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Geometrical Aspect of Pointwise Semi-Slant Conformal Submersions

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Abstract. The aim of this paper is to define pointwise semi-slant conformal submersions from locally product Riemannian manifolds onto Riemannian manifolds. We investigated the conditions under which the distributions are integrable and the leaves of the distributions defines totally geodesic foliation. Additionally, we examined the concept of pluriharmonicity of pointwise semi-slant conformal submersions. In support of the results we obtained, we present non-trivial examples.

1. Introduction

In this contribution, T. W. Lee and B. Sahin [19] expanded the notion of slant submersion to include pointwise slant submersions from almost Hermitian manifolds onto Riemannian manifolds, thus taking it a step further. In the process, they found a way to illustrate examples of this type of submersions. They also established the characterizations of pointwise slant submersions.

As a generalized case of Riemannian submersion, B. Fuglede [13] and T. Ishihara [17] defined conformal submersions and studied their geometric properties. It is clear that conformal submersion with dilation $\lambda = 1$ is a Riemannian submersion. A step forward, conformal holomorphic submersion defined by Gudmundsson and Wood [14] as generalization of holomorphic submersions. They also studied the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. In this contribution, Akyol and Sahin investigated conformal anti-invariant submersions [28], [24], conformal semi-invariant submersions [4], conformal slant submersions [2], and conformal semi-slant submersions [1]. Conformal hemi-slant submersions [30], [31], conformal bi-slant submersions [5] and quasi bi-slant conformal submersions [6]

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have all been studied geometrically recently and several decomposition theorems have been covered.

The focus of this study is on pointwise semi-slant conformal submersions from locally product Riemannian manifold onto a Riemannian manifold. The sequence of the paper is as follows. We introduce almost product Riemannian manifolds in Section 2, specifically the locally product Riemannian manifold with the required properties for our study. We define pointwise semi-slant conformal submersion in the third section of the paper and uncover some interesting results. The requirements for distribution integrability and the total geodesicness of its leaves were thoroughly covered in Section 4.

Note: In this paper, we will use abbreviation as follows:
Pointwise semi-slant conformal submersion- $\mathcal{PWSSCS}$
Locally product Riemannian manifold- $\mathcal{LPRM}$
Riemannian manifold-$\mathcal{RM}$
Almost product Riemannian manifold- $\mathcal{APRM}$

2. Preliminaries

This section will deal with the concept of an $\mathcal{APRM}$, in addition to Riemannian submersions and pointwise semi-slant conformal submersions between two $\mathcal{RM}$s, along with a few essential facts and findings. We cited these ideas because they were previously addressed in prior studies in this area, and appropriate citations were included to recognize their contributions. Moreover, the definitions have been restated here to ensure clarity and facilitate a comprehensive understanding of the concepts presented in this study.

"An n-dimensional manifold $\bar{B}$ with $(1,1)$ type tensor field F such that

$$F^2 = I, (F \neq I), \tag{2.1}$$

is called an almost product manifold with almost structure F. Let g is a Riemannian metric on an almost product $\mathcal{RM}$ $\bar{B}$ such that

$$g(FX, FY) = g(X, Y), \tag{2.2}$$

for $X, Y \in \Gamma(T\bar{B})$, then $(\bar{B}, g, F)$ is said to be an $\mathcal{APRM}$. The covariant derivative of F is defined by

$$(\nabla_X FY) = \nabla_X FY - F\nabla_X Y, \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(T\bar{B})$. If F is parallel with respect to connection ∇, then the manifold $\bar{B}$ is said to be a $\mathcal{LPR}$ manifold, i.e.,

$$(\nabla_X F)Y = 0, \tag{2.4}$$

for any $X, Y \in \Gamma(T\bar{B})$.

Definition 2.1. *[34] "A smooth map β between two $\mathcal{RM}$s $(\bar{B}_1, g)$ and $(\bar{B}_2, g')$ with dimensions m and n is said to be a horizontally weakly conformal or semi conformal at $x \in \bar{B}_1$ if either*

(i) $\beta_{*x} = 0$, or

(*ii*) *β_{*x} maps horizontal space $h_x = (\ker(fi_{*x}))^{\perp}$ conformally onto $T_{\beta*}(\bar{B}_2)$ i.e., β_{*x} is surjective and there exits a number $\Lambda(x) \neq 0$ such that*

$$g'(\beta_{*x}X, \beta_{*x}Y) = \Lambda(x)g(X, Y), \tag{2.5}$$

for any $X, Y \in h_x$.

Equation (2.5) can be re-written as

$$(\beta_* g')_x \mid_{h_x \times h_x} = \Lambda(x)g(x) \mid_{h_x \times h_x}.$$

A point x is a critical point of β if and only if it satisfies (i) in the definition above and at that point, β_{*x} has rank 0. A point is called a regular point if (ii) holds good at which the rank of β_{*x} has rank n. The square dilation (of β at x) is denoted by the number $\lambda(x)$, which is inescapably non-negative. $\lambda(x) = \sqrt{\Lambda(x)}$, its square root, is referred to as the dilation of β at x. If the map β is horizontally weakly conformal at each point of $\bar{B}_1$, it is referred to as semi-conformal or horizontally weakly conformal on $\bar{B}_1$. It is evident that β is a (horizontally) conformal submersion if it is missing any critical points."

Definition 2.2. *[7] Let β be a Riemannian submersions between two $\mathcal{RM}$s. If λ is a positive function, then β is referred to as a horizontally conformal submersion such that*

$$g_1(U_1, V_1) = \frac{1}{\lambda^2} g'(\beta_* U_1, \beta_* V_1), \tag{2.6}$$

for any $U_1, V_1 \in \Gamma(\ker\beta_*)^{\perp}$. It is easy to seen that Riemannian submersions is particularly horizontally conformal submersions with $\lambda = 1$.

Let $\beta : (\bar{B}_1, g_1) \to (\bar{B}_2, g')$ be a Riemannian submersion. A vector field X on $\bar{B}_1$ is called a basic vector field if $X \in \Gamma(\ker\beta_*)^{\perp}$ and β-related with a vector field X on $\bar{B}_2$ i.e $\beta_*(X(q)) = X\beta(q)$ for $q \in \bar{B}_1$.

The two formulae of $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are given by O'Neill as:

$$\mathcal{A}_{E_1} F_1 = h\nabla_{hE_1} vF_1 + v\nabla_{hE_1} hF_1, \tag{2.7}$$

$$\mathcal{T}_{E_1} F_1 = h\nabla_{vE_1} vF_1 + v\nabla_{vE_1} hF_1, \tag{2.8}$$

for any $E_1, F_1 \in \Gamma(T\bar{B}_1)$ and ∇ is Levi-Civita connection of g. From equations (2.7) and (2.8), we can deduce that

$$\nabla_{U_1} V_1 = \mathcal{T}_{U_1} V_1 + v\nabla_{U_1} V_1 \tag{2.9}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + h\nabla_{U_1} X_1 \tag{2.10}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + v_1\nabla_{X_1} U_1 \tag{2.11}$$

$$\nabla_{X_1} Y_1 = h\nabla_{X_1} Y_1 + \mathcal{A}_{X_1} Y_1 \tag{2.12}$$

for any vector fields $U_1, V_1 \in \Gamma(\ker\beta_*)$ and $X_1, Y_1 \in \Gamma(\ker\beta_*)^\perp$ [11].

It is obvious that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is

$$g(\mathcal{A}_X E_1, F_1) = -g(E_1, \mathcal{A}_X F_1),\ g(\mathcal{T}_V E_1, F_1) = -g(E_1, \mathcal{T}_V F_1), \tag{2.13}$$

for any vector fields $E_1, F_1 \in \Gamma(T\bar{B}_1)$. The Riemannian submersion β has totally geodesic fibres if and only if $\mathcal{T} = 0$. For the special case when β is horizontally conformal submersion, we have

Proposition 2.1. *Let $\beta : (\bar{B}_1, g) \to (\bar{B}_2, g')$ be a horizontally conformal submersion with dilation λ and X, Y be the horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{v[X, Y] - \smile^2 g(X, Y) grad_v(\frac{1}{\smile^2})\} \tag{2.14}$$

measures the obstruction integrability of the horizontal distribution.

