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Global Properties of a Discrete SARS-CoV-2/HIV Co-Dynamics Model

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Abstract. Coronavirus disease 2019 (COVID-19), which is caused by the virus known as severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), is a respiratory disease. In this paper, we analyze the global stability of a discrete SARS-CoV-2/HIV co-dynamics model. We create the discrete model by applying a nonstandard finite difference (NSFD) method. We demonstrate that NSFD retains essential solution properties, including positivity and boundedness. We determine the fixed points and identify their existence conditions. We investigate the global stability of these fixed points through the application of the Lyapunov method. To complement our analytical findings, we present numerical simulations.

1. Introduction

Coronavirus disease 2019 (COVID-19), a new epidemic that surfaced in late 2019 in China, is a respiratory illness caused by a virus known as severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). SARS-CoV-2 is indeed an RNA virus, and it belongs to the Coronaviridae family [1]. The primary target of SARS-CoV-2 is the epithelial cells (ECs) in the respiratory system. Specifically, the virus primarily infects cells in the upper respiratory tract, including those in the nasal passages and throat, as well as cells in the lower respiratory tract, such as those in the lungs [2]. This is why COVID-19 primarily presents with respiratory symptoms.

Human Immunodeficiency Virus (HIV), belongs to the family of viruses known as lentiviruses [3]. It specifically targets the immune system, primarily infecting $CD4^+$ T cells, which play a crucial role in the body's immune response [3]. Infected $CD4^+$ T cells begin generating new virus particles, thereby amplifying the infection and diminishing the population of uninfected $CD4^+$ T

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cells. This reduction of healthy CD4$^+$ T cells weakens the immune system, rendering the body incapable of defending against various other infections and illnesses. HIV has the ability to remain in a dormant state within certain viral reservoirs, such as latently infected CD4$^+$ T cells. These latently infected cells do not produce new viral particles, but they have the potential to be activated and transformed into actively infected cells. This presents a significant challenge for the immune system.

There are signs of a rising trend in HIV transmission, particularly in conjunction with other blood-borne infections such as hepatitis B (and C) viruses, as well as sexually transmitted diseases like human papillomavirus (HPV), syphilis, Herpes simplex virus (HSV), and chlamydia. The first reported case of SARS-CoV-2/HIV coinfection involved a 61-year-old man from China and subsequent cases were documented in Spain, Italy and the USA [3]. Common symptoms of coinfection included fever, cough, and shortness of breath [3]. Research indicates that HIV patients face an elevated risk of experiencing severe COVID-19 symptoms [4]. This risk is further heightened in HIV patients who are not undergoing antiretroviral therapy or have low CD4+ T cell counts [4,5]. Additionally, the severity risk increases in the presence of other comorbidities such as hypertension, diabetes, respiratory disease, cardiovascular disease, and chronic kidney disease [4, 6, 7]. According to WHO recommendations [8], numerous COVID-19 vaccines are considered safe for individuals living with HIV.

Studying mathematical modeling of the dynamics of viral infection within the host may be very useful in understanding the dynamic behavior of the virus and its target cells as well as immune cells. This study also helps in understanding the effectiveness of medications, whether individually or in combination. Coinfection models are essential for comprehending the dynamics of SARSCoV-2 infection in individuals with HIV, elucidating the role of the immune system, supporting medical research and identifying improved approaches to treat this vulnerable patient group. In [9] HIV and SARS-CoV-2 co-dynamics model was formulated as:

