the same time, we also give a Cesàro-Hypercyclicity criterion and offer an example of this criterion.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the scalar field $\mathbb{C}$. As usual, $\mathbb{N}$ is the set of all non-negative integers, $\mathbb{Z}$ is the set of all integers, and $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called hypercyclic if there is some vector $x \in \mathcal{H}$ such that $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where such a vector x is said hypercyclic for T.

The first example of hypercyclic operator was given by Rolewicz in [12]. He proved that if B is a backward shift on the Banach space l^p, then λB is hypercyclic if and only if $|\lambda| > 1$.

Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$. If $\{w_n\}_{n\in\geq 1}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the unilateral backward weighted shift $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ is defined by $Te_n = w_n e_{n-1}, \quad n \geq 1, Te_0 = 0$, and let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $l^2(\mathbb{Z})$. If $\{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the bilateral weighted shift $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by $Te_n = w_n e_{n-1}$.

The definition and the properties of supercyclicity operators were introduced by Hilden and Wallen [10]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic.

A bounded linear operator $T \in B(\mathcal{H})$ is called supercyclic if there is some vector $x \in \mathcal{H}$ such that the projective orbit $\mathbb{C}.Orb(T,x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. Such a vector x is said supercyclic for T. Refer to ([1], [9], [4], [15]) for more informations about hypercyclicity and

Received: Feb. 7, 2024.

2020 *Mathematics Subject Classification.* 47A16, 47A25.

Key words and phrases. hypercyclic; Cesàro-hypercyclic; Cesàro-mixing; Cesàro-hypercyclicity criterion.

https://doi.org/10.28924/2291-8639-22-2024-49

ISSN: 2291-8639

supercyclicity.

A nice criterion namely Hypercyclicity Criterion, was developed independently by Kitai [8] and, Gethner and Shapiro [7]. The Hypercyclicity Criterion has been widely used to show that many different types of operators are hypercyclic. For instance hypercyclic operators arise in the classes of composition operators [3], adjoints of multiplication operators [7], cohyponormal operators [6], and weighted shifts [13].

For the following theorem, see ([1], [9]).

Theorem 1.1. *(Hypercyclicity Criterion). Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets X_0 and Y_0 in $\mathcal{H}$ and an increasing sequence n_j of positive integer such that:*

(1) $T^{n_j}x \rightarrow 0$ *for each* $x \in X_0$, *and*
(2) *there exist mappings* $S_{n_j} : Y_0 \longrightarrow \mathcal{H}$ *such that* $S_{n_j}y \rightarrow 0$, *and* $T^{n_j}S_{n_j}y \rightarrow y$ *for each* $y \in Y_0$,

then T is hypercyclic.

In [13] and [14], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. In [5], N. Feldman gave a characterization of the invertible bilateral weighted shifts that are hypercyclic or supercyclic.

For the following theorem, see [5, Theorem 4.1].

Theorem 1.2. *Suppose that $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$ and either $w_n \geq m > 0$ for all $n < 0$ or $w_n \leq m$ for all $n > 0$. Then:*

(1) *T is hypercyclic if and only if there exists a sequence of integers $n_k \rightarrow \infty$ such that $\lim_{k\rightarrow\infty} \prod_{j=1}^{n_k} w_j = 0$ and $\lim_{k\rightarrow\infty} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0$.*
(2) *T is supercyclic if and only if there exists a sequence of integers $n_k \rightarrow \infty$ such that $\lim_{k\rightarrow\infty}(\prod_{j=1}^{n_k} w_j)(\prod_{j=1}^{n_k} \frac{1}{w_{-j}}) = 0$.*

Let $\mathcal{M}_n(T)$ denote the arithmetic mean of the powers of $T \in B(\mathcal{H})$, that is

$$\mathcal{M}_n(T) = \frac{I + T + T^2 + ... + T^{n-1}}{n}, n \in \mathbb{N}^*.$$

If the arithmetic means of the orbit of x are dense in $\mathcal{H}$ then the operator T is said to be Cesàro-hypercyclic. In [11], Fernando León-Saavedra proved that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq1}$ is dense in $\mathcal{H}$ and characterized the bilateral weighted shifts that are Cesàro-hypercyclic.

For the following proposition, see [11, Proposition 3.4].