The formula of second fundamental form of smooth map β is given by

$$(\nabla\beta_*)(X, Y) = \nabla^\beta_X \beta_* Y - \beta_* \nabla_X Y, \tag{2.15}$$

and the map be totally geodesic if $(\nabla\beta_*)(X, Y) = 0$ for all $X, Y \in \Gamma(T\bar{B}_1)$ where ∇ and ∇^β are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\beta : \bar{B}_1 \to \bar{B}_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla\beta_*)(X_1, Y_1) = X_1(ln\lambda)\beta_*(Y_1) + Y_1(ln\lambda)\beta_*(X_1) - g_1(X_1, Y_1)\beta_*(\text{grad } ln\lambda)$

(ii) $(\nabla\beta_*)(U_1, V_1) = -\beta_*(\mathcal{T}_{U_1} V_1)$

(iii) $(\nabla\beta_*)(X_1, U_1) = -\beta_*(\nabla_{X_1} U_1) = -\beta_*(\mathcal{A}_{X_1} U_1)$

for any horizontal vector fields X_1, Y_1 and vertical vector fields U_1, V_1 [7].

Now, we recall the definition of pointwise slant submersion defined by S. A. Sepet and M. Ergut [29].

Definition 2.3. *Let β be a Riemannian submersion from almost contact metric manifold $(\bar{B}_1, F, g)$ onto $\mathcal{RM}$ $(\bar{B}_2, g')$. If at each given point $q \in \bar{B}_1$, the wirtinger angle $\theta(X)$ between FX and the space* $\ker\beta_*$ *is independent of choice of the non-zero vector field $X \in \Gamma(\ker\beta_*)$, then we say that β is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on $\bar{B}_1$, which is called slant function of the pointwise slant submersion.*

A pointwise slant submersion called slant submersion if its slant function θ is independent of the choice of the point on $\bar{B}_1$. Then θ is called the slant angle of the slant submersions.

Now, we extended the concept of F-pluriharmonicity from almost Hermitian manifolds to $\mathcal{APRM}$ which was once studied and defined by Y. Ohnita [21]. Let β be a $\mathcal{PWSSCS}$ from $\mathcal{APRM}$ $(\bar{B}_1, F, g)$ onto a $\mathcal{RM}$ $(\bar{B}_2, g')$. Then $\mathcal{PWSSCS}$ is F-pluriharmonic, $\mathfrak{D}$-F-pluriharmonic, $\mathfrak{D}^\theta$-F-pluriharmonic, $(\mathfrak{D} - \mathfrak{D}^\theta)$-$F$ pluriharmonic, $\ker\beta_*$-F-pluriharmonic, $(\ker\beta_*)^\perp$-F-pluriharmonic and $((\ker\beta_*)^\perp - \ker\beta_*)$-$F$-pluriharmonic if

$$(\nabla\beta_*)(X, Y) + (\nabla\beta_*)(FX, FY) = 0, \tag{2.16}$$

for any $X, Y \in \Gamma(\mathfrak{D})$, for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$, for any $X \in \Gamma(\mathfrak{D}), Y \in \Gamma(\mathfrak{D}^\theta)$, for any $X, Y \in \Gamma(\ker\beta_*)$, for any $X, Y \in \Gamma(\ker\beta_*)^\perp$ and for any $X \in \Gamma(\ker\beta_*)^\perp, Y \in \Gamma(\ker\beta_*)$.

3. Pointwise semi-slant conformal submersions ($\mathcal{PWSSCS}$)

The definitions necessary to comprehend and explore the idea of pointwise semi-slant conformal submersions from $\mathcal{APRM}$s onto $\mathcal{RM}$s will be covered in this section. We'll also talk about a few fundamental outcomes and findings that are pertinent to our research paper.

Definition 3.1. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a horizontal conformal submersion where $(\bar{B}_1, F, g)$ is an $\mathcal{APRM}$ and $(\bar{B}_2, g')$ is a $\mathcal{RM}$. A horizontal conformal submersion β is called a pointwise semi-slant conformal submersion if there exists a distribution $\mathfrak{D}$ such that $(\ker\beta_*) = \mathfrak{D} \oplus \mathfrak{D}^\theta$, $F(\mathfrak{D}) = \mathfrak{D}$ and for any given point $q \in \bar{B}_1$ and $X \in (\mathfrak{D}^\theta)_q$, the angle $\theta = \theta(X)$ between FX and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $X \in (\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}$ in $(\ker\beta_*)$. In this case, the angle θ can be regarded as a semi-slant function and called pointwise semi-slant function of submersion.*

If we suppose m_1 and m_2 are the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, then we have the following:

(i) If $m_1 = 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then β is a pointwise slant submersion.
(i) If $m_1 \neq 0$ and $m_2 = 0$, then β is a invariant submersion
(ii) If $m_1 \neq 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then β is a pointwise semi-slant submersion.

Let β be a $\mathcal{PWSSCS}$ from an $\mathcal{APRM}$ $(\bar{B}_1, F, g)$ onto a $\mathcal{RM}$ $(\bar{B}_2, g')$. Then, for any $U \in (\ker\beta_*)$, we have

$$U = \mathbb{R}U + \bar{\mathbb{R}}U \tag{3.1}$$

where $\mathbb{R}$ and $\bar{\mathbb{R}}$ are the projections morphism onto $\mathfrak{D}$ and $\mathfrak{D}^\theta$. Now, for any $U \in (\ker\beta_*)$, we have

$$FU = \phi U + \eta U \tag{3.2}$$

where $\phi U \in \Gamma(\ker\beta_*)$ and $\eta U \in \Gamma(\ker\beta_*)^\perp$. From equations (3.1) and (3.2), we have

$$\begin{aligned} FU =& \phi(\mathbb{R}U) + \phi(\bar{\mathbb{R}}U) \\ =& \phi(\mathbb{R}U) + \eta(\mathbb{R}U) + \phi(\bar{\mathbb{R}}U) + \eta(\bar{\mathbb{R}}U). \end{aligned}$$

Since $F\mathfrak{D} = \mathfrak{D}$, we have $\eta(\mathbb{R}U) = 0$, we have

$$FU = \phi(\mathbb{R}U) + \phi(\bar{\mathbb{R}}U) + \eta(\bar{\mathbb{R}}U).$$

Now, we have the following decomposition

$$(\ker\beta_*)^\perp = \eta\mathfrak{D}^\theta \oplus \mu, \tag{3.3}$$

where μ is the orthogonal complement to $\eta\mathfrak{D}^\theta$ in $(\ker\beta_*)^\perp$ such that μ is invariant with respect to ϕ. Now, for any $X \in \Gamma(\ker\beta_*)^\perp$, we have

$$FX = \mathfrak{B}X + \mathfrak{C}X \tag{3.4}$$

where $\mathfrak{B}X \in \Gamma(\ker\beta_*)$ and $\mathfrak{C}X \in \Gamma(\ker\beta_*)^\perp$.

Lemma 3.1. *Let $(\bar{B}_1, F, g)$ be $\mathcal{APRM}$ and $(\bar{B}_2, g')$ be a $\mathcal{RM}$. If $\beta : \bar{B}_1 \to \bar{B}_2$ is a $\mathcal{PWSSCS}$, then we have*

$$U = \phi^2 U + \mathfrak{B}\eta U,\ \eta\phi U + \mathfrak{C}\eta U = 0,\ X = \eta\mathfrak{B}X + \mathfrak{C}^2 X,\ \phi\mathfrak{B}X + \mathfrak{B}\mathfrak{C}X = 0,$$

for any $U \in \Gamma(\ker\beta_*)$ and $X \in \Gamma(\ker\beta_*)^\perp$.