$$\dot{X} = \xi - \phi_X X - \varrho V X, \tag{1.1}$$

$$\dot{N} = \varrho V X - (\rho + \phi_N) N, \tag{1.2}$$

$$\dot{Y} = \rho N - \phi_Y Y - \mu Y Q, \tag{1.3}$$

$$\dot{V} = a Y - \phi_V V, \tag{1.4}$$

$$\dot{Q} = \eta + \kappa Y Q - \phi_Q Q - \theta D Q, \tag{1.5}$$

$$\dot{Z} = (1 - b)\theta D Q - (\alpha + \phi_Z) Z, \tag{1.6}$$

$$\dot{E} = b\theta D Q + \alpha Z - \phi_E E, \tag{1.7}$$

$$\dot{D} = \lambda E - \phi_D D, \tag{1.8}$$

where $(X, N, Y, V, Q, Z, E, D) = (X(t), N(t), Y(t), V(t), Q(t), Z(t), E(t), D(t))$ represent the concentrations of uninfected ECs, latently infected ECs, actively infected ECs, free SARS-CoV-2 particles, uninfected CD4$^+$T cells, latently infected CD4$^+$T cells, actively infected CD4$^+$T cells and free HIV particles at time t, respectively. ECs are produced from a source at a constant rate ξ, die at rate $\phi_X X$ and get infected by SARS-CoV-2 at rate ϱVX. Latently infected ECs proliferate at rate ϱVX, turn into actively infected cells at rate ρN and die at rate $\phi_N N$. Actively infected ECs die at rate $\phi_Y Y$ and are indirectly eliminated by $CD4^+T$ cells at rate μYQ. SARS-CoV-2 particles are produced from infected ECs at rate aY and die at rate $\phi_V V$. Uninfected $CD4^+T$ cells are produced at a constant rate η, stimulated by infected ECs at rate κYQ, die at rate $\phi_Q Q$, and get infected by HIV at rate θDQ. A fraction $b \in [0,1]$ of new infected $CD4^+T$ cells will be active and rest $1-b$ will be latent. Latently infected $CD4^+T$ cells are transmitted into active cells at rate αZ and die at rate $\phi_Z Z$. Actively infected $CD4^+T$ die at rate $\phi_E E$. HIV particles are produced by infected $CD4^+T$ cells at rate λE and die at rate $\phi_D D$.

Model (1.1)-(1.8) is given by nonlinear continuous-time system. The exact analytical solution for this system remains unknown. Consequently, it is only possible to derive an approximate discrete-time solution for such a system. Furthermore, real-world experimental data from patients can only be collected at specific discrete-time points. Classical discretization methods, such as Euler, Runge-Kutta, and others, can lead to numerical instability and bias when dealing with large step sizes when used to discretize nonlinear continuous-time models, as pointed out in [10]. Conversely, it is crucial to employ a discretization method that preserves the fundamental qualitative characteristics of the original continuous-time model. Mickens [11] introduced a nonstandard finite difference (NSFD) scheme for solving differential equations, which has been effectively applied to various models for within-host viral mono-infection in references [12]- [21]. Furthermore, in [22] and [23], NSFD was employed to solve HIV-1/HTLV-I co-dynamics models. However, the discretization of continuous-time SARS-CoV-2/HIV co-dynamics models has not been previously investigated.

The aim of our current research is to employ the NSFD method for discretizing the SARS-CoV-2/HIV co-dynamics model. We demonstrate that the solutions of the discretized SARS-CoV-2/HIV co-dynamics model are both positive and ultimately bounded. We find four fixed points of the model and establish their existence conditions, which depend on four threshold parameters (TP_i, $i = 1, 2, 3, 4$). To demonstrate the global stability of all fixed points, we apply the Lyapunov method for discrete-time system. We also conduct numerical simulations to validate the theoretical findings we have obtained.

2. The Discrete SARS-CoV-2/HIV co-dynamics model

In this section, we discretize model (1.1)-(1.8) by using NSFD approach as:

$$\frac{X_{n+1}-X_n}{\Pi(d)} = \xi - \phi_X X_{n+1} - \varrho V_n X_{n+1}, \tag{2.1}$$

$$\frac{N_{n+1}-N_n}{\Pi(d)} = \varrho V_n X_{n+1} - (\rho + \phi_N) N_{n+1}, \tag{2.2}$$

$$\frac{Y_{n+1}-Y_n}{\Pi(d)} = \rho N_{n+1} - \phi_Y Y_{n+1} - \mu Y_{n+1} Q_{n+1}, \tag{2.3}$$

$$\frac{V_{n+1}-V_n}{\Pi(d)} = aY_{n+1} - \phi_V V_{n+1}, \tag{2.4}$$

$$\frac{Q_{n+1}-Q_n}{\Pi(d)} = \eta + \kappa Y_{n+1}Q_{n+1} - \phi_Q Q_{n+1} - \theta D_n Q_{n+1}, \tag{2.5}$$

$$\frac{Z_{n+1}-Z_n}{\Pi(d)} = (1-b)\theta D_n Q_{n+1} - (\alpha+\phi_Z)Z_{n+1}, \tag{2.6}$$

