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On the Analysis of Environmental and Engineering Data Using Alpha Power Transformed Cosine Moment Exponential Model

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Abstract. This article introduces a new model using the alpha power cosine transformed method for modeling complex data used in hydrology and engineering studies. The alpha power novel distribution transformed the cosine moment exponential model with two parameters. Its probability density function can be skewed and unimodal. Various statistical and mathematical properties are established, and the unknown parameters of the suggested model are determined using numerous estimation procedures. Also, the potential of these estimation techniques is calculated via some simulation studies. In the end, two real data sets are made using the proposed model to make a practical application in environmental and survival fields. The potential and utility of the recommended distribution are verified with other well known models and it shows great superiority in fitting the proposed data sets.

1. Introduction

The transformation of classical model and the proposal of a novel version of the existing probability distributions are famed and motivating research topics in the literature. Further, the approaches of the probability models using different techniques, including trigonometric, power transformed, and compounding methods, have received great attention in the last few decades. These new extensions of models represent more efficiency in fitting and modeling data in many applied sciences areas, particularly hydrology, engineering, survival analysis, finance, economics, and medical sciences. In this context, different techniques for obtaining a novel family of models have been provided, for example, Hamedani et al. [12], Eugene et al. [10], Marshall and Olkin [16], Cordeiro and Castro [7], Almetwally et al. [20], BuHamra et al. [6], and Alizadeh et al. [1].

The probability distribution functions (PDFs) defined on $\mathbb{R}^+$ are extensively implemented to fit the period until a particular event or phenomenon occurs. The Moment exponential (ME)

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distribution has a special place provided by Dara and Ahmad [24]. The ME model is one of the first selections of authors to implement for modeling data in the expiry times of patients, time to performance failure, and recovery time after health injury see El Gazar et al. [8], Almetwally et al. [3], Abonongo et al. [22], Iqbal et al. [32], Zafar Iqbal et al. [31], and Salem et al. [27].

Let X has the ME distribution denoted by $X \sim ME(\theta)$, with cumulative distribution function (CDF) and PDF can be obtained to be

$$H(x;\theta) = 1 - e^{-\frac{x}{\theta}}\left(\frac{x}{\theta}+1\right),\ x,\ \theta > 0, \tag{1.1}$$

and

$$h(x;\theta) = \frac{xe^{-\frac{x}{\theta}}}{\theta^2}. \tag{1.2}$$

Newly, Mahdavi and Kundu [15] proposed a novel extension family of probability distributions, which are more efficient for exploring more data sets. These ewe family of distributions referred to alpha power transformation (APT) family with CDF and PDF are defined, respectively, as

$$F(x) = \frac{\lambda^{K(x)} - 1}{\lambda - 1}, \quad x \in \mathbb{R}, \lambda > 0;\ \lambda \neq 1, \tag{1.3}$$

and

$$f(x) = \frac{\log \lambda}{\lambda - 1}\,\lambda^{K(x)}\,k(x), \tag{1.4}$$

where $K(x)$ and $k(x)$ represent the CDF and PDF of the baseline model. It is well documented that different authors utilize the APT family of distributions to generate some exciting models. In this way, Eissa and Sonar [11] defined the APT Extended power Lindley (APT-EPL) model by taking the extended power Lindley baseline distribution, and they derived various distributional properties. Hassan et al. [13] proposed APT Power Lindley (APT-PL) distribution. Also, the APT Extended Exponential (APT-EE) distribution is studied by Hassan et al. [14] and they proved that the new model is better than some other well-known distributions for modeling different kinds of data sets. In the same way, Shivanshi et al. [29] provided the APT Xgamma (APT-XG) distribution and applied the suggested model to the reliability, survival, and environmental data sets. Sin extension of the exponential distribution has been introduced by [19]. Reyad et al. [21] introduced the APT Dagum (APT-D) model and established the various characterizations of the recommended distributions. Abonongo et al. [22] discussed cosine Fréchet loss distribution with actuarial measures and insurance applications.

Many researchers attempted to analyze and explain different data by employing generalized structures of the ME distribution. However, the results were not credible. To overcome the issue and based on the trigonometric function with the APT family, this article offers two main objectives: firstly, it introduced a new version of ME distribution that can be applied in various applications, such as fitting the environmental and engineering data sets. We referred to this novel suggested model as the alpha power transformed cosine moment exponential (APCos-ME) model with two parameters. The APCos-ME distribution can be skewed and unimodal. Secondly, the model parameters of APCos-ME have been estimated using various estimation procedures.

Suppose T follows the APTCos-T family. The corresponding CDF and PDF of T are expressed by

$$F(t;\lambda,\eta)=\frac{\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi H(t;\eta)}{2}\right)}-1}{\lambda-1},\ t,\ \lambda,\ \eta>0,\ \lambda\neq 1, \tag{1.5}$$

and

$$f(t;\lambda,\eta)=\frac{\pi\log\lambda\ h(t;\eta)\ \sin\left(\frac{\pi}{2}-\frac{\pi H(t;\eta)}{2}\right)}{2(\lambda-1)}\ \lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi H(t;\eta)}{2}\right)}. \tag{1.6}$$

The article is structured and arranged in the following ways. Section 2 introduced the new version of ME distribution and its corresponding reliability measures. Numerous statistical properties of the proposed model are established in Section 3. Section 4 contains the estimation of the unknown parameters by applying various procedures. Section 5 considers a Monte Carlo simulation study to conduct the comparison and consistency properties of different proposed estimation methods. Finally, in Section 6, two real data sets representing environmental and reliability areas are illustrated for validation. In the last section, closing remarks are devoted.

2. Model formulation

In this part of the work, we establish numerous distributional properties of the APTCos-ME model, likely CDF, PDF, and some reliability functions. Let T be a random variable following the APT-Cos-ME with parameters λ and θ denoted by $T\sim$ APTCos-ME(λ,θ). According to Eq. (1.6) and (1.2), the CDF and PDF of the APTCos-ME model are, respectively, given by

$$G(t)=\frac{\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}-1}{\lambda-1},\ t,\ \lambda,\ \theta>0,\ \lambda\neq 1, \tag{2.1}$$

and

$$g(t)=\frac{1}{2(\lambda-1)}\left\{\pi\log(\lambda)\frac{te^{-\frac{t}{\theta}}}{\theta^2}\sin\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}\right\}. \tag{2.2}$$

From Eq (2.1), it can be deduced that the proposed APCos-ME model reduces to ME distribution if λ tends to be 1. Figure 1 depicts the PDF curves of the APCos-ME model. The PDF can take numerous shapes; it is right-skewed, left-skewed, and always unimodal.