Proof. By considering the equations (2.1), (3.2) and (3.4), the proof of Lemma exists. □

Let us now present some beneficial results that will be used throughout the study since $\beta : \bar{B}_1 \to \bar{B}_2$ is a $\mathcal{PWSSCS}$.

Lemma 3.2. *Let β be a $\mathcal{PWSSCS}$ from an $\mathcal{APRM}$ $(\bar{B}_1, F, g)$ onto a $\mathcal{RM}$ $(\bar{B}_2, g')$, then we have*

$$\phi^2 W = (\cos^{2}\grave{\ })W,$$

for any $W \in \Gamma(\mathfrak{D}^\theta)$.

Lemma 3.3. *Let β be a $\mathcal{PWSSCS}$ from an $\mathcal{APRM}$ $(\bar{B}_1, F, g)$ onto a $\mathcal{RM}$ $(\bar{B}_2, g')$, then we have*

(i) $g(\phi X, \phi Y) = \cos^2\grave{\ }\, g(X, Y)$,
(ii) $g(\eta X, \eta Y) = \sin^2\theta g(X, Y)$,

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The proof of the earlier Lemmas is precisely the same as the proof of [9]'s Theorem (2.2). As a result, the proofs were deleted. □

Assuming that $(\bar{B}_1, F, g)$ is an $\mathcal{APRM}$ and $(\bar{B}_2, g')$ is a $\mathcal{RM}$. The effect of the $\mathcal{APRM}$ on the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of $\mathcal{PWSSCS}$ $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ is presently being examined.

Lemma 3.4. *Let $\beta : \bar{B}_1 \to \bar{B}_2$ be $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{B}_1, F, g)$ $\mathcal{LPRM}$ and $(\bar{B}_2, g')$ be a $\mathcal{RM}$, then we have*

(i) $\mathcal{A}_X\mathfrak{C}Y + v\nabla_X\mathfrak{B}Y = \mathfrak{B}h\nabla_X Y + \text{Œ}\mathcal{A}_X Y$
(ii) $h\nabla_X\mathfrak{C}Y + \mathcal{A}_X\mathfrak{B}Y = \mathfrak{C}h\nabla_X Y + \jmath\mathcal{A}_X Y$
(iii) $v\nabla_X\text{Œ}W + \mathcal{A}_X\jmath W = \mathfrak{B}\mathcal{A}_X W + \text{Œ}v\nabla_X W$
(iv) $\mathcal{A}_X\phi W + h\nabla_X\jmath W = \mathfrak{C}\mathcal{A}_X W + \jmath v\nabla_X W$
(v) $v\nabla_W\mathfrak{B}X + \mathcal{T}_W\mathfrak{C}X = \text{Œ}\mathcal{T}_W X + \mathfrak{B}h\nabla_W X$
(vi) $\mathcal{T}_W\mathfrak{B}X + h\nabla_W\mathfrak{C}X = \jmath\mathcal{T}_W X + \mathfrak{C}h\nabla_W X$
(vii) $v\nabla_Z\text{Œ}W + \mathcal{T}_Z\jmath W = \text{Œ}v\nabla_Z W + \mathfrak{B}\mathcal{T}_Z W$
(viii) $\mathcal{T}_Z\phi W + h\nabla_Z\jmath W = \mathfrak{C}\mathcal{T}_Z W + \jmath v\nabla_Z W$,

for any vector fields $Z, W \in \Gamma(\ker\beta_*)$ and $X, Y \in \Gamma(\ker\beta_*)^\perp$.

Proof. By using (2.4), (2.3) and (2.12) (3.4), we get first two relations (i) and (ii). Similarly, by considering equations (2.4), (2.3) (2.12), (2.9)-(2.12) and (3.2) (3.4), the desired results holds good. □

We will now go through some key conclusions that can be utilised to examine the geometry of $\mathcal{PWSSCS}$ $\beta : \bar{B}_1 \to \bar{B}_2$. From the direct calculations, we can conclude the following:

(a) $(\nabla_Z \phi)W = v\nabla_Z ŒW - Œv\nabla_Z W$
(b) $(\nabla_Z \eta)W = h\nabla_Z \jmath W - \jmath v\nabla_Z W$
(c) $(\nabla_X \mathfrak{B})Y = v\nabla_X \mathfrak{B}Y - \mathfrak{B}h\nabla_X Y$
(d) $(\nabla_X \mathfrak{C})Y = h\nabla_X \mathfrak{C}Y - \mathfrak{C}h\nabla_X Y,$

for any $Z, W \in \Gamma(\ker \beta_*)$ and $X, Y \in \Gamma(\ker \beta_*)^\perp$.

Lemma 3.5. *Let $\beta : \bar{B}_1 \to \bar{B}_2$ be a $\mathcal{PWSSCS}$ with semi-slant function θ from $\mathcal{APRM}$s onto a $\mathcal{RM}$s, then we have*

(i) $(\nabla_Z \phi)W = \mathfrak{B}\mathcal{T}_Z W - \mathcal{T}_Z \eta W$
(i) $(\nabla_Z \eta)W = \mathfrak{C}\mathcal{T}_Z W - \mathcal{T}_Z \phi W$
(i) $(\nabla_X \mathfrak{B})Y = \phi\mathcal{A}_X Y - \mathcal{A}_X \mathfrak{C}Y$
(i) $(\nabla_X \mathfrak{C})Y = \eta\mathcal{A}_X Y - \mathcal{A}_X \mathfrak{B}Y,$

for all $Z, W \in \Gamma(\mathrm{kerfi}_*)$ and $X, Y \in \Gamma(\mathrm{kerfi}_*)^\perp$.

Proof. By using equations (2.3), (2.9)- (2.12) and formulae $(a) - (d)$ from above, we can obtained the results. □

The tensor fields ϕ and η, if they are parallel with regard to the Levi-Civita connection ∇ of $\bar{B}_1$, then we obtain

$$\mathfrak{B}\mathcal{T}_U V = \mathcal{T}_U \eta V,\ \mathfrak{C}\mathcal{T}_U V = \mathcal{T}_U \phi V$$

for any $U, V \in \Gamma(T\bar{B}_1)$.

4. Integrability and totally geodesicness

This section examines the integrability of both slant and invariant distributions when analyzing the $\mathcal{PWSSCS}$ from $\mathcal{LPRM}$s onto $\mathcal{RM}$s. Furthermore, we examine the necessary and sufficient conditions under which the distributions' leaves characterize complete geodesic foliation. :

Theorem 4.1. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a $\mathcal{PWSSCS}$ where, $(\bar{B}_1, F, g)$ is a $\mathcal{LPRM}$ and $(\bar{B}_2, g')$ is a $\mathcal{RM}$ with semi-slant function θ. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$(v\nabla_U ŒW + \mathcal{T}_U \jmath W) \in \Gamma(\grave{\mathfrak{D}}) \text{ and } (v\nabla_V ŒW + \mathcal{T}_V \jmath W) \in \Gamma(\grave{\mathfrak{D}}), \tag{4.1}$$

for any $U, V \in \Gamma(\mathfrak{D})$ and $W \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For all vector fields $U, V \in \Gamma(\mathfrak{D})$, $W \in \Gamma(\mathfrak{D}^\theta)$ and by using equations (2.1), (2.4) and (2.3), we have

$$g([U, V], W) = g(\nabla_V FW, FU) - g(\nabla_U FW, FV).$$

By using (2.9), (2.10) and (3.2), we get

$$g([U, V], W) = g(v\nabla_V ŒW + \mathcal{T}_V \jmath W, ŒU) - g(v\nabla_U ŒW + \mathcal{T}_U \jmath W, ŒV).$$

From this, we get desired result. □

Theorem 4.2. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a $\mathcal{PWSSCS}$ from a Kenmotsu manifold onto a $\mathcal{RM}$ with semi-slant function θ. Then $\mathfrak{D}^\theta$ is integrable if and only if*

$$\phi(\mathcal{T}_Z\eta W - \mathcal{T}_W\eta Z) = (\mathcal{T}_W\eta\phi Z + \mathcal{T}_Z\eta\phi W), \tag{4.2}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $U \in \Gamma(\mathfrak{D})$.