$$\frac{E_{n+1}-E_n}{\Pi(d)} = b\theta D_n Q_{n+1} + \alpha Z_{n+1} - \phi_E E_{n+1}, \tag{2.7}$$

$$\frac{D_{n+1}-D_n}{\Pi(d)} = \lambda E_{n+1} - \phi_D D_{n+1}. \tag{2.8}$$

where $d > 0$ is step size $(X_{n,}, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n)$ be an are approximation of the solution of system (1.1)-(1.8) at the time instants $t_n = nd$ and $n = \{0, 1, 2, ...\}$. Function $\Pi(d)$ is selected such that $\Pi(d) = d + O(d^2)$ [24] We consider

$$\Pi(d) = \frac{1-e^{-\phi_X d}}{\phi_X}. \tag{2.9}$$

The initial conditions of system (2.1)-(2.8) are

$$(X_0, N_0, Y_0, V_0, Q_0, Z_0, E_0, D_0) \in \mathbb{R}_+^8 = \{(X, N, Y, V, Q, Z, E, D) \mid X > 0, N > 0, Y > 0, \quad (2.10)$$
$$V > 0, Q > 0, Z > 0, E > 0, D > 0\}.$$

3. Preliminaries

Let $\sigma = \min\{\phi_X, \phi_N, \frac{\phi_Y}{2}, \phi_V, \phi_Q, \phi_Z, \frac{\phi_E}{2}, \phi_D\}$ and $W_1 = \xi + \frac{\mu}{\kappa}\eta, W_2 = \frac{2a}{\phi_Y}W_1, W_3 = \frac{\kappa}{\mu}W_1$, and $W_4 = \frac{2\kappa\lambda}{\mu\phi_E}W_1$ and define the region $\Gamma = \{(X, N, Y, V, Q, Z, E, D) : 0 < X, N, Y < W_1, 0 < V < W_2, 0 < Q, Z, E < W_3, 0 < D < W_4\}$.

Lemma 1. Any solution of the discrete model (2.1)-(2.8) under initial conditions (2.10) is positive and ultimately bounded.

Proof Eqs. (2.1)-(2.8) yield

$$X_{n+1} = \frac{\Pi(d)\xi + X_n}{1+\Pi(d)(\phi_X + \varrho V_n)}, \tag{3.1}$$

$$N_{n+1} = \frac{\Pi(d)\varrho V_n X_{n+1} + N_n}{1+\Pi(d)(\rho+\phi_N)}, \tag{3.2}$$

$$Y_{n+1} = \frac{\Pi(d)\rho N_{n+1} + Y_n}{1+\Pi(d)(\phi_Y + \mu Q_{n+1})}, \tag{3.3}$$

$$V_{n+1} = \frac{\Pi(d)aY_{n+1} + V_n}{1+\Pi(d)\phi_V}, \tag{3.4}$$

$$Q_{n+1} = \frac{\Pi(d)\eta + Q_n}{1+\Pi(d)(\phi_Q - \kappa Y_{n+1} + \theta D_n)}, \tag{3.5}$$

$$Z_{n+1} = \frac{\Pi(d)(1-b)\theta D_n Q_{n+1} + Z_n}{1+\Pi(d)(\alpha+\phi_Z)}, \tag{3.6}$$

$$E_{n+1} = \frac{\Pi(d)b\theta D_n Q_{n+1} + \Pi(d)\alpha Z_{n+1} + E_n}{1 + \Pi(d)\phi_E} \tag{3.7}$$

$$D_{n+1} = \frac{\Pi(d)\lambda E_{n+1} + D_n}{1 + \Pi(d)\phi_D}. \tag{3.8}$$

We show that $(X_1, N_1, Y_1, V_1, Q_1, Z_1, E_1, D_1)$ exists uniquely and is positive. If $n = 0$, then from equations Eqs. (3.1) and (3.2) and the initial conditions Eq. (2.10), we get $X_1 > 0$ and $N_1 > 0$. From Eqs. (3.3) and (3.5), we get

$$Q_1 = \frac{[\Pi(d)\eta + Q_0]\,[1 + \Pi(d)(\phi_Y + \mu Q_1)]}{\left[1 + \Pi(d)(\phi_Q + \theta D_0)\right][1 + \Pi(d)(\phi_Y + \mu Q_1)] - \kappa\Pi(d)\,[\Pi(d)\rho N_1 + Y_0]}. \tag{3.9}$$