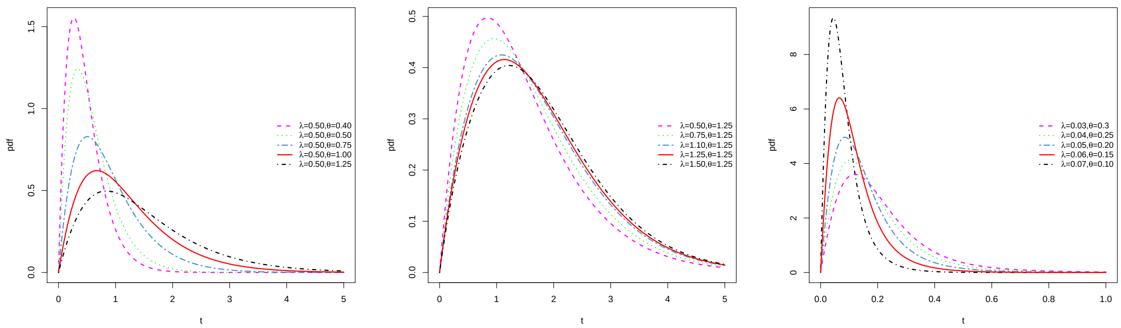

Figure 1. Density Plots for APCos-ME distribution under different selected parameter values.

2.1. **Reliability Measures.** Suppose $T \sim$ APTCos-ME(λ, θ). The survival (SF) and hazard rate (HRF) functions of T are written as

$$S(t) = \frac{\lambda - \lambda^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}}{\lambda - 1}, \tag{2.3}$$

and

$$\begin{aligned} h(t) &= 2^{-1}\left[\lambda - \lambda^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}\right]^{-1}\left(\pi\log(\lambda)\frac{te^{-\frac{t}{\theta}}}{\theta^2}\sin\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)\right) \\ &\times \lambda^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}. \end{aligned} \tag{2.4}$$

The APCos-ME's cumulative hazard rate function (CHRF) is defined by

$$H(x) = = -\log\left\{\frac{\lambda - \lambda^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}}{\lambda - 1}\right\}. \tag{2.5}$$

The HRF plots of the proposed APCos-ME model are plotted in Figure 2. From this Figure, it can be seen that the hazard function of the suggested model is increasing and reverse-J curves.

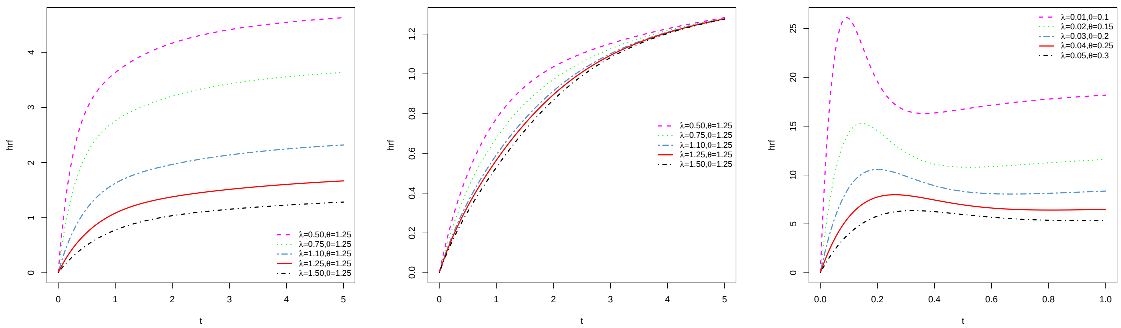

Figure 2. Hazard Rate Plots for APCos-ME distribution under different selected parameter values.

3. Mathematical properties

Here, in this Section, we provide different significant statistical features of the APCos-ME model, including quantile function, k-moments, moment generating function (MGF), Coefficient of variation (CV), Lorenz, and Bonferroni and order statistics distribution.

3.1. **Quantile Function of APCos-ME model.** Let $T \sim$ APTCos-ME(λ, θ). Based on inverting Eq (2.1), the quantile function is expressed as

$$p = \frac{\lambda^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)} - 1}{\lambda - 1}, \tag{3.1}$$

for $0 \leq p \leq 1$, that is,

$$Q(p) = -\theta\left\{1 + W\left(-\frac{2e^{-1}}{\pi}\cos^{-1}\left[\frac{1+p(\lambda-1)}{\log(\lambda)}\right]\right)\right\}, \tag{3.2}$$

where $W(.)$ represents the Lambert function.
Consequently, for generating random numbers from the APCos-ME model, Eq (3.2) can be used.

3.2. **Moments with Related Connects.** The moments of density function are of importance in statistical analysis. They help us to find the mean, variance (Var), skewness (Skw), kurtosis (Kurt), and shape of any given data set. The k^{th} moment of the APCos-ME is presented by

$$\mu_k = \frac{\pi}{2\theta^2(\lambda-1)} \sum_{i=1}^{\infty} \frac{(\log \lambda)^{i+1}}{i!}\left[\Phi_{i,k}(t) - \Phi_{i+1,k}(t)\right], \tag{3.3}$$

where,

$$\Phi_{i,k}(t) = \int_0^{\infty} t^{k+1} e^{-t/\theta} \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\left[1 - e^{-t/\theta}(1+\frac{t}{\theta})\right]\right)^i dt,$$

and

$$\Phi_{i+1,k}(t) = \int_0^{\infty} t^{k+1}e^{-t/\theta} \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\left[1 - e^{-t/\theta}(1+\frac{t}{\theta})\right]\right)^{i+1} dt.$$

Consequently, the mean (μ_1), Var, and CV of T can be expressed as

$$\mu_1 = \frac{\pi}{2\theta^2(\lambda-1)} \sum_{i=1}^{\infty} \frac{(\log\lambda)^{i+1}}{i!}\left[\Phi_{i,1}(t) - \Phi_{i+1,1}(t)\right], \tag{3.4}$$

$$\text{Var} = \mu_2 - \mu_1^2,$$

and

$$\text{CV} = \frac{\sqrt{\mu_2 - \mu_1^2}}{\mu_1}.$$

The coefficients of Skw and Kurt measures of the proposed APCos-ME model are given as

$$Skw = \frac{\mu_3 - 3\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}},$$

and

$$Kurt = \frac{\mu_4 - 4\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}.$$

Table (1) summarizes numerous proposed statistical measures of the APCos-ME by applying varied parameter selections λ and θ. Clearly from Table (1) as θ is growing, the μ_1 and Var of the APCos-ME model are diminishing, whereas the CV, Skew, and Kurt are fixed, which ensures that these values are free parameters of θ. Another remark from Table (1) is that both CV, Skw, and Kurt amounts are diminishing as λ is growing. Hence, the APCos-ME is a flexible distribution for explaining more complex data. All these conclusions are confirmed in Figure (3).