Proposition 1.1. *Let $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ be a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$. Then T is Cesàro-hypercyclic if and only if there exists an increasing sequence n_k of positive integers such that for any integer q,*

$$\lim_{k\rightarrow\infty} \prod_{i=1}^{n_k} \frac{w_{i+q}}{n_k} = \infty \text{ and } \lim_{k\rightarrow\infty} \prod_{i=0}^{n_k-1} \frac{w_{q-i}}{n_k} = 0.$$

In this paper we will characterize some properties of the Cesàro-Hypercyclic and mixing operators. Furthermore, we give a Cesàro-Hypercyclicity criterion and offer an example of this criterion.

2. Main results

Suppose $\{n^{-1}T^n : n \geq 1\}$ is a sequence of bounded linear operators on $\mathcal{H}$

Definition 2.1. *An operator $T \in B(\mathcal{H})$ is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense in $\mathcal{H}$*

The following example gives an operator which is Cesàro-hypercyclic but not hypercyclic.

Example 2.1. *[11] Let T the bilateral backward shift with the weight sequence*

$$w_n = \begin{cases} 1 & \text{if } n \leq 0, \\ 2 & \text{if } n \geq 1. \end{cases}$$

Then T is not hypercyclic, but it is Cesàro-hypercyclic.

Now, we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic.

Example 2.2. *Let T the bilateral backward shift with the weight sequence*

$$w_n = \begin{cases} 2 & \text{if } n < 0, \\ \frac{1}{2} & \text{if } n \geq 0. \end{cases}$$

Then T is not Cesàro-hypercyclic, but it is hypercyclic and supercyclic.

Definition 2.2. *We say that $T \in B(\mathcal{H})$ is Cesàro-topologically transitive if for every nonempty open subsets U and V of $\mathcal{H}$ there exists $n \geq 1$ such that $\left(n^{-1}T^n\right)(U) \cap V \neq \emptyset$.*

Definition 2.3. *We say that $T \in B(\mathcal{H})$ is Cesàro-mixing if for every nonempty open subsets U and V of $\mathcal{H}$ there exists $m \geq 1$ such that $\left(n^{-1}T^n\right)(U) \cap V \neq \emptyset, \ \forall n \geq m$.*

The set of Cesàro-hypercyclic vectors for T is denoted by $CH(T)$.

Theorem 2.1. *Let T be a cesàro-hypercyclic operator. Then*

$$CH(T) = \bigcap_{k\geq 1} \bigcup_{n\geq 1} \left(n^{-1}T^n\right)^{-1}(B_k),$$

where $(B_k)_{k\geq 1}$ is a countable open basis for $\mathcal{H}$.

Proof. Let $(B_k)_{k\geq 1}$ is a countable open basis for $\mathcal{H}$. We have $x \in CH(T)$ if and only if $\{n^{-1}T^n x : n \geq 1\}$ is dense in $\mathcal{H}$ if and only if for each $k \geq 1$, there exist $n \geq 1$ such that $n^{-1}T^n x \in B_k$ if and only if $x \in \bigcap_{k\geq 1} \bigcup_{n\geq 1} \left(n^{-1}T^n\right)(B_k)$.

□

Corollary 2.1. *If T is cesàro-topologically transitive operator, then $CH(T)$ is a dense set in $\mathcal{H}$.*

Proof. For every non-empty open U of $\mathcal{H}$ and for all $k \geq 1$, there exist $n \geq 1$ such that the set $\left(n^{-1}T^n\right)^{-1}(U) \bigcap B_k$ is nonempty and open. Hence the set

$$A_k = \bigcup_n \left(n^{-1}T^n\right)^{-1}(B_k)$$

is nonempty and open. Furthermore, $U \cap A_k \neq \emptyset$ for all $k \geq 1$. Thus each A_k is dense in $\mathcal{H}$ and so by the Baire category theorem and theorem 2.1 $CH(T)$ is also dense in $\mathcal{H}$. □

Theorem 2.2. *(Cesàro-Hypercyclicity Criterion). Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets M_0 and M_1 in $\mathcal{H}$ and an increasing sequence (n_j) of positive integer such that:*

(1) $\frac{T^{n_j}}{n_j}x \to 0$ *for each $x \in M_0$, and*

(2) *there exist mappings $S_{n_j} : M_1 \longrightarrow \mathcal{H}$ such that $S_{n_j}y \to 0$, and $\frac{T^{n_j}}{n_j}S_{n_j}y \to y$ for each $y \in M_1$,*

then T is Cesàro-hypercyclic.