Then

$$A_1 Q_1^2 + B_1 Q_1 + C_1 = 0, \tag{3.10}$$

where

$$\begin{aligned} A_1 &= \mu\Pi(d)(1 + \Pi(d)\phi_Q + \Pi(d)\theta D_0), \\ B_1 &= \left[1 + \Pi(d)\phi_Q + \Pi(d)\theta D_0\right][1 + \Pi(d)\phi_Y] - \kappa\Pi(d)\,[\Pi(d)\rho N_1 + Y_0] - \mu\Pi(d)\,[\Pi(d)\eta + Q_0]\,, \\ C_1 &= -\left[(1 + \Pi(d)\phi_Y)(\Pi(d)\eta + Q_0)\right]. \end{aligned} \tag{3.11}$$

Since $A_1 > 0$ and $C_1 < 0$, then $B_1^2 - 4A_1C_1 > 0$, and hence there exists a unique positive root of Eq. (3.10) $Q_1 > 0$. From Eqs. (3.3),(3.4),(3.6),(3.7) and (3.8), we get

$$Y_1 = \frac{\Pi(d)\rho N_1 + Y_0}{1 + \Pi(d)(\phi_Y + \mu Q_1)} > 0, \tag{3.12}$$

$$V_1 = \frac{\Pi(d)aY_1 + V_0}{1 + \Pi(d)\phi_V} > 0, \tag{3.13}$$

$$Z_1 = \frac{\Pi(d)(1 - b)\theta D_0 Q_1 + Z_0}{1 + \Pi(d)(\alpha + \phi_Z)} > 0, \tag{3.14}$$

$$E_1 = \frac{\Pi(d)b\theta D_0 Q_1 + \Pi(d)\alpha Z_1 + E_0}{1 + \Pi(d)\phi_E} > 0, \tag{3.15}$$

$$D_1 = \frac{\Pi(d)\lambda E_1 + D_0}{1 + \Pi(d)\phi_D} > 0. \tag{3.16}$$

Therefore, the solution $(X_1, N_1, Y_1, V_1, Q_1, Z_1, E_1, D_1)$ exists uniquely and is positive. Repeating the above process for $n = 1$ we can prove that $(X_2, N_2, Y_2, V_2, Q_2, L_2, A_2)$ exists uniquely and is positive. Mathematical induction provides that for all $n \geq 0$, $(X_n, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n)$ exists uniquely and is positive. By induction, we get $X_n > 0, N_n > 0, Y_n > 0, V_n > 0, Q_n > 0, Z_n > 0, E_n > 0$, and $D_n > 0\ \forall n \geq 0$. Define a sequence sequence M_n:

$$M_n = X_n + N_n + Y_n + \frac{\phi_Y}{2a}V_n + \frac{\mu}{\kappa}(Q_n + Z_n + E_n) + \frac{\mu\phi_E}{2\kappa\lambda}D_n.$$

Then

$$\begin{aligned}
M_{n+1}-M_n &= (X_{n+1}-X_n)+(N_{n+1}-N_n)+(Y_{n+1}-Y_n)+\frac{\phi_Y}{2a}(V_{n+1}-V_n)+\frac{\mu}{\kappa}\left[(Q_{n+1}-Q_n)\right.\\
&\quad \left.+(Z_{n+1}-Z_n)+(E_{n+1}-E_n)\right]+\frac{\mu\phi_E}{2\kappa\lambda}(D_{n+1}-D_n)\\
&= \Pi(d)\left[\xi-\phi_X X_{n+1}-\phi_N N_{n+1}-\frac{\phi_Y}{2}Y_{n+1}-\frac{\phi_Y}{2a}\phi_V V_{n+1}+\frac{\mu}{\kappa}\eta-\frac{\mu}{\kappa}\phi_Q Q_{n+1}-\frac{\mu}{\kappa}\phi_Z Z_{n+1}\right.\\
&\quad \left.-\frac{\mu}{2\kappa}\phi_E E_{n+1}-\frac{\mu\phi_E}{2\kappa\lambda}\phi_D D_{n+1}\right]\\
&\le \Pi(d)\left(\xi+\frac{\mu}{\kappa}\eta\right)-\Pi(d)\sigma\left[X_{n+1}+N_{n+1}+Y_{n+1}+\frac{\phi_Y}{2a}V_{n+1}+\frac{\mu}{\kappa}(Q_{n+1}+Z_{n+1}+E_{n+1})\right.\\
&\quad \left.+\frac{\mu\phi_E}{2\kappa\lambda}D_{n+1}\right]\\
&= \Pi(d)\left(\xi+\frac{\mu}{\kappa}\eta\right)-\Pi(d)\sigma M_{n+1}.
\end{aligned}$$