The k^{th} incomplete moments of the APCos-ME is

$$\begin{aligned} \varphi_k(x) &= \frac{\pi}{2\theta^2(\lambda-1)} \sum_{j=1}^{\infty} \frac{(\log\lambda)^{j+1}}{j!}\left\{\int_0^{x} t^{k+1}e^{-t/\theta} \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\left[1 - e^{-t/\theta}\left(1+\frac{t}{\theta}\right)\right]\right)^{j} dt\right. \\ &- \left.\int_0^{x} t^{k+1}e^{-t/\theta} \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\left[1 - e^{-t/\theta}\left(1+\frac{t}{\theta}\right)\right]\right)^{j+1} dt\right\} \\ &= \frac{\pi}{2\theta^2(\lambda-1)} \sum_{j=1}^{\infty} \frac{(\log\lambda)^{j+1}}{j!}\left[\Psi_{j,k}(x) - \Psi_{j+1,k}(x)\right]. \end{aligned} \tag{3.5}$$

Finally, the Bonferroni and Lorenz curves of T are defined by

$$B(p) = \frac{1}{p\mu}\int_0^{t_p} tg(t)dt = \frac{1}{p\mu}\varphi_1(t_p),\ G(t_p) = p,$$

and

$$L(p) = \frac{1}{\mu}\int_0^{t_p} tg(t)dt = \frac{1}{\mu}\varphi_1(t_p).$$

Now, the MGF of the APCos-ME model is given by

$$M(x) = E(e^{xt}) = \frac{\pi}{2\theta^2(\lambda-1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^j(\log\lambda)^{i+1}}{i!j!}\left[\Phi_{i,k}(t) - \Phi_{i+1,k}(t)\right]. \tag{3.6}$$

3.3. **Order Statistics of APCos-ME model.** Let $T \sim$ APCos-ME(λ, θ) and $t_{(1)} < \ldots < t_{(n)}$ represent the order statistics of the random sample from T. Then the r^{th} PDF of T is written as

Table 1. Different statistical properties of APCos-ME with various parameter values.

	θ	μ_1	Var	CV	Skw	$Kurt$
λ=0.25	0.5	0.526	0.1380	0.7063	1.4990	3.3105
	1.0	1.052	0.5521	0.7063	1.4990	3.3105
	1.5	1.578	1.2422	0.7063	1.4990	3.3105
	2.0	2.104	2.2084	0.7063	1.4990	3.3105
λ=0.5	0.5	0.5975	0.1624	0.6744	1.3173	2.4816
	1.0	1.1950	0.6495	0.6744	1.3173	2.4816
	1.5	1.7925	1.4614	0.6744	1.3173	2.4816
	2.0	2.3900	2.5981	0.6744	1.3173	2.4816
λ=0.75	0.5	0.6425	0.1757	0.6524	1.2159	2.0923
	1.0	1.2850	0.7029	0.6524	1.2159	2.0923
	1.5	1.9276	1.5814	0.6524	1.2159	2.0923
	2.0	2.5701	2.8114	0.6524	1.2159	2.0923
λ=1.25	0.5	0.7015	0.1907	0.6224	1.0976	1.7014
	1.0	1.4031	0.7626	0.6224	1.0976	1.7014
	1.5	2.1046	1.7159	0.6224	1.0976	1.7014
	2.0	2.8061	3.0505	0.6224	1.0976	1.7014
λ=1.5	0.5	0.7230	0.1954	0.6113	1.0586	1.5874
	1.0	1.446	0.7814	0.6113	1.0586	1.5874
	1.5	2.1690	1.7582	0.6113	1.0586	1.5874
	2.0	2.8919	3.1256	0.6113	1.0586	1.5874

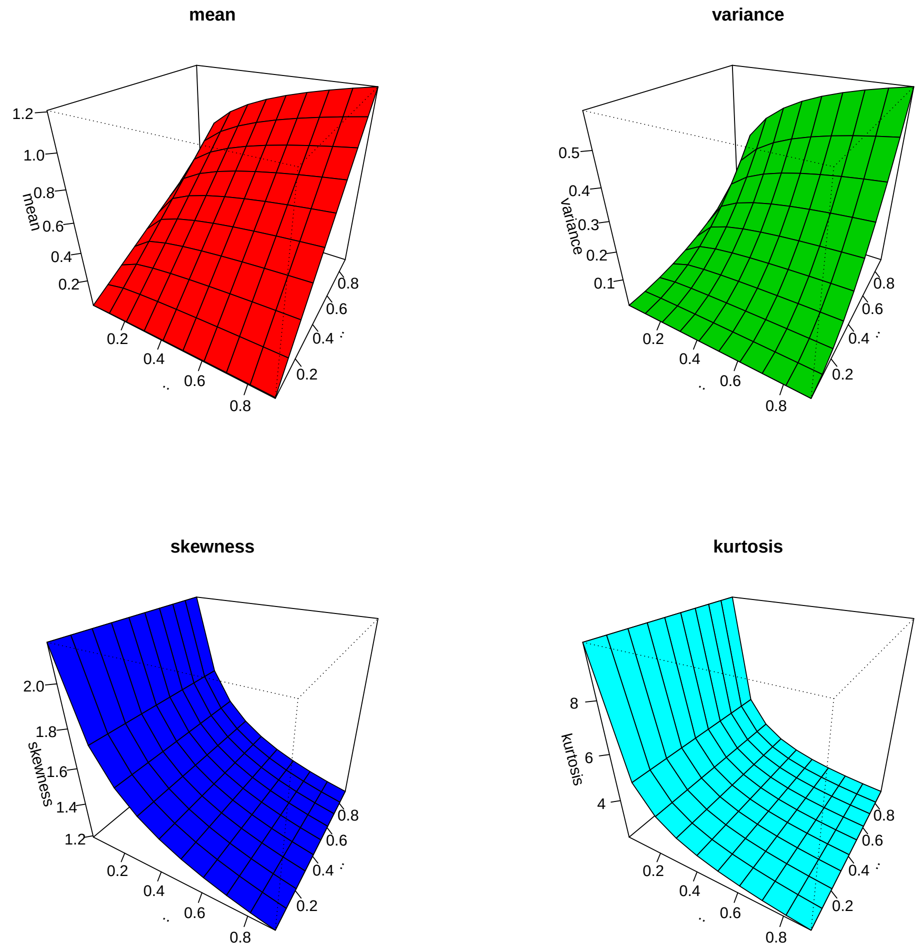

Figure 3. 3D plots of the recommended mathematical measures of the APCos-ME distribution using different parameter values of λ and θ.

$$
\begin{aligned}
g_{r:n}(t) &= \frac{n!g(t)}{(r-1)!(n-r)!}[G(t)]^{r-1}[1-G(t)]^{n-r} \\
&= \frac{n!g(t)}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}[G(t)]^{l+r-1} \\
&= \frac{n!}{2(r-1)!(n-r)!}\sum_{l=0}^{n-r}\frac{(-1)^l}{(\lambda-1)^{l+r-2}}\binom{n-r}{l}\left[\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}-1\right]^{l+r-1} \\
&\quad\times\pi\log(\lambda)\frac{te^{-\frac{t}{\theta}}}{\theta^2}\sin\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}.
\end{aligned}
$$

Respectively, we can be defined the first and latest order statistics of the random variable T as $g_{1:n}(t) = \min\{T_1, T_2, ..., T_n\}$ and $g_{n:n}(t) = \max\{T_1, T_2, ..., T_n\}$. Its pdf are given by

$$
g_{1:n}(t) = \frac{n}{2(\lambda-1)^{n-2}}\pi\log(\lambda)\frac{te^{-\frac{t}{\theta}}}{\theta^2}\sin\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)\lambda^{n\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)},
$$

and

$$
\begin{aligned}
g_{n:n}(t) &= \frac{n}{2(\lambda-1)^{n-2}}\pi\log(\lambda)\frac{te^{-\frac{t}{\theta}}}{\theta^2}\sin\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)} \\
&\times\left[\lambda^{\cos\left(\frac{\pi}{2}-\frac{\pi\left(1-e^{-\frac{t}{\theta}}\left(\frac{t}{\theta}+1\right)\right)}{2}\right)}-1\right]^{n-1}
\end{aligned}
$$

4. Estimation methods of APCos-ME model

Here, various estimation techniques for determining the APCos-ME's parameters are covered. For additional information concerning the application of estimation techniques, see Alshawarbeh et al. [4], Rahman et al. [18], Rodrigues et al. [23], Shama et al. [28], and Almetwally and Meraou [2].