Proof. By using equation (2.1), (2.4) and (2.3), we may yield

$$g([Z, W], U) = g(\nabla_Z FW, FU) - g(\nabla_W FZ, FU),$$

for every $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $U \in \Gamma(\mathfrak{D})$. In the light of equation (3.2), we can write

$$g([Z, W], U) = g(\nabla_Z\phi W, FU) - g(\nabla_W\phi Z, FU) + g(\nabla_Z\eta W, FU) - g(\nabla_W\eta Z, FU).$$

By using (2.1) and (2.10) in third and fourth terms, above equation can be written as

$$g([Z, W], U) = g(\nabla_Z F\phi W, U) - g(\nabla_W F\phi Z, U) + g(\mathcal{T}_Z\eta W, FU) - g(\mathcal{T}_W\eta Z, FU).$$

Taking account the fact from equation (3.2) and Lemma 3.2, finally above equations takes the form

$$\sin^2\theta g([Z, W], U) = g(\mathcal{T}_W\eta\phi Z, U) - g(\mathcal{T}_Z\eta\phi W, U) + g(\mathcal{T}_Z\eta W - \mathcal{T}_W\eta Z, FU).$$

From which, we can conclude the result. □

After talking about the integrability condition of the distributions, we will examine the necessary and sufficient conditions under which the leaves of the distributions can form a totally geodesic foliation on $\bar{B}_1$.

Theorem 4.3. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be $\mathcal{PWSSCS}$ from $\mathcal{LPRM}$ $(\bar{B}_1, F, g)$ onto a $\mathcal{RM}$ $(\bar{B}_2, g')$ with semi-slant function θ. Then $\mathfrak{D}$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

$$\mathcal{T}_U\eta\phi Z = \phi(\mathcal{T}_U\eta Z) \text{ and } g(v\nabla_U Œ V, \mathfrak{B}X) + g(\mathcal{T}_U Œ V, \mathfrak{C}X) = 0, \tag{4.3}$$

for any $U, V \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\ker\beta_*)^\perp$.

Proof. For any $U, V \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^\theta)$ and by using orthogonality of V and Z, we get $g(\nabla_U V, Z) = -g(\nabla_U Z, V)$. Further, in the light of equations (2.1), (2.4), (2.3) and (3.2) (2.10), we get

$$g(\nabla_U V, Z) = g(\nabla_U\phi^2 Z, V) + g(\nabla_U\eta\phi Z, V) + g(\mathcal{T}_U\eta Z, FV).$$

Since, β is a $\mathcal{PWSSCS}$ with semi-slant function θ, then by using Lemma 3.2 in first term of above equation, finally this will takes the form

$$\sin^2\theta g(\nabla_U V, Z) = g(\nabla_U\eta\phi Z, V) + g(\mathcal{T}_U\eta Z, FV).$$

From this we can get the first part of theorem. Now, for any vector fields $U, V \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker\beta_*)^\perp$ with using equation (2.1), (2.4), (2.3) (2.9) and (3.4), (3.2), we can write

$$g(\nabla_U V, X) = g(v\nabla_U Œ V, \mathfrak{B}X) + g(\mathcal{T}_U Œ V, \mathfrak{C}X).$$

from which the second part of theorem holds good. This concludes the proof of theorem. □

The invariant distribution and the slant distribution are mutually orthogonal. Following the discussion of the geometry of the leaves of the invariant distribution, it is fascinating to investigate the leaves of the slant distribution from a geometric perspective in the way that follows.

Theorem 4.4. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{B}_1, F, g)$ is a $\mathcal{LPRM}$ and $(\bar{B}_2, g')$ a $\mathcal{RM}$. Then $\mathfrak{D}^\theta$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

$$\phi(\mathcal{T}_Z\eta\bar{R}W) = \mathcal{T}_Z\eta\phi\mathbb{R}W \text{ and}$$

$$\begin{aligned}&\sin 2\theta X(\theta)g(\bar{R}Z, W) - \cos^2\theta g(\nabla_X\mathbb{R}Z, W) + g(\nabla_X\eta\phi\bar{R}Z, W) + g(\mathcal{A}_X\eta\bar{R}Z, \phi W)\\ &= -g(X, \operatorname{grad}\ln\lambda)g(\eta\bar{R}Z, \eta W) - g(\eta\bar{R}Z, \operatorname{grad}\ln\lambda)g(X, \eta W) + g([Z, X], W)\\ &+ g(\eta W, \operatorname{grad}\ln\lambda)g(X, \eta\bar{R}Z) + \frac{1}{\lambda^2}g(\nabla^\beta_X\beta_*(\eta\bar{R}Z), \beta_*(\eta W)).\end{aligned}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta), U \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker\beta_*)^\perp$.

Proof. Let us consider for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $U \in \Gamma(\mathfrak{D})$. In light of equation (2.1), (2.4), (2.3) with decomposition (3.1) and (3.2), we have

$$\begin{aligned}g(\nabla_Z W, U) &= g(\nabla_Z\phi\mathbb{R}W, FU) + g(\nabla_Z\eta\mathbb{R}W, FU)\\ &+ g(\nabla_Z\phi\bar{R}W, FU) + g(\nabla_Z\eta\bar{R}W, FU).\end{aligned}$$

From equations (2.9), (2.10) with $\mathfrak{D}$ is invariant, i.e., $F\mathfrak{D} = \mathfrak{D}$ and $W = \mathbb{R}W$ if $W \in \Gamma(\mathfrak{D}^\theta)$, we may yields

$$g(\nabla_Z W, U) = g(\nabla_Z\phi^2\bar{R}W, U) + g(\nabla_Z\eta\bar{R}W, FU) + g(\nabla_Z\eta\phi\mathbb{R}W, U). \tag{4.4}$$

By using Lemma 3.2 in third term of above equation, we can write as: $g(\nabla_Z\phi^2\bar{R}W, U) = g(\nabla_Z(\cos^2\theta)\bar{R}W, U)$. Then the equation (4.4) and using the orthogonality of W and U, will takes the form as

$$g(\nabla_Z W, U) = g(\nabla_Z\eta\bar{R}W, FU) + \cos^2\theta g(\nabla_Z\bar{R}W, U) + g(\nabla_Z\eta\phi\mathbb{R}W, U).$$

From which the first part of theorem holds good. For other part of theorem, let us suppose for any vector fields $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\ker\beta_*)^\perp$. By using equation (2.1), (2.4), (2.9) and (2.3), we arrive at

$$g(\nabla_Z W, X) = -g([Z, X], W) - g(\nabla_X\phi\mathbb{R}Z, FW) - g(\nabla_X\phi\bar{R}Z, FW) - g(\nabla_X\eta\bar{R}Z, FW).$$

By using equations (2.1), (2.9) and (2.10) with fact that $\mathbb{R}Z = 0$ if $Z \in \Gamma(\mathfrak{D})^\theta$, we have

$$\begin{aligned}g(\nabla_Z W, X) = &- g([Z, X], W) - g(\nabla_X\phi^2\bar{R}Z, W) - g(\nabla_X\eta\phi\bar{R}Z, W)\\ &- g(\mathcal{A}_X\eta\bar{R}Z, \phi W) - g(h\nabla_X\jmath\bar{R}Z, \jmath W).\end{aligned} \tag{4.5}$$

Since, β is a $\mathcal{PWSSCS}$ with semi-slant function θ then with simple steps of calculations, we can write

$$g(\nabla_Z W, X) = -\sin 2\theta X(\theta) g(\bar{R}Z, W) - g([Z,X], W) + \cos^2\theta g(\nabla_X \bar{R}Z, W) \\ - g(\nabla_X \eta\phi\bar{R}Z, W) - g(\mathcal{A}_X \eta\bar{R}Z, \phi W) - g(h\nabla_X \jmath\bar{R}Z, \jmath W). \tag{4.6}$$