Hence

$$M_{n+1}\le \frac{M_n}{1+\Pi(d)\sigma}+\frac{\Pi(d)\left(\frac{\xi\kappa+\mu\eta}{\kappa}\right)}{1+\Pi(d)\sigma}.$$

Lemma 2.2 in [25] gives

$$M_n\le\left(\frac{1}{1+\Pi(d)\sigma}\right)^n M_0+\left(\frac{\xi\kappa+\mu\eta}{\kappa\sigma}\right)\left[1-\left(\frac{1}{1+\Pi(d)\sigma}\right)^n\right].$$

This gives, $\limsup_{n\to\infty} M_n\le W_1$, $\limsup_{n\to\infty} X_n\le W_1$, $\limsup_{n\to\infty} N_n\le W_1$, $\limsup_{n\to\infty} Y_n\le W_1$, $\limsup_{n\to\infty} V_n\le W_2$, $\limsup_{n\to\infty} Q_n \le W_3$, $\limsup_{n\to\infty} Z_n \le W_3$, $\limsup_{n\to\infty} E_n \le W_3$ and $\limsup_{n\to\infty} D_n \le W_4$. Therefore, $(X_{n,}, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n)$ converges to Γ as $n\to\infty$.

4. Fixed points

The fixed points of (2.1)-(2.8) satisfy the following:

$$\begin{aligned}
0 &= \xi-\phi_X X-\varrho VX,\\
0 &= \varrho VX-(\rho+\phi_N)N,\\
0 &= \rho N-\phi_Y Y-\mu YQ,\\
0 &= aY-\phi_V V,\\
0 &= \eta+\kappa YQ-\phi_Q Q-\theta DQ,\\
0 &= (1-b)\theta DQ-(\alpha+\phi_Z)Z,\\
0 &= b\theta DQ+\alpha Z-\phi_E E,\\
0 &= \lambda E-\phi_D D.
\end{aligned}$$

We find that the system admits four fixed points.

1) Infection-free fixed point $FP_0 = (X^0, 0, 0, 0, Q^0, 0, 0, 0)$, where $X^0 = \frac{\xi}{\phi_X}$ and $Q^0 = \frac{\eta}{\phi_Q}$, which always exists.

2) HIV mono-infection fixed point $FP_1 = (\hat{X}, 0, 0, 0, \hat{Q}, \hat{Z}, \hat{E}, \hat{D})$, where

$$\begin{aligned}
\hat{X} &= X^0 = \frac{\xi}{\phi_X}, \quad \hat{Q} = \frac{\phi_E \phi_D (\phi_z + \alpha)}{\theta \lambda (\alpha + \phi_z b)} = \frac{Q^0}{TP_1}, \\
\hat{Z} &= (1-b)\left[\frac{\eta}{\phi_Z + \alpha} - \frac{\phi_Q \phi_E \phi_D}{\theta \lambda (\alpha + \phi_z b)}\right] = \frac{\phi_Q \phi_E \phi_D (1-b)}{\theta \lambda (\alpha + \phi_z b)}(TP_1 - 1), \\
\hat{E} &= -\frac{\phi_Q \phi_D}{\theta \lambda} + \frac{\eta(\alpha + \phi_z b)}{\phi_E(\phi_Z + \alpha)} = \frac{\phi_Q \phi_D}{\theta \lambda}(TP_1 - 1), \\
\hat{D} &= -\frac{\phi_Q}{\theta} + \frac{\lambda \eta (\alpha + \phi_z b)}{\phi_E \phi_D (\phi_Z + \alpha)} = \frac{\phi_Q}{\theta}(TP_1 - 1).
\end{aligned}$$

Where $TP_1 = \frac{\eta \theta \lambda (\alpha + \phi_z b)}{\phi_Q \phi_E \phi_D (\phi_z + \alpha)}$. Here, the threshold parameter TP_1 represents the basic reproduction of HIV mono-infection. It determines the establishment of HIV infection. We see that $\hat{X}$ and $\hat{Q}$ is always positive, while $\hat{Z}, \hat{E}$ and $\hat{D}$ are positive if $TP_1 > 1$. Therefore, FP_1 exists when $TP_1 > 1$.