4.1. Maximum Likelihood Estimation.

Let $\{t_1, \ldots, t_n\}$ be observed random sample (RS) taken from APCos-ME(λ, θ). The corresponding log-likelihood function may be expressed as

$$\begin{aligned} l(t, \Psi) &= \sum_{i=1}^{n} \log g(t, \Psi) \\ &- n\log(\lambda - 1) - 2n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} t_i + \log\lambda \sum_{i=1}^{n} \cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t_i}{\theta}}\left(\frac{t_i}{\theta} + 1\right)\right)}{2}\right) \\ &+ \sum_{i=1}^{n} \log\sin\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t_i}{\theta}}\left(\frac{t_i}{\theta} + 1\right)\right)}{2}\right), \end{aligned} \tag{4.1}$$

with $\Psi = (\lambda, \theta)$. Suppose $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$ are the MLEs of λ and θ. They are obtained, respectively, by solving the two non-linear equations

$$\frac{\partial l(t, \Psi)}{\partial \lambda} = -\frac{n}{\lambda - 1} + \frac{1}{\lambda}\sum_{i=1}^{n} \cos\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t_i}{\theta}}\left(\frac{t_i}{\theta} + 1\right)\right)}{2}\right) = 0,$$

and

$$\begin{aligned} \frac{\partial l(t, \Psi)}{\partial \theta} &= -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} t_i - \frac{\log\lambda\pi}{2}\sum_{i=1}^{n}\left[\frac{t_i}{\theta^2}e^{-\frac{t_i}{\theta}}(1 + \frac{t_i}{\theta}) - e^{-\frac{t_i}{\theta}}\frac{t_i}{\theta^2}\right]\sin\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t_i}{\theta}}\left(\frac{t_i}{\theta} + 1\right)\right)}{2}\right) \\ &+ \frac{\log\lambda\pi}{2}\sum_{i=1}^{n}\left[\frac{t_i}{\theta^2}e^{-\frac{t_i}{\theta}}(1 + \frac{t_i}{\theta}) - e^{-\frac{t_i}{\theta}}\frac{t_i}{\theta^2}\right]\cot\left(\frac{\pi}{2} - \frac{\pi\left(1 - e^{-\frac{t_i}{\theta}}\left(\frac{t_i}{\theta} + 1\right)\right)}{2}\right) = 0. \end{aligned}$$

4.2. Least Square and Weighted Least Square Estimators.

Let $t_1, ..., t_n$ be an observed RS taking from the APCos-ME model. The ordinary least square estimator(OLS) of λ and θ (note that, $\hat{\lambda}_{OLS}$ and $\hat{\theta}_{OLS}$) are resulted with minimize the function

$$\sum_{i=1}^{n}\left[G(t_{(i)}|\Psi) - \frac{i}{n+1}\right]^2,$$

where $G(t|\Psi)$ is (2.1). As a result, the estimate of λ and θ based on LSE can be obtained by resolving the non-linear equations

$$\sum_{i=1}^{n}\left[G(t_{(i)}|\Psi) - \frac{i}{n+1}\right]\Theta_1(t_{(i)}|\Psi) = 0,$$

and

$$\sum_{i=1}^{n}\left[G(t_{(i)}|\Psi)-\frac{i}{n+1}\right]\Theta_2(t_{(i)}|\Psi)=0,$$

where

$$\Theta_1(t_{(i)}|\Psi)=\frac{\partial}{\partial\lambda}G(t_{(i)}|\Psi), \tag{4.2}$$

and

$$\Theta_2(t_{(i)}|\Psi)=\frac{\partial}{\partial\theta}G(t_{(i)}|\Psi). \tag{4.3}$$

Further, the ordinary weighted least square estimators (OWLS) of λ and θ, note $\hat{\lambda}_{PWLS}$ and $\hat{\theta}_{OWLS}$ are defined with minimize the function

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(t_{(i)}|\Psi)-\frac{i}{n+1}\right]^2.$$

Consequently, $\hat{\lambda}_{OWLS}$ and $\hat{\theta}_{OWLS}$ can be obtained as solution of

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(t_{(i)}|\Psi)-\frac{i}{n+1}\right]\Theta_1(t_{(i)}|\Psi)=0,$$

and

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(t_{(i)}|\Psi)-\frac{i}{n+1}\right]\Theta_2(t_{(i)}|\Psi)=0.$$

4.3. **Maximum product of Spacings.** The maximum product spacing (MPS) can be described as follows. Let

$$\mathcal{MP}_i(\Psi)=G(t_{(i)}|\Psi)-G(t_{(i-1)}|\Psi);\;\; i=1,...,n+1,$$

with

$$G(t_{(0)}|\Psi)=0,\;\text{ and }\; G(t_{(n+1)}|\Psi)=1.$$

Evidently, $\sum_{i=1}^{n+1}\mathcal{MP}_i(\Psi)=1$.

The MPS estimators of λ, and θ ($\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$), can be obtained by maximizing

$$\mathcal{P}(\Psi)=\left[\prod_{i=1}^{n+1}\mathcal{MP}_i(\Psi)\right]^{\frac{1}{n+1}}. \tag{4.4}$$

Also, they result by maximizing

$$\mathcal{R}(\Psi)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\mathcal{MP}_i(\Psi). \tag{4.5}$$

The estimates $\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$ are obtained by solving the non-linear equations

$$\frac{\partial\mathcal{R}(\Psi)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\mathcal{MP}_i(\psi)}\left\{\Theta_1(t_{(i)}|\Psi)-\Theta_1(t_{(i-1)}|\Psi)\right\}=0,$$

and

$$\frac{\partial\mathcal{R}(\Psi)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\mathcal{MP}_i(\psi)}\left\{\Theta_2(t_{(i)}|\Psi)-\Theta_2(t_{(i-1)}|\Psi)\right\}=0.$$

with $\Theta_i(.|\Psi)$ for $i=1,2$ are given in (4.2)-(4.3).