Proof. Let U and V be non-empty open subsets of $\mathcal{H}$. By topologically transitive, it is enough to prove that there exist $n \geq 1$ such that

$$(n_k^{-1}T^{n_k})^{-1}(U) \bigcap V \text{ is non-empty.}$$

Since M_0 and M_1 are dense in $\mathcal{H}$, there exist $x \in M_0 \cap V, y \in M_1 \cap U$. And since U and V are nonempty open subsets, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq V$ and $B(y,\varepsilon) \subseteq U$. By assumption, there exist (n_k) such that

$$||n_k^{-1}T^{n_k}x|| \leq \tfrac{\varepsilon}{2}, ||S_{n_k}y|| \leq \tfrac{\varepsilon}{2} \text{ and } ||n_k^{-1}T^{n_k}S_{n_k}y - y|| \leq \tfrac{\varepsilon}{2}.$$

Define $u = x + S_{n_k}y$. We know that $u \in \mathcal{H}$ and $u \in V$, since $||u - x|| = ||S_{n_k}y|| \leq \frac{\varepsilon}{2}$. Since

$$||n_k^{-1}T^{n_k}u - y|| = ||n_k^{-1}T^{n_k}x|| + ||n_k^{-1}T^{n_k}y - y|| < \varepsilon,$$

we have that $n_k^{-1}T^{n_k}u \in U$. Then $(n_k^{-1}T^{n_k})^{-1}(U) \bigcap V \neq \emptyset$ and T is cesàro-hypercyclic. □

Suppose $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be a unilateral weighted shift given by $Te_n = w_n e_{n-1}, \ n \geq 1, Te_0 = 0$. Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$. Define the sequence of linear mappings S_k as $S_k e_j = n_k\left(\prod_{i=1}^{n_k} w_{i+j}\right)^{-1} e_{j+n_k}$.

Example 2.3. *Taking $n_j = n \geq 1$ and suppose $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{w_{j+i}}{n} = \infty$ and $\lim_{n\to\infty} \prod_{i=0}^{n-1} \frac{w_{j-i}}{n} = 0$. Let $M_0 = M_1 = span\{e_j : j \in \mathbb{N}\}$ and $S_n = S^n$, where S_n is the right inverse of $n^{-1}T^n$. So we get*

$$\frac{T^n}{n}e_j = \prod_{i=0}^{n-1} \frac{w_{j-i}}{n} e_{j-n} \to 0 \ \ forall j \in \mathbb{N}.$$

Furthermore, we have

$$S_n e_j = S^n e_j = \frac{n}{\prod_{i=1}^{n} w_{j+i}} \to 0,$$

and

$$\|\frac{T^n}{n}S_ne_j - e_j\| = \|\frac{T^n}{n}.\frac{n}{\prod_{i=1}^{n} w_{j+i}}e_{j+n} - e_j\| \to 0.$$

Hence $\frac{T^n}{n}S_ne_j \to e_j$ *for all* $j \in \mathbb{N}$. *Thus* T *satisfies the Cesàro-Hypercyclicity Criterion with respect to* $(n_j) = (n)$.

Theorem 2.3. *Let* $T \in B(\mathcal{H})$. *Then the following* (1) *and* (2) *are equivalent:*

(1) *T satisfies Cesàro-Hypercyclic Criterion.*

(2) *(Outer Cesàro-Hypercyclic Criterion) There exist an increasing sequence* (n_k) *of positive integer , a dense linear subspace* $Y_0 \subseteq \mathcal{H}$ *and, for each* $y \in Y_0$, *a dense linear subspace* X_0 *of* $\mathcal{H}$ *such that:*

 (a) *There exists a sequence of mappings* $S_{n_k} : Y_0 \to \mathcal{H}$, $k \in \mathbb{N}^*$ *such that* $(n_k^{-1}T^{n_k} \circ S_{n_k})y \to y$, *For each* $y \in Y_0$ *and*

 (b) $\|n_k^{-1}T^{n_k}x\| \, \|S_{n_k}y\| \to 0$ *For each* $y \in Y_0$ *and* $x \in X_0$,