Now, using the conformality of β from Lemma 2.1 and equation (2.15), we get

$$\begin{aligned} & g(\nabla_Z W, X) \\ &= -g([Z,X], W) + \cos^2\theta g(\nabla_X \bar{\mathbb{R}}Z, W) - g(\nabla_X \eta\phi\bar{R}Z, W) - g(\mathcal{A}_X \eta\bar{R}Z, \phi W) \\ & - g(X, \operatorname{grad}\ln\lambda) g(\eta\bar{R}Z, \eta W) - g(\eta\bar{R}Z, \operatorname{grad}\ln\lambda) g(X, \eta W) + g(\eta W, \operatorname{grad}\ln\lambda) g(X, \eta\bar{R}Z) \\ & - \sin 2\theta X(\theta) g(\bar{R}Z, W) + \frac{1}{\lambda^2} g(\nabla^\beta_X \beta_*(\eta\bar{R}Z), \beta_*(\eta W)). \end{aligned}$$

Hence, this proves the theorem completely. □

We start our discussion with necessary and sufficient conditions for vertical distributions $\ker\beta_*$ is totally geodesic.

Theorem 4.5. *Let us suppose that $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{B}_1, F, g)$ a $\mathcal{LPRM}$ and $(\bar{B}_2, g')$ a $\mathcal{RM}$. Then $\ker\beta_*$ defines totally geodesic foliation if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2} g'(\nabla^\beta_X \beta_* \eta\bar{\mathbb{R}}U, \beta_*\eta V) + g(\mathcal{A}_X \phi\mathbb{R}U, \eta V) + g(v\nabla_X Œ\mathbb{R}U, ŒV) + g(\mathcal{A}_X \jmath\bar{\mathbb{R}}U, ŒV) \\ &= \sin 2\theta X(\theta) g(\bar{\mathbb{R}}U, V) - \cos^{2`} g(\nabla_X \bar{R}U, V) + g(\eta V, \operatorname{grad}\ln\lambda) g(X, \eta\bar{\mathbb{R}}U) \\ & - g(X, \operatorname{grad}\ln\lambda) g(\eta\bar{\mathbb{R}}U, \eta V) - g(\eta\bar{\mathbb{R}}U, \operatorname{grad}\ln\lambda) g(X, \eta V) - g([U,X], V) - g(\mathcal{A}_X \eta\phi\bar{\mathbb{R}}U, V), \end{aligned}$$

for any $U, V \in \Gamma(\ker\beta_*)$ and $X \in \Gamma(\ker\beta_*)^\perp$.

Proof. From simple steps of calculations with using (2.1), (2.4), (2.3) and decompositions (3.1), (3.2), we can write

$$g(\nabla_U V, X) = -g([U,X], V) - g(\nabla_X \phi\mathbb{R}U, FV) - g(\nabla_X \phi\bar{\mathbb{R}}U, FV) - g(\nabla_X \eta\bar{\mathbb{R}}U, FV), \tag{4.7}$$

for any $U, V \in \Gamma(\ker\beta_*)$ and $X \in \Gamma(\ker\beta_*)^\perp$. In the light of equation (3.2) and (2.11), second term of above equation become $g(\nabla_X \phi\mathbb{R}U, FV) = g(\mathcal{A}_X \phi\mathbb{R}U, \eta V) + g(v\nabla_X Œ\mathbb{R}U, ŒV)$. Similarly, by using equation (2.1), (2.4) and (2.11), third term turns into: $-g(\nabla_X \phi\bar{\mathbb{R}}U, FV) = -g(\nabla_X \phi^2\bar{\mathbb{R}}U, V) - g(\nabla_X \eta\phi\bar{\mathbb{R}}U, V)$. In last term, taking account the fact from decomposition (3.2) and equation (2.12), this will take place as $-g(\nabla_X \eta\bar{\mathbb{R}}U, FV) = -g(h\nabla_X \jmath\bar{\mathbb{R}}U, \jmath V) - g(\mathcal{A}_X \jmath\bar{\mathbb{R}}U, ŒV)$. By using all these facts in equation (4.7), we get

$$g(\nabla_U V, X) = -g([U,X], V) - g(\mathcal{A}_X \phi\mathbb{R}U, \eta V) - g(v\nabla_X Œ\mathbb{R}U, ŒV) - g(\nabla_X Œ^2\bar{\mathbb{R}}U, V) \\ - g(\nabla_X \eta\phi\bar{\mathbb{R}}U, V) - g(h\nabla_X \jmath\bar{\mathbb{R}}U, \jmath V) - g(\mathcal{A}_X \jmath\bar{\mathbb{R}}U, ŒV).$$

Since, β is a $\mathcal{PWSSCS}$ with semi-slant function θ, using Lemma 3.2 in fourth term and considering equation (2.15) in second last term, above equation finally turns into

$$\begin{aligned} g(\nabla_U V, X) = & - g([U,X],V) - g(\mathcal{A}_X \phi \mathbb{R}U, \eta V) - g(v\nabla_X \text{Œ}\mathbb{R}U, \text{Œ}V) - g(\mathcal{A}_X \jmath \text{Œ}\bar{\mathbb{R}}U, V) \\ & + \sin 2\theta X(\theta) g(\bar{\mathbb{R}}U, V) - \cos^2\theta g(\nabla_X \bar{R}U, V) - g(\mathcal{A}_X \eta \bar{\mathbb{R}}U, \phi V) \\ & - g(X, \operatorname{grad}\ln\lambda) g(\eta \bar{\mathbb{R}}U, \eta V) - g(\eta\bar{\mathbb{R}}U, \operatorname{grad}\ln\lambda) g(X, \eta V) \\ & + g(X, \eta\bar{\mathbb{R}}U) g(\eta V, \operatorname{grad}\ln\lambda) - \frac{1}{\lambda^2} g'(\nabla^{\beta}_X \beta_* \eta \bar{R}U, \beta_* \eta V), \end{aligned}$$

from which we can get the result. □

Theorem 4.6. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{B}_1, F, g)$ a locally product Riemannian manifold and $(\bar{B}_2, g')$ a $\mathcal{RM}$. Then the map β is totally geodesic map if and only if*

(i) $g(\mathcal{T}_Z \phi^2 W, X) + \frac{1}{\lambda^2} g'(\nabla^{\beta}_Z \beta_* \eta\phi W, \beta_* X) = 0,$

(ii) $\cos^2\theta g(v\nabla_Y U, V) - \sin 2\text{`}Y(\text{`}) g(U,V) + g(\mathcal{A}_Y \jmath \text{Œ}U, V) + g([Y,U],V)$
$= -g(\mathcal{A}_Y \jmath U, \text{Œ}V) - \frac{1}{\text{˘}2} g'(Y(\ln\text{˘})\text{fi}_* \jmath U + \jmath U(\ln\text{˘})\text{fi}_* Y - \text{fi}_*(\operatorname{grad}\ln\text{˘}), \text{fi}_* Y),$

(iii) $g(\mathcal{A}_X \phi \mathbb{R}U, \mathfrak{C}Y) + g(v\nabla_X \text{Œ}\mathbb{R}U, \mathfrak{B}Y) + g(\mathcal{A}_X \jmath \bar{\mathbb{R}}U, \mathfrak{B}Y) + \cos^2\text{`} g(\nabla_X \bar{\mathbb{R}}U, Y)$
$= -\frac{1}{\text{˘}2} g'(X(\ln\text{˘})\text{fi}_* \jmath \bar{\mathbb{R}}U + \jmath\bar{\mathbb{R}}U(\ln\text{˘})\text{fi}_* X - g(X, \jmath\bar{\mathbb{R}}U)\text{fi}_*(\operatorname{grad}\text{˘}), \text{fi}_* \mathfrak{C}Y)$
$- g_1(h\nabla_X \jmath \text{Œ}\bar{R}U, Y) + \frac{1}{\text{˘}2} g'(\nabla^{\text{fi}}_X \text{fi}_* \jmath \bar{\mathbb{R}}U, \text{fi}_* \mathfrak{C}Y),$

for any $U, V \in \Gamma(\mathfrak{D}^\theta), X, Y \in \Gamma(\ker \text{fi}_*)^\perp$ and $Z, W \in \Gamma(\mathfrak{D}), U_1 \in \Gamma(\ker\beta_*)$.