4.4. **Cramer-Von Mises Minimum Distance Estimators.** The Cramer-von Mises-type minimum distance estimates (CVMs) $\hat{\lambda}_{CVM}$ and $\hat{\theta}_{CVM}$ are obtained by minimizing

$$\mathcal{CV}(\Psi) = \frac{1}{12n} + \sum_{i=1}^{n}\left[G(t_{(i)}|\Psi) - \frac{2i-1}{2n}\right]^2. \tag{4.6}$$

These estimates can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[G(x_{(i)}|\Psi) - \frac{2i-1}{2n}\right]\Theta_1(t_{(i)}|\Psi) = 0,$$

and

$$\sum_{i=1}^{n}\left[G(x_{(i)}|\Psi) - \frac{2i-1}{2n}\right]\Theta_2(t_{(i)}|\Psi) = 0.$$

4.5. **Anderson-Darling Estimators.** Anderson-Darling estimators (ADs) $\hat{\lambda}_{AD}$, and $\hat{\theta}_{AD}$ of λ and θ are calculated by minimizing

$$\mathcal{AD}(\Psi) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\ln G(t_{(i)}|\Psi) + \ln S(t_{(n+1)}|\Psi)\right\}.$$

$\hat{\lambda}_{AD}$, and $\hat{\theta}_{AD}$ are calculated by resolving

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Theta_1(t_{(i)}|\Psi)}{G(t_{(i)}|\Psi)} - \frac{\Theta_1(t_{(n+1)}|\Psi)}{S(t_{(n+1)}|\Psi)}\right] = 0,$$

and

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Theta_2(t_{(i)}|\Psi)}{G(t_{(i)}|\Psi)} - \frac{\Theta_2(t_{(n+1)}|\Psi)}{S(t_{(n+1)}|\Psi)}\right] = 0.$$

5. Numerical simulation

Here, we provide some results from a Monte Carlo (MC) simulation study to see how the suggested estimating technique working in the practice. Under selected values of λ and θ and over on $N = 1000$ times, We generate an observed sample from the APCos-ME model of size n using the quantile function (3.2) and we calculate the average estimates (AEs), average biases (ABs), and the associated mean squared errors (MSEs). The results are reported in Tables (2)-(4) represent the result. Tables (2)-(4) show that as the sample size increases, the AEs, ABs and MSEs decrease based on all estimation methods. This guarantees the consistency and asymptotic properties of all techniques. Further the MPSEs procedure can be considered best technique of estimate for the APCos-ME model since it has a smaller MSE among other techniques.

Table 2. AEs, ABs, and MSEs of (λ, θ)=(0.75, 0.25) considering different sample sizes.

n	Method	$\hat{\lambda}$			$\hat{\theta}$		
		AE	AB	MSE	AE	AB	MSE
300	MLE	0.9366	0.1866	0.2433	0.2656	0.0156	0.0042
	OLS	0.8870	0.1370	0.4396	0.2647	0.0147	0.0052
	OWLS	0.8019	0.0519	0.3359	0.2608	0.0108	0.0049
	MPS	0.7790	0.0290	0.1964	0.2641	0.0141	0.0029
	CVM	0.9466	0.1966	0.3996	0.2532	0.0032	0.0017
	AD	0.8305	0.0805	0.2446	0.2572	0.0072	0.0012
500	MLE	0.8230	0.0730	0.1435	0.2519	0.0019	0.0007
	OLS	0.8203	0.0703	0.1922	0.2567	0.0067	0.0014
	OWLS	0.8105	0.0605	0.1525	0.2538	0.0038	0.0009
	MPS	0.7257	0.0243	0.0962	0.2565	0.0065	0.0004
	CVM	0.9074	0.1574	0.3008	0.2523	0.0023	0.0008
	AD	0.7716	0.0216	0.1325	0.2564	0.0064	0.0006
700	MLE	0.8012	0.0512	0.0789	0.2498	0.0001	0.0002
	OLS	0.7955	0.0455	0.1202	0.2530	0.0030	0.0005
	OWLS	0.8064	0.0564	0.0826	0.2515	0.0015	0.0004
	MPS	0.7380	0.012	0.0690	0.2548	0.0048	0.0003
	CVM	0.783	0.0330	0.0992	0.2534	0.0034	0.0005
	AD	0.7437	0.0063	0.0764	0.2563	0.0063	0.0005
1000	MLE	0.7612	0.0112	0.0409	0.2515	0.0015	0.0001
	OLS	0.7605	0.0105	0.0767	0.2537	0.0037	0.0003
	OWLS	0.7881	0.0381	0.0588	0.2509	0.0009	0.0002
	MPS	0.7548	0.0048	0.0398	0.2542	0.0042	0.0001
	CVM	0.7429	0.0071	0.0736	0.2548	0.0048	0.0004
	AD	0.7671	0.0171	0.0573	0.2526	0.0026	0.0002

Table 3. AEs, ABs, and MSEs of (λ, θ)=(0.5, 0.5) considering different sample sizes.

n	Method	$\hat{\lambda}$			$\hat{\theta}$		
		AE	AB	MSE	AE	AB	MSE
300	MLE	0.5626	0.0626	0.1051	0.5210	0.0210	0.1050
	OLS	0.5557	0.0557	0.1495	0.5278	0.0278	0.1495
	OWLS	0.5236	0.0236	0.1349	0.5328	0.0328	0.1348
	MPS	0.4442	0.0558	0.0897	0.5677	0.0677	0.0896
	CVM	0.6193	0.1193	0.3023	0.5237	0.0237	0.0105
	AD	0.6084	0.1084	0.1619	0.5114	0.0114	0.0789
500	MLE	0.5518	0.0518	0.0552	0.5014	0.0014	0.0552
	OLS	0.5439	0.0439	0.1221	0.5232	0.0232	0.1221
	OWLS	0.5449	0.0449	0.0851	0.5083	0.0083	0.0851
	MPS	0.4867	0.0133	0.0498	0.5435	0.0435	0.0498
	CVM	0.5451	0.0451	0.1196	0.5223	0.0223	0.0775
	AD	0.5091	0.0091	0.0699	0.5138	0.0138	0.0324
700	MLE	0.5540	0.0540	0.0368	0.4979	0.0021	0.0367
	OLS	0.5440	0.0440	0.0814	0.5081	0.0081	0.0813
	OWLS	0.5311	0.0311	0.0678	0.5073	0.0073	0.0678
	MPS	0.4673	0.0327	0.0296	0.5234	0.0234	0.0296
	CVM	0.4909	0.0091	0.0834	0.5242	0.0242	0.0599
	AD	0.5402	0.0402	0.0666	0.5113	0.0113	0.0577
1000	MLE	0.5181	0.0181	0.0314	0.5050	0.0050	0.0313
	OLS	0.5007	0.0007	0.0579	0.5163	0.0163	0.0578
	OWLS	0.5320	0.0320	0.0529	0.5077	0.0077	0.0529
	MPS	0.4938	0.0062	0.0210	0.5083	0.0083	0.0210
	CVM	0.5317	0.0317	0.0514	0.5053	0.0053	0.0278
	AD	0.5198	0.0198	0.0407	0.5039	0.0039	0.0447

Table 4. AEs, ABs, and MSEs of (λ, θ)=(1.3, 1.5) considering different sample sizes.