Proof. It is obvious that any operator satisfying the Cesàro-Hypercyclic Criterion also satisfie the criteria of (2). It suffices to show that (2) implies (1). Let $U_i, V_i \subseteq \mathcal{H}$ non-empty open sets with $i = 1, 2$. The same argument as in the proof of [2, Theorem 3.2] and [4, Theorem 2.5] can be used to show that there exist (n_k) of positive integer such that

$$\left(n_k^{-1}T^{n_k}\right)^{-1}(U_i) \bigcap V_i \neq \emptyset, \text{ for } i = 1, 2.$$

Then we can know that $(T \oplus T)$ is cesàro-hypercyclic for $\mathcal{H} \oplus \mathcal{H}$ and (x, y) is cesàro-hypercyclic vector for $(T \oplus T)$. In particular, x is cesàro-hypercyclic vector for T and $CH(T)$ is a dense G_δ of $\mathcal{H}$. Let (U_k) be a base of 0-neighborhoods in $\mathcal{H}$. Then there exist (n_k) of positive integer such that

$$n_k^{-1}T^{n_k}x \in U_k \text{ and } n_k^{-1}T^{n_k}y \in x + U_k \text{ for all } k \geq 1.$$

This implies that $n_k^{-1}T^{n_k}x \to 0$ and $n_k^{-1}T^{n_k}y \to x$. Let $M_0 = M_1 = Orb(T, x)$, which is dense in $\mathcal{H}$. Also for all $k \geq 1$ define

$$S_{n_k}(n^{-1}T^nx) = n^{-1}T^ny.$$

Note that

$$n_k^{-1}T^{n_k}S_{n_k}(n^{-1}T^nx) = n_k^{-1}T^{n_k}(n^{-1}T^ny) = n^{-1}T^n\left(n_k^{-1}T^{n_k}y\right) \to n^{-1}T^nx.$$

Hence (1) holds. We complete the proof. □

Proposition 2.1. *Let* $T \in B(\mathcal{H})$. *The following statements are equivalent.*

(1) $T \in CH(\mathcal{H})$.

(2) *T is cesàro-topologically transitive.*

(3) *For each* $x, y \in \mathcal{H}$, *there exist sequences* (x_k) *in* $\mathcal{H}$, (n_k) *in* $\mathbb{N}^*$, *such that* $x_k \to x$ *and* $n_k^{-1}T^{n_k}x_k \to y$.

(4) *For each* $x, y \in \mathcal{H}$, *and each neighborhood W of the zero in* $\mathcal{H}$, *there exist* $z \in \mathcal{H}$, $n \geq 1$ *such that* $x - z \in W$ *and* $n^{-1}T^nz - y \in W$.

Proof.

1 ⇔ 2: By Theorem 2.1 and Corollairy 2.1.

2 ⇒ 3: Let $x, y \in \mathcal{H}$, and let $B(x, \frac{1}{k})$, $B(y, \frac{1}{k})$ for all $k \geq 1$. Then, there exist (n_k) in $\mathbb{N}^*$ and (x_k) in $\mathcal{H}$ such that $x_k \in B(x, \frac{1}{k})$ and $n_k^{-1}T^{n_k}x_k \in B(y, \frac{1}{k})$ for all $k \geq 1$. Then $\|x_k - x\| < \frac{1}{k}$ and $\|n_k^{-1}T^{n_k}x_k - y\| < \frac{1}{k}$ for all $k \geq 1$.

3 ⇒ 4: Follows immediately from part (3).

4 ⇒ 2: Let U and V be two non-empty open subset of $\mathcal{H}$. Let W be a neighborhood for zero, pick $x \in U$ and $y \in V$, so there exist $z \in \mathcal{H}$, $n \geq 1$ such that $x - z \in W$ and $n^{-1}T^n z - y \in W$. It follows immediately that $z \in U$ and $n^{-1}T^n z \in V$.

□

Definition 2.4. *Let $T \in B(\mathcal{H})$. For every $x_0 \in \mathcal{H}$ and $n \geq 1$ the sets*

$$L^{mix}(x_0) := \{x_1 \in \mathcal{H} : \; n^{-1}T^n x_0 \to x_1 \}$$

$J^{mix}(x_0) = \{x_1 \in \mathcal{H}$: *for every neighborhood V_0, V_1 of x_0, x_1 respectively, there exists $m \geq 1$ such that $\left(n^{-1}T^n\right)(V_0) \cap V_1 \neq \emptyset$ for every $n \geq m\}$*

will be called the cesàro-mixing limit set of x_0 under T and cesàro-mixing extended limit set of x_0 under T respectively.