3) SARS-CoV-2 mono-infection fixed point $FP_2 = (\tilde{X}, \tilde{N}, \tilde{Y}, \tilde{V}, \tilde{Q}, 0, 0, 0)$ where

$$\tilde{Y} = \frac{\phi_V}{a}\tilde{V}, \ \tilde{Q} = \frac{\eta}{\phi_Q - \kappa \tilde{Y}}, \ \tilde{X} = \frac{(\rho + \phi_N)(\tilde{Y}\phi_Y + \tilde{Q}\tilde{Y}\mu)}{\rho \varrho \tilde{V}}, \ \tilde{N} = \frac{\tilde{Y}\phi_Y + \tilde{Q}\tilde{Y}\mu}{\rho}, \tag{4.1}$$

and $\tilde{V}$ satisfies the following equation:

$$\frac{T_1 \tilde{V}^2 + T_2 \tilde{V} + T_3}{a \rho \varrho (a \phi_Q - \kappa \phi_V \tilde{V})} = 0, \tag{4.2}$$

where

$$\begin{aligned}
T_1 &= \kappa \phi_Y \phi_V^2 \varrho (\rho + \phi_N), \\
T_2 &= \phi_X \phi_Y \phi_V^2 \kappa (\rho + \phi_N) - a \varrho \phi_Y \phi_V \phi_Q (\rho + \phi_N) - a \varrho \eta \mu \phi_V (\rho + \phi_N) - \xi a \varrho \rho \kappa \phi_V, \\
T_3 &= -a \phi_X \phi_Q \phi_V \phi_Y (\rho + \phi_N) - a \mu \eta \phi_X \phi_V (\rho + \phi_N) + a^2 \xi \varrho \rho \phi_Q.
\end{aligned}$$

We show that Eq. (4.2) has a positive root. We define

$$F(V) = \frac{T_1 V^2 + T_2 V + T_3}{a \rho \varrho (a \phi_Q - \kappa \phi_V V)}.$$

Then

$$\begin{aligned}
F(0) &= \frac{-a \phi_X \phi_Q \phi_V \phi_Y (\rho + \phi_N) - a \phi_X \phi_V \mu \eta (\rho + \phi_N) + a^2 \phi_Q \varrho \rho \xi}{a^2 \phi_Q \varrho \rho} \\
&= \frac{(\rho + \phi_N)(\phi_X \phi_V \phi_Q \phi_Y + \phi_X \phi_V \mu \eta)}{a \phi_Q \varrho \rho}(TP_2 - 1),
\end{aligned}$$

where $TP_2 = \frac{a \xi \varrho \rho \phi_Q}{\phi_X \phi_V (\rho + \phi_N)(\phi_Y \phi_Q + \eta \mu)}$. This shows that $F(0) > 0$ when $TP_2 > 1$, and

$$\limsup_{V \to \frac{a \phi_Q}{\kappa \phi_V}^-} F(V) = -\infty.$$

It follows that there exists a unique $\tilde{V} \in \left(0, \frac{a\phi_Q}{\kappa\phi_V}\right)$ such that $F(\tilde{V}) = 0$. From Eq. (4.1), we get that $\tilde{Y} > 0$, $\tilde{Q} > 0$, $\tilde{X} > 0$ and $\tilde{N} > 0$. As a result, FP_2 exists when $TP_2 > 1$. The parameter TP_2 represents the basic reproduction number of SARS-CoV-2 mono-infection. It determines the establishment of SARS-CoV-2 single infection.