n	Method	$\hat{\lambda}$			$\hat{\theta}$		
		AE	AB	MSE	AE	AB	MSE
300	MLE	1.3730	0.0730	0.4496	1.5393	0.0393	0.0393
	OLS	1.2754	0.0246	0.6000	1.5798	0.0798	0.0426
	OWLS	1.4171	0.1171	0.5646	1.5323	0.0323	0.0419
	MPS	1.3197	0.0197	0.4026	1.5392	0.0392	0.0213
	CVM	1.4010	0.1010	0.4656	1.5408	0.0408	0.0411
	AD	1.4749	0.1749	0.8196	1.5215	0.0215	0.0633
500	MLE	1.4860	0.1860	0.2339	1.4863	0.0137	0.0079
	OLS	1.4633	0.1633	0.5181	1.5049	0.0049	0.0203
	OWLS	1.4749	0.1749	0.3690	1.4917	0.0083	0.0114
	MPS	1.3334	0.0334	0.2017	1.5245	0.0245	0.0070
	CVM	1.3486	0.0486	0.2724	1.5200	0.0200	0.0110
	AD	1.4381	0.1381	0.6913	1.4972	0.0028	0.0309
700	MLE	1.3575	0.0575	0.2125	1.5036	0.0036	0.0071
	OLS	1.2644	0.0356	0.3080	1.5294	0.0294	0.0110
	OWLS	1.4335	0.1335	0.2235	1.4881	0.0119	0.0078
	MPS	1.2501	0.0499	0.1378	1.531	0.0310	0.0065
	CVM	1.4173	0.1173	0.2556	1.4937	0.0063	0.0075
	AD	1.3871	0.0871	0.4120	1.4975	0.0025	0.0096
1000	MLE	1.3879	0.0879	0.1331	1.4919	0.0081	0.0053
	OLS	1.3199	0.0199	0.1602	1.5119	0.0119	0.0078
	OWLS	1.3005	0.0005	0.1365	1.5152	0.0152	0.0068
	MPS	1.2342	0.0658	0.1029	1.5322	0.0322	0.0047
	CVM	1.3359	0.0359	0.1349	1.5065	0.0065	0.0060
	AD	1.3239	0.0239	0.1820	1.5089	0.0089	0.0089

6. Real data analysis

6.1. **Snowfall Application.** Here, the data set contains the monthly maximum snowfall records for the month February 2018. The values of the data set are picked from the National Centers for Environmental Information (NCEI) https://www.ncdc.noaa.gov/cdoweb/datatools/records. The considered data is studied by Meraou and Raqab [17] and its records are summarized in Table 5.

Table 5. The values for the monthly maximum snowfall data set.

7.99	5.98	2.52	5.98	7.99	7.01	7.01	7.99	4.21	8.5
7.99	7.99	7.99	10	3.5	6.30	10	9.02	12.01	15.98
7.52	7.01	12.01	9.09	4.41	10.71	7.99	5.98	7.01	7.99
12.01	5	7.99	12.01	12.99	12.01	7.99	10.12	5.98	4.69
10	0.98	7.99	12.01	12.01	7.01	5.98	14.02	5.51	2.99
2.52	15.98	17.01							

6.2. **Reliability Application.** This data set provided the strengths measurements of 69 single carbon fibers which is obtained in GPa. It is considered by workers at the UK National Physical Laboratory and it is studeid by different reaserches such as Bader and Priest [5], Wani and Shafi [30], Alsadat [25] and Alsadat et al. [26]. The considered data is given as

Table 6. The values of the second data set.

0.312	0.314	0.479	0.552	0.700	0.803	0.861	0.865	0.944	0.958	0.966	0.977	1.006	1.021	1.027
1.055	1.063	1.098	1.140	1.179	1.224	1.240	1.253	1.270	1.272	1.274	1.301	1.301	1.359	1.382
1.382	1.426	1.434	1.435	1.478	1.490	1.511	1.514	1.535	1.554	1.566	1.570	1.586	1.629	1.633
1.642	1.648	1.684	1.697	1.726	1.770	1.773	1.800	1.809	1.818	1.821	1.848	1.880	1.954	2.012
2.067	2.084	2.090	2.096	2.128	2.233	2.433	2.585	2.585						

For checking the efficiency of the suggested model, the APCos-ME is compared with numerous models including ME, Gompertz (Gomp), XLindley (XL), Two parameters Mira (TPM), and Extended Exponential models. The PDFs of competing distributions are

(1) Gomp:

$$h(t) = \alpha\beta e^{-\beta(e^{\alpha t}-1)+\alpha t}, \quad t > 0,\ \alpha,\beta > 0.$$

(2) XL:

$$h(t) = \frac{\theta^2 e^{-\theta t}(\theta + t + 2)}{(\theta+1)^2}; \ t > 0, \quad \theta > 0.$$

(3) TPM:

$$h(t) = \frac{\delta^3\left(\alpha t^2 + 2\right)e^{-\delta t}}{2\left(\alpha + \delta^2\right)}., \quad t > 0,\ \alpha,\delta > 0.$$

(4) EE:

$$h(t) = \alpha\theta e^{-\theta t}(1 - e^{-\theta t})^{\alpha-1}; \ t > 0, \ \alpha,\theta > 0.$$

TABLE 7. Parameter estimation for fitting models using the two considered datasets.

Data	Model	Par.		Llik
1	APCos-ME	$\hat{\lambda}$=14.065	$\hat{\theta}$=4.1707	-143.411
	ME	$\hat{\theta}$=4.1250		-152.951
	Gomp	$\hat{\alpha}$ = 0.2228	$\hat{\beta}$=0.0288	-146.194
	XL	$\hat{\theta}$=0.2040		-159.084
	TPM	$\hat{\alpha}$=0.3627	$\hat{\delta}$=41.970	-147.183
	EE	$\hat{\alpha}$=0.2858	$\hat{\theta}$=5.6428	-145.950
2	APCos-ME	$\hat{\lambda}$=242.670	$\hat{\theta}$=0.6042	-50.825
	ME	$\hat{\theta}$=0.7254		-73.087
	Gomp	$\hat{\alpha}$ = 1.8940	$\hat{\beta}$=0.0825	-51.471
	XL	$\hat{\theta}$=0.8759		-92.176
	TPM	$\hat{\alpha}$=2.0238	$\hat{\delta}$=102.936	-64.736
	EE	$\hat{\alpha}$=1.8629	$\hat{\theta}$=8.3802	-56.705

Table (7) reported the obtained results of the MLEs for fitting model parameters with its negative log likelihood Function (Llik). Now for checking the model validity, Table (8) summarized the values of certain statistical measures notably Akaike Information Criterion ($\mathcal{A}_1$), Hannan Quinn Information Criterion ($\mathcal{A}_3$), Akaike Information Criterion corrected ($\mathcal{A}_2$), Bayesian Information Criterion ($\mathcal{A}_4$) and Kolmogorov-Smirnov ($\mathcal{KS}$) statistics with associated p-values ($\mathcal{P}$) which are reported in . Accordingly to these results, the recommended APCos-ME distribution can be considered as the best choice i for modeling the data set. In Figures (6)-(9), the estimated PDF, CDF, and SF of suggested models are sketched, while the scaled total time on the test (TTT), the probability-probability (PP), and box plots are plotted in Figures (4)-(5) for the two data sets.

TABLE 8. Comparison of suggested statistical measures for the two proposed datasets.