Proof. Let us consider $g'((\nabla\beta_*)(Z,W), \beta_*(X))$, for any $Z, W \in \Gamma(\mathfrak{D})$ and $X \in \Gamma(\ker\beta_*)^\perp$. By using (2.9), (2.10), (2.15), (2.1) and (3.2) with definition 2.2, we get

$$\frac{1}{\lambda^2} g'((\nabla\beta_*)(Z,W), \beta_*(X)) = -g(\mathcal{T}_Z \phi^2 W, X) - g(h\nabla_Z \jmath \text{Œ}W, X).$$

Since β is a $\mathcal{PWSSCS}$, by using definition 2.2, the second term in the right hand side of above equation can be turn into: $g(h\nabla_Z \jmath \text{Œ}W, X) = -\frac{1}{\text{˘}2} g'((\nabla\text{fi}_*)(Z, \jmath\text{Œ}W), \text{fi}_* X) + \frac{1}{\text{˘}2} g'(\nabla^{\text{fi}}_Z \text{fi}_* \jmath \text{Œ}W, \text{fi}_* X)$. By using this in above equation, we may have

$$\begin{aligned} \frac{1}{\lambda^2} g'((\nabla\beta_*)(Z,W), \beta_*(X)) = & - g(\mathcal{T}_Z \phi^2 W, X) + \frac{1}{\lambda^2} g'((\nabla\beta_*)(Z, \eta\phi W), \beta_*(X)) \\ & - \frac{1}{\lambda^2} g'(\nabla^{\beta}_Z \beta_* \eta\phi W, \beta_*(X)). \end{aligned}$$

Finally using the conformality of β with Lemma 3.3, we get

$$\begin{aligned} & \frac{1}{\lambda^2} g'((\nabla\beta_*)(Z,W), \beta_*(X)) \\ & = \frac{1}{\lambda^2} g'(Z(\ln\lambda)\beta_* \eta\phi W + \eta\phi W(\ln\lambda)\beta_* Z - g(Z, \eta\phi W)\beta_*(\operatorname{grad}\ln\lambda), \beta_* X) \\ & - g(\mathcal{T}_Z \phi^2 W, X) - \frac{1}{\lambda^2} g'(\nabla^{\beta}_Z \beta_* \eta\phi W, \beta_*(X)), \end{aligned}$$

from which we can get the is part (i) of the theorem. For part (ii), take into consideration $g'((\nabla\beta_*)(U,V), \beta_*(Y))$, for any $U, V \in \Gamma(\mathfrak{D}^\theta)$ and $Y \in \Gamma(\ker\beta_*)^\perp$. From equations (2.15) with definition 2.2, we can write $g'((\nabla\beta_*)(U,V),\beta_*(Y)) = -\lambda^2 g(\nabla_U V, Y)$. In the light of relation (2.1), (2.4), (3.4) and (3.2), we get

$$\frac{1}{\lambda^2}g'((\nabla\beta_*)(U,V),\beta_*(Y)) = -g([Y,U],V) - g(\nabla_Y\phi^2 U, V) - g(\mathcal{A}_Y\eta\phi U, V) - g(h\nabla_Y JU, JV) - g(\mathcal{A}_Y JU, ŒV).$$

Taking account the fact from equation (2.10) with Lemma 3.3, we may have

$$\frac{1}{\lambda^2}g'((\nabla\beta_*)(U,V),\beta_*(Y)) = -g([Y,U],V) + \sin 2\theta Y(\theta)g(U,V) - \cos^2\theta g(v\nabla_Y U, V) - g(\mathcal{A}_Y\eta\phi U, V) - g(\mathcal{A}_Y\eta U, \phi V) - g(h\nabla_Y JU, JV).$$

By using equation (2.15), Lemma 2.1 and horizontal conformality of β, we may have

$$\frac{1}{\lambda^2}g'((\nabla\beta_*)(U,V),\beta_*(Y)) = - g([Y,U],V) + \sin 2\theta Y(\theta)g(U,V) - \cos^2\theta g(v\nabla_Y U, V) - g(\mathcal{A}_Y\eta\phi U, V) - g(\mathcal{A}_Y\eta U, \phi V) - \frac{1}{\lambda^2}g'(Y(\ln\lambda)\beta_*\eta U + \eta U(\ln\lambda)\beta_* Y - g(\eta U, Y)\beta_*(\mathrm{grad}\ln\lambda), \beta_* Y).$$

This is the proof of part (ii). For (iii) part, by using equations (2.1), (2.4), (3.1), (3.2) and consider Lemma 3.3, we can write

$$\frac{1}{\lambda^2}g'((\nabla\beta_*)(X,U_1),\beta_* Y) = -g(\mathcal{A}_X\phi\mathbb{R}U, \mathfrak{C}Y) - g(v\nabla_X Œ\mathbb{R}U, \mathfrak{B}Y) - g(\mathcal{A}_X J\bar{\mathbb{R}}U, \mathfrak{B}Y) - g(\nabla_X F\phi\bar{\mathbb{R}}U, Y) - g(h\nabla_X JŒ\bar{\mathbb{R}}U, Y),$$

for any $U \in \Gamma(\mathfrak{D}^\theta)$ and $X, Y \in \Gamma(\mathrm{kerfi}_*)^\perp$. Since $\mathcal{PWSSCS}$, then by using Lemma 3.3 and definition of conformality 2.2, the above equation turn into

$$\begin{aligned}
&\frac{1}{\lambda^2}g'((\nabla\beta_*)(X,U_1),\beta_* Y)\\
&= -g(\mathcal{A}_X\phi\mathbb{R}U, \mathfrak{C}Y) - g(v\nabla_X Œ\mathbb{R}U, \mathfrak{B}Y) - g(\mathcal{A}_X J\bar{\mathbb{R}}U, \mathfrak{B}Y)\\
&\quad + \sin 2\theta X(\theta)g(\bar{\mathbb{R}}U, Y) - \cos^2\theta g(\nabla_X\bar{\mathbb{R}}U, Y) - g_1(\nabla_X\eta\phi\bar{R}U, Y)\\
&\quad - \frac{1}{\lambda^2}g'(X(\ln\lambda)\beta_*\eta\bar{\mathbb{R}}U + \eta\bar{\mathbb{R}}U(\ln\lambda)\beta_* X - g(X, \eta\bar{\mathbb{R}}U)\beta_*(\mathrm{grad}\lambda), \beta_*\mathfrak{C}Y)\\
&\quad + \frac{1}{\lambda^2}g'(\nabla^\beta_X\beta_*\eta\bar{\mathbb{R}}U, \beta_*\mathfrak{C}Y).
\end{aligned}$$

This completes the proof of theorem. □

Theorem 4.7. *Let $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ be a $\mathcal{PWSSCS}$ with emi-slant function θ such that $\mathcal{LPRM}$ $(\bar{B}_1, F, g)$ and $(\bar{B}_2, g')$ a $\mathcal{RM}$. Suppose that β is $\mathfrak{D}^\theta$-F-pluriharmonic. Then $\mathfrak{D}^\theta$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

$$\begin{aligned}
\nabla^\beta_{FX_1}\beta_* FY_1 + \nabla^\beta_{\eta X_1}\beta_*\eta Y_1 =& \beta_*(h\nabla_{ŒX_1}JY_1 + \mathcal{A}_{JX_1}ŒY_1 + \mathcal{T}_{ŒX_1}Œ^2\mathbb{R}ŒY_1 + h\nabla_{ŒX_1}JŒ\mathbb{R}ŒY_1)\\
&+ \beta_*(\mathcal{T}_{\phi X_1}\eta^2\bar{\mathbb{R}}\phi Y_1 + h\nabla_{ŒX_1}JŒ\bar{\mathbb{R}}ŒY_1 + \mathcal{T}_{ŒX_1}ŒJ\bar{\mathbb{R}}ŒY_1)\\
&- \cos^2\theta\beta_*(\nabla_{\phi X_1}\bar{\mathbb{R}}\phi Y_1),
\end{aligned}$$

for any $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and using the pluriharmonicity of F with equation (2.15), we get

$$\beta_* \nabla_{X_1} Y_1 = \nabla^{\beta}_{FX_1} \beta_* F Y_1 - \beta_* \nabla_{FX_1} F Y_1. \tag{4.8}$$