Proposition 2.2. *An equivalent definition for the set $J^{mix}(x_0)$ is the following:*

$$J^{mix}(x_0) = \{x_1 \in \mathcal{H} : \textit{there exists a sequence } (x_n)_{n\geq 1} \textit{ in } \mathcal{H} \textit{ such that } x_n \to x_0 \textit{ and } n^{-1}T^n x_0 \to x_1\}$$

Proof. Let us prove that
$J^{mix}(x_0) \subset \{x_1 \in \mathcal{H}$: there exists a sequence $(x_n)_{n\geq 1}$ in $\mathcal{H}$ such that $x_n \to x_0$ and $n^{-1}T^n x_0 \to x_1\}$
Let $x_1 \in J^{mix}(x_0)$ and consider the open balls
$V_0 = B(x_0, \frac{1}{n}), V_1 = B(x_1, \frac{1}{n})$ centered at $x_0, x_1 \in \mathcal{H}$ and with radius $1/n$ for $n \geq 1$. Then there exists $m \geq 1$ so that $\left(n^{-1}T^n\right)(V_0) \cap V_1 \neq \emptyset$ for every $n \geq m$. Hence there exists $x_n \in V_0 = B(x_0, \frac{1}{n})$ such that $n^{-1}T^n(x_n) \in V_1$. Therefore there exists a sequence (x_n) in $\mathcal{H}$ such that $x_n \to x_0$ and $Tx_n \to x_1$. The converse is obvious. □

Proposition 2.3. *Let $T \in B(\mathcal{H})$. For every $x_0 \in \mathcal{H}$, $J^{mix}(x_0) = X$. Then T is cesàro-mixing.*

Proof. Let V_0, V_1 the nonempty open. Consider $x_1 \in V_1$. Since $J^{mix}(x_0) = X$. There exists $m \geq 1$ such that $\left(n^{-1}T^n\right)(V_0) \cap V_1 \neq \emptyset$ for every $n \geq m$. By definition T is cesàro-mixing.

□

Theorem 2.4. *Let $S \in B(\mathcal{H})$ and $S^n = n^{-1}T^n$. If S is power bounded then $J^{mix}(x_0) = L^{mix}(x_0)$ For every $x_0 \in \mathcal{H}$.*

Proof. Since S is power bounded, there exists a positive number M such that $\|S^n\| \leq M$ for every positive integer $n \geq 1$. Let $x_0 \in \mathcal{H}$. If $J^{mix}(x_0) = \emptyset$ there is nothing to prove. Therefore assume that

$J^{mix}(x_0) \neq \emptyset$. Since the inclusion $L^{mix}(x_0) \subset J^{mix}(x_0)$ is always true, it suffices to show that $J^{mix}(x_0) \subset L^{mix}(x_0)$. Take $x_1 \in J^{mix}(x_0)$. There exist a sequence (x_n) in $\mathcal{H}$ such that $x_n \to x_0$ and $S^n x_n \to x_1$. Then we have

$$\begin{aligned} \|S^n x_0 - x_1\| &\leq \|S^n x_0 - S^n x_n\| + \|S^n x_n - x_1\| \\ &\leq M\|x_0 - x_n\| + \|S^n x_n - x_1\| \end{aligned}$$

and letting n goes to infinity to the above inequality, we get that $x_1 \in L^{mix}(x_0)$.

□

Theorem 2.5. *Let T and S in $B(\mathcal{H})$ and $T \oplus S$ is cesàro-mixing operator, then T and S are cesàro-mixing operators, respectively.*

Proof. let U_1, U_2 ,V_1 and V_2 be open sets in $\mathcal{H}$, then $U_1 \oplus V_1$ and $U_2 \oplus V_2$ are open in $\mathcal{H} \oplus \mathcal{H}$. So there exists an $n_0 \geq 1$ such that

$$\left(n^{-1}\left(T \oplus S\right)\right)^{-1}(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \emptyset.$$

Then

$$\left(n^{-1}T^n\right)^{-1}(U_1) \bigcap U_2 \neq \emptyset, \ \left(n^{-1}S^n\right)^{-1}(V_1) \bigcap V_2 \neq \emptyset.$$

Therefore, T and S are cesàro-mixing operators, respectively. □

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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