Data	Model	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{KS}$	$\mathcal{P}$
1	APCos-ME	290.830	291.065	292.364	294.808	0.1063	0.5742
	ME	307.902	307.979	308.669	309.891	0.2214	0.0100
	Gomp	296.389	296.624	297.923	300.367	0.1576	0.1367
	XL	320.169	320.246	320.936	322.158	0.2527	0.0020
	TPM	298.366	298.601	299.900	302.344	0.1668	0.0987
	EE	295.900	296.135	297.434	299.878	0.1267	0.3505
2	APCos-ME	105.650	105.832	107.423	110.119	0.0615	0.9565
	ME	148.175	148.235	149.061	150.409	0.2576	0.0002
	Gomp	106.943	107.124	108.715	111.411	0.0790	0.7815
	XL	186.353	186.412	187.239	188.587	0.3439	1.63×10^{-07}
	TPM	133.472	133.654	135.245	137.940	0.2043	0.0062
	EE	117.418	117.600	119.190	121.886	0.1150	0.3202

Next Table (9) considered estimates of λ and θ employing the various estimation procedures.

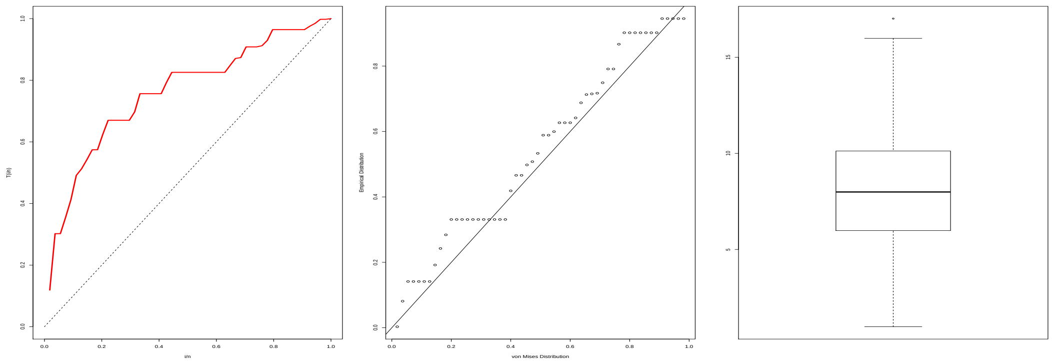

Figure 4. TTT, PP, and box curves for first data for different fitting models.

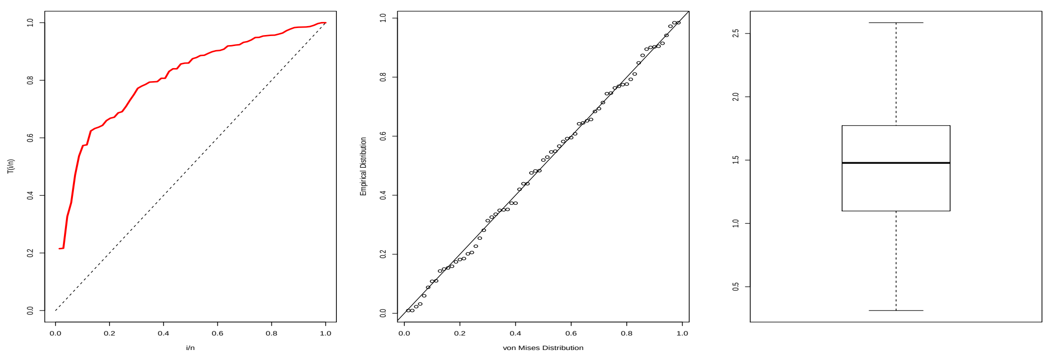

Figure 5. TTT, PP, and box curves for second data.

Table 9. The estimates of unknown parameters for the APCos-ME model under various methods of estimation.

Data Set	Par	MLE	LSE	WLSE	MPS	CME	ADE
1	$\hat{\lambda}$	14.065	39.916	34.080	36.894	33.193	40.311
	$\hat{\theta}$	4.1707	3.8390	3.8993	3.9217	3.7448	3.8433
2	$\hat{\lambda}$	242.670	460.561	491.637	463.641	441.162	387.705
	$\hat{\theta}$	0.6042	0.5949	0.5881	0.5972	0.5854	0.5947

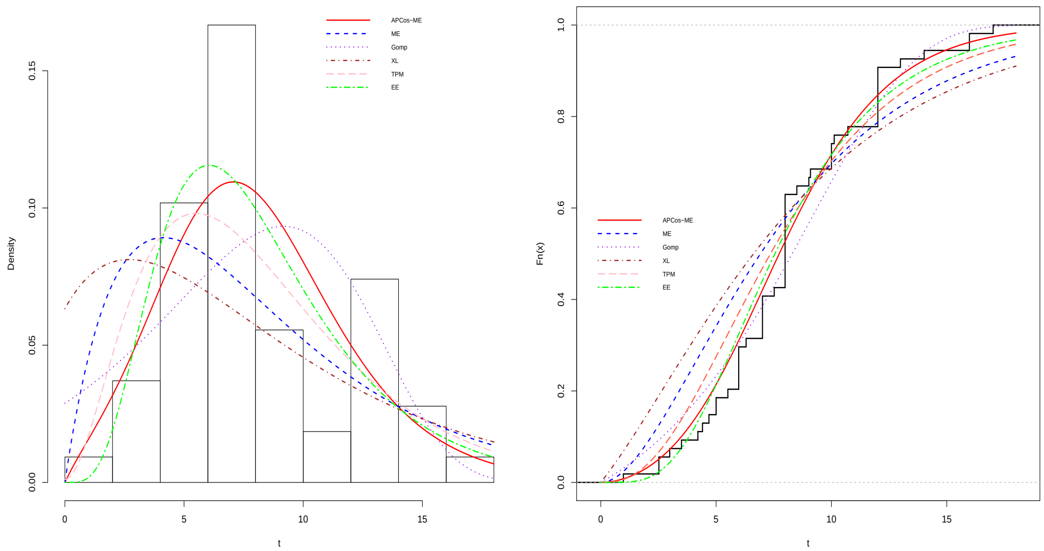

FIGURE 6. Curves of density and cumulative function of the suggested fitting model employing dataset 1.