By using equation (3.2), second term reduces as: $\beta_*\nabla_{\phi X_1}\phi Y_1 + \beta_*\nabla_{\phi X_1}\eta Y_1 + \beta_*\nabla_{\eta X_1}\phi Y_1 + \beta_*\nabla_{\phi X_1}\eta Y_1$. Now, equation (4.8) can be write as

$$\begin{aligned}\beta_*\nabla_{X_1}Y_1 =&\nabla^{\beta}_{FX_1}\beta_* FY_1 - \beta_*\nabla_{\phi X_1}\phi Y_1 - \beta_*\nabla_{\phi X_1}\eta Y_1\\ &- \beta_*\nabla_{\eta X_1}\phi Y_1 - \beta_*\nabla_{\phi X_1}\eta Y_1.\end{aligned}$$

Taking account the fact that β is $\mathcal{PWSSCS}$ with using equations (2.10), (2.11), (2.15) and (3.1), we have

$$\begin{aligned}\beta_*\nabla_{X_1}Y_1 =& -\beta_*(\mathcal{T}_{\phi X_1}\eta Y_1 + h\nabla_{ŒX_1}\jmath Y_1 + \mathcal{A}_{\jmath X_1}ŒY_1 + v\nabla_{\jmath X_1}ŒY_1)\\ &+ \{\eta X_1(\ln\lambda)\beta_*\eta Y_1 + \eta Y_1(\ln\lambda)\beta_*\eta X_1 - g(\eta X_1, \eta Y_1)\beta_*(\mathrm{grad}\ln\lambda)\}\\ &- \nabla^{\beta}_{FX_1}\beta_* FY_1 - \nabla^{\beta}_{\eta X_1}\beta_*\eta Y_1 + \beta_*(F\nabla_{\phi X_1}F(\mathbb{R}\phi Y_1 + \bar{\mathbb{R}}\phi Y_1)).\end{aligned}$$

In the last term in the right hand side of above equation with Lemma 3.2 and equations (2.10) and (2.11), we may have

$$\begin{aligned}\beta_*\nabla_{X_1}Y_1 =&\{\eta X_1(\ln\lambda)\beta_*\eta Y_1 + \eta Y_1(\ln\lambda)\beta_*\eta X_1 - g(\eta X_1,\eta Y_1)\beta_*(\mathrm{grad}\ln\lambda)\}\\ &+ \beta_*(\mathcal{T}_{\phi X_1}\phi^2\mathbb{R}\phi Y_1 + v\nabla_{ŒX_1}Œ^2\mathbb{R}ŒY_1 + \mathcal{T}_{ŒX_1}\jmath Œ\mathbb{R}ŒY_1 + h\nabla_{ŒX_1}\jmath Œ\mathbb{R}ŒY_1)\\ &+ \sin 2\theta\phi X_1(\theta)\beta_*(\bar{\mathbb{R}}\phi Y_1) - \cos^2\theta\beta_*(\nabla_{\phi X_1}\bar{\mathbb{R}}\phi Y_1) + \beta_*(\mathcal{T}_{\phi X_1}\eta\phi\bar{\mathbb{R}}\phi Y_1\\ &+ h\nabla_{ŒX_1}\jmath Œ\bar{\mathbb{R}}ŒY_1) + ﬁ_*(\mathcal{T}_{ŒX_1}Œ\jmath\bar{\mathbb{R}}ŒY_1 + v\nabla_{ŒX_1}Œ\jmath\bar{\mathbb{R}}ŒY_1 + \mathcal{T}_{ŒX_1}\jmath^2\bar{\mathbb{R}}ŒY_1\\ &+ v\nabla_{ŒX_1}\jmath^2\bar{\mathbb{R}}ŒY_1) + ﬁ_*(\mathcal{T}_{ŒX_1}\jmath Y_1 + h\nabla_{ŒX_1}\jmath Y_1 + \mathcal{A}_{\jmath X_1}ŒY_1 + v\nabla_{\jmath X_1}ŒY_1)\\ &- \nabla^{\beta}_{FX_1}\beta_* FY_1 - \nabla^{\beta}_{\eta X_1}\beta_*\eta Y_1.\end{aligned}$$

□

Theorem 4.8. *Let* $\beta : (\bar{B}_1, F, g) \to (\bar{B}_2, g')$ *be a* $\mathcal{PWSSCS}$ *with semi-slant function* θ *such that* $(\bar{B}_1, F, g)$ *a* $\mathcal{LPRM}$ *and* $(\bar{B}_2, g')$ *a* $\mathcal{RM}$. *Suppose that* β *is* $((\ker\beta_*)^\perp - \ker\beta_*)$*-F-pluriharmonic. Then the horizontal distribution* $(\ker\beta_*)^\perp$ *defines totally geodesic foliation on* $\bar{B}_1$ *if and only if*

$$\begin{aligned}&\nabla^{\beta}_{\mathfrak{C}X}\beta_*\eta\phi\mathbb{R}U + \cos^2\theta(\eta v\nabla_{\mathfrak{C}X}\bar{\mathbb{R}}U - \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\bar{\mathbb{R}}U) - \sin 2`\mathfrak{C}X(`)ﬁ_*\jmath\bar{\mathbb{R}}U\\ &= -\nabla^{\beta}_{\mathfrak{C}X}\beta_*\eta\phi\bar{\mathbb{R}}U + \mathfrak{C}X(\ln\lambda)\beta_*\eta\phi\mathbb{R}U + \eta\phi\mathbb{R}U(\ln\lambda)\beta_*\mathfrak{C}X - g(\mathfrak{C}X, \eta\phi\mathbb{R}U)\beta_*(\mathrm{grad}\ln\lambda)\\ &+ \mathfrak{C}X(\ln\lambda)\beta_*\eta\phi\bar{\mathbb{R}}U + \eta\phi\bar{\mathbb{R}}U(\ln\lambda)\beta_*\mathfrak{C}X - g(\mathfrak{C}X, \eta\phi\bar{\mathbb{R}}U)\beta_*(\mathrm{grad}\ln\lambda)\\ &+ \beta_*\{\mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\phi^2\mathbb{R}U + \eta v\nabla_{\mathfrak{C}X}Œ^2\mathbb{R}U + \jmath\mathcal{A}_{\mathfrak{C}X\jmath}Œ\mathbb{R}U + \jmath\mathcal{A}_{\mathfrak{C}X\jmath}Œ\bar{\mathbb{R}}U\}\\ &+ \beta_*\{\mathcal{A}_X U + h\nabla_{\mathfrak{B}X}ŒU + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\jmath U + \jmath v\nabla_{\mathfrak{B}X}\mathfrak{B}\jmath U\} + \nabla^{ﬁ}_{FX}ﬁ_*\jmath U,\end{aligned}$$

for any $X \in \Gamma(\ker\beta_*)^\perp$ *and* $U \in \Gamma(\ker\beta_*)$.