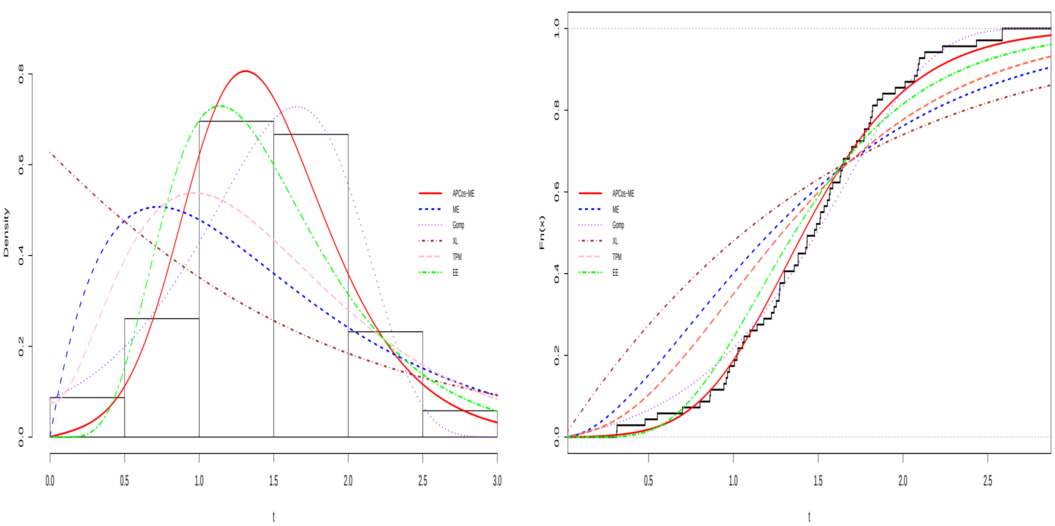

FIGURE 7. Curves of density and cumulative function of the suggested fitting model employing dataset 2.

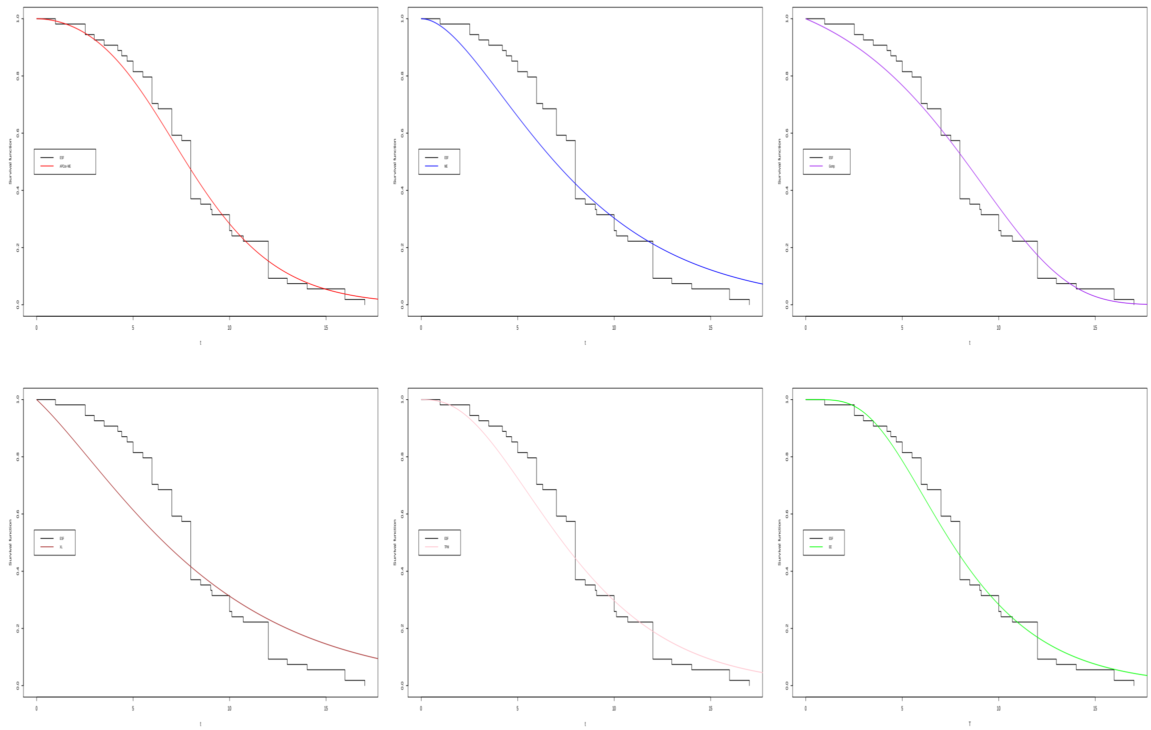

Figure 8. Estimated ESF using first data for different fitting models.

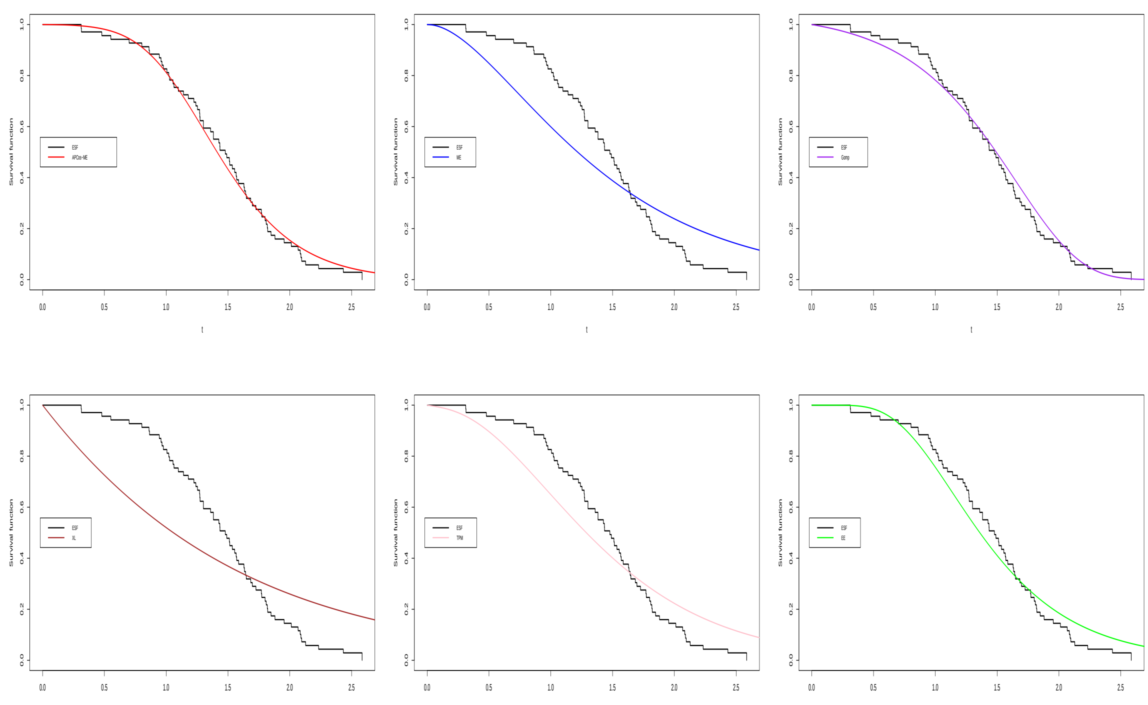

Figure 9. Estimated ESF using second data for different fitting models.

7. Conclusion

This work defined a novel extension of the moment exponential model using alpha power transformed with trigonometric function technique. We have obtained numerous characteriscs of the proposed model. Henceforth, Several estimation technique are applied to obtain the estimation of the unknown parameters. We conducted some experiment studies for the simulation experiment and to check the utility and effectiveness of suggested estimation techniques. At the end, two applications drown from environmental and engineering fields to demonstrate the applicability of the recommended model, and it is shown that the proposed APCos-ME is more appropriate to modeling the two considered data sets.

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