Proof. For any $X \in \Gamma(\ker\beta_*)^\perp, U \in \Gamma(\ker\beta_*)$ and using equations (2.15), (3.1), (3.2), (2.9) with considering the fact of pluriharminicity of F, we can write

$$\begin{aligned}\beta_*(\nabla_{\mathfrak{C}X}\eta U) = &-\beta_*\nabla_X U + \nabla^\beta_{FX}\beta_* FU - \beta_*(\mathcal{T}_{\mathfrak{B}X}\phi U + h\nabla_{\mathfrak{B}X}Œ U)\\ &- \beta_*(\nabla_{\mathfrak{B}X}\eta U + \nabla_{\mathfrak{C}X}\phi U).\end{aligned} \tag{4.9}$$

The second last term of the above equation, by using the equations (2.1) and (2.2) turn into: $\beta_*(\nabla_{\mathfrak{B}X}\eta U) = \beta_*(F\nabla_{\mathfrak{B}X}F\eta U)$ whereas, the last term reduces into: $\beta_*(F\nabla_{\mathfrak{C}X}\phi U) = \beta_*(F\nabla_{\mathfrak{C}X}F\phi U)$. By using these facts into (4.9) reduces to

$$\begin{aligned}\beta_*(\nabla_{\mathfrak{C}X}\eta U) = &-\beta_*\nabla_X U + \nabla^\beta_{FX}\beta_* FU - \beta_*(\mathcal{T}_{\mathfrak{B}X}\phi U + h\nabla_{\mathfrak{B}X}Œ U)\\ &+ \beta_*(F\nabla_{\mathfrak{C}X}F\phi U) + \beta_*(F\nabla_{\mathfrak{B}X}F\eta U).\end{aligned}$$

Now, by using equation (3.1), (3.2), (3.4), (2.15) with Lemma 3.2, we can write

$$\begin{aligned}\beta_*(\nabla_{\mathfrak{C}X}\eta U) =&\beta_*(\mathfrak{A}_X U + v\nabla_X U - \mathcal{T}_{\mathfrak{B}X}Œ U + h\nabla_{\mathfrak{B}X}Œ U + \jmath\mathcal{A}_{\mathfrak{C}X}\jmath Œ\bar{\mathbb{R}}U)\\ &+ \beta_*\{\mathfrak{B}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\eta U + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\eta U + \phi v\nabla_{\mathfrak{B}X}\mathfrak{B}\jmath U + \jmath v\nabla_{\mathfrak{B}X}\mathfrak{B}\jmath U\}\\ &+ \beta_*\{\mathfrak{B}\mathcal{A}_{\mathfrak{C}X}\phi^2\mathbb{R}U + \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\phi^2\mathbb{R}U + \phi v\nabla_{\mathfrak{C}X}Œ^2\mathbb{R}U + \jmath v\nabla_{\mathfrak{C}X}Œ^2\mathbb{R}U\}\\ &+ \beta_*\{\phi\mathcal{A}_{\mathfrak{C}X}\eta\phi\mathbb{R}U + \eta\mathcal{A}_{\mathfrak{C}X}\eta\phi\mathbb{R}U + \mathfrak{B}h\nabla_{\mathfrak{C}X\jmath}Œ\mathbb{R}U + Œ\mathcal{A}_{\mathfrak{C}X\jmath}Œ\bar{\mathbb{R}}U\}\\ &+ \nabla^\beta_{\mathfrak{C}X}\beta_*\eta\phi\mathbb{R}U + (\nabla\beta_*)(\mathfrak{C}X, \eta\phi\mathbb{R}U) + \nabla^\beta_{\mathfrak{C}X}\beta_*\eta\phi\bar{\mathbb{R}}U + (\nabla\beta_*)(\mathfrak{C}X, \eta\phi\bar{\mathbb{R}}U)\\ &+ \beta_*\{-\sin 2\grave{}\,\mathfrak{C}X(\theta)\eta\bar{\mathbb{R}}U + \cos^2\grave{}\,F\nabla_{\mathfrak{C}X}\bar{\mathbb{R}}U\} + \nabla^\beta_{\mathfrak{C}X}\beta_*\eta\phi\mathbb{R}U\nabla^\beta_{FX}\beta_* FU.\end{aligned}$$

Since β is a $\mathcal{PWSSCS}$, then by using Lemma 2.1, above equation finally turn into

$$\begin{aligned}&\beta_*(\nabla_{\mathfrak{C}X}\eta U)\\ &= \cos^2\theta(\eta v\nabla_{\mathfrak{C}X}\bar{\mathbb{R}}U + \mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\bar{\mathbb{R}}U) - \sin 2\grave{}\,\mathfrak{C}X(\grave{}\,)ﬁ_*\jmath\bar{\mathbb{R}}U - \nabla^{ﬁ}_{\mathfrak{C}X}ﬁ_*\jmath Œ\mathbb{R}U - \nabla^{ﬁ}_{\mathfrak{C}X}ﬁ_*\jmath Œ\bar{\mathbb{R}}U\\ &+ \mathfrak{C}X(\ln\lambda)\beta_*\eta\phi\mathbb{R}U + \eta\phi\mathbb{R}U(\ln\lambda)\beta_*\mathfrak{C}X - g(\mathfrak{C}X, \eta\phi\mathbb{R}U)\beta_*(\mathrm{grad}\ln\lambda)\\ &+ \mathfrak{C}X(\ln\lambda)\beta_*\eta\phi\bar{\mathbb{R}}U + \eta\phi\bar{\mathbb{R}}U(\ln\lambda)\beta_*\mathfrak{C}X - g(\mathfrak{C}X, \eta\phi\bar{\mathbb{R}}U)\beta_*(\mathrm{grad}\ln\lambda)\\ &+ \beta_*\{\mathfrak{C}\mathcal{A}_{\mathfrak{C}X}\phi^2\mathbb{R}U + \eta v\nabla_{\mathfrak{C}X}Œ^2\mathbb{R}U + \jmath\mathcal{A}_{\mathfrak{C}X\jmath}Œ\mathbb{R}U + \jmath\mathcal{A}_{\mathfrak{C}X\jmath}Œ\bar{\mathbb{R}}U\}\\ &+ \beta_*\{\mathcal{A}_X U + h\nabla_{\mathfrak{B}X}Œ U + \mathfrak{C}\mathcal{T}_{\mathfrak{B}X}\mathfrak{B}\jmath U + \jmath v\nabla_{\mathfrak{B}X}\mathfrak{B}\jmath U\} + \nabla^{ﬁ}_{FX}ﬁ_*\jmath U,\end{aligned}$$

from which we can get the desired result. □

Now, at last we show how to prove the existence of $\mathcal{PWSSCS}$ from $\mathcal{APRM}$ onto a $\mathcal{RM}$ using non-trivial example.

Example 4.1. *Consider a map $\beta : \mathbb{R}^{10} \to \mathbb{R}^4$ such that*

$$\beta(w_1, w_2, ..., w_{10}) = \sqrt{\pi}\left(\frac{w_4 - w_6}{\sqrt{2}}, w_9, \frac{w_5 - w_7}{\sqrt{2}}, w_{10}\right)$$

Then it follows that $\mathfrak{D} = \{U_1 = \frac{\partial}{\partial w_1}, U_2 = \frac{\partial}{\partial w_2}\}$ and

$$\mathfrak{D}^\theta = \left\langle U_3 = \frac{\partial}{\partial w_2}, U_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w_4} - \frac{\partial}{\partial w_6}\right), U_5 = \frac{\partial}{\partial w_8}, U_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w_5} - \frac{\partial}{\partial w_7}\right)\right\rangle.$$

Moreover,

$$(\ker f_*)^{\perp} = \left\langle X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w_4} + \frac{\partial}{\partial w_6}\right), X_2 = \frac{\partial}{\partial w_1}, X_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w_4} + \frac{\partial}{\partial w_6}\right), X_4 = \frac{\partial}{\partial w_{10}}\right\rangle.$$

Then β is a $\mathcal{PWSSCS}$ from $\mathcal{LPRM}$an manifold onto Riemannian manifold with semi-slant angle $\theta = \frac{\pi}{4}$ and dilation $\lambda = \sqrt{\pi}$.

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