4) SARS-CoV-2/HIV co-dynamics fixed point $FP_3 = (\bar{X}, \bar{N}, \bar{Y}, \bar{V}, \bar{Q}, \bar{Z}, \bar{E}, \bar{D})$, where

$$
\begin{aligned}
\bar{X} &= \frac{\phi_V(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}{a\rho\varrho\lambda\theta\left(\alpha+\phi_Z b\right)},\\
\bar{N} &= -\frac{\phi_X\phi_V\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}{a\rho\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}+\frac{\xi}{\rho+\phi_N},\\
\bar{Y} &= -\frac{\phi_X\phi_V}{a\varrho}+\frac{\lambda\theta\xi\rho\left(\alpha+\phi_Z b\right)}{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]},\\
\bar{V} &= -\frac{\phi_X}{\varrho}+\frac{\theta a\lambda\xi\rho\left(\alpha+\phi_Z b\right)}{\phi_V(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]},\\
\bar{Q} &= \frac{\phi_E\phi_D\left(\phi_Z+\alpha\right)}{\lambda\theta\left(\alpha+\phi_Z b\right)},\\
\bar{Z} &= \frac{\phi_E\phi_D(1-b)(\phi_X\phi_V\kappa+a\varrho\phi_Q)}{a\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}\times\\
&\left[\frac{a\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}{a\varrho\phi_Q+\phi_X\phi_V\kappa}\left(\frac{\eta}{(\phi_E\phi_D\left(\phi_Z+\alpha\right)}+\frac{\kappa\xi\rho}{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}\right)-1\right],\\
\bar{E} &= \frac{\phi_D(a\varrho\phi_Q+\phi_X\phi_V\kappa)}{a\varrho\lambda\theta}\times\\
&\left[\frac{a\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}{a\varrho\phi_Q+\phi_X\phi_V\kappa}\left(\frac{\eta}{(\phi_E\phi_D\left(\phi_Z+\alpha\right)}+\frac{\kappa\xi\rho}{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}\right)-1\right],\\
\bar{D} &= \frac{a\varrho\phi_Q+\phi_X\phi_V\kappa}{a\varrho\theta}\times\\
&\left[\frac{a\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}{a\varrho\phi_Q+\phi_X\phi_V\kappa}\left(\frac{\eta}{(\phi_E\phi_D\left(\phi_Z+\alpha\right)}+\frac{\kappa\xi\rho}{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}\right)-1\right].
\end{aligned}
$$

It follows that $\bar{Z} > 0$, $\bar{E} > 0$ and $\bar{D} > 0$ only when $\frac{a\varrho\lambda\theta(\alpha+\phi_Z b)}{a\varrho\phi_G+\phi_X\phi_V\kappa}\left(\frac{\eta}{(\phi_E\phi_D(\phi_Z+\alpha)}+\frac{\kappa\xi\rho}{(\rho+\phi_N)[\phi_Y\lambda\theta(\alpha+\phi_Z b)+\mu\phi_E\phi_D(\phi_Z+\alpha)]}\right) > 1$. On other hand, $\bar{N} > 0$, $\bar{Y} > 0$ and $\bar{V} > 0$ if $\frac{a\varrho\lambda\theta\xi\rho(\alpha+\phi_Z b)}{\phi_X\phi_V(\rho+\phi_N)[\phi_Y\lambda\theta(\alpha+\phi_Z b)+\mu\phi_E\phi_D(\phi_Z+\alpha)]} > 1$. Therefore, we can rewrite the components of FP_3 as

$$
\begin{aligned}
\bar{X} &= \frac{X^0}{TP_4}, \quad \bar{N} = \frac{\phi_X\phi_V\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}{a\rho\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}(TP_4-1),\\
\bar{Y} &= \frac{\phi_X\phi_V}{a\varrho}(TP_4-1), \quad \bar{V} = \frac{\phi_X}{\varrho}(TP_4-1),\\
\bar{Q} &= \frac{\phi_E\phi_D\left(\phi_Z+\alpha\right)}{\lambda\theta\left(\alpha+\phi_Z b\right)}, \quad \bar{Z} = \frac{\phi_E\phi_D(1-b)(\phi_X\phi_V\kappa+a\varrho\phi_Q)}{a\varrho\lambda\theta\left(\alpha+\phi_Z b\right)}(TP_3-1),
\end{aligned}
$$

$$\bar{E} = \frac{\phi_D(a\varrho\phi_Q + \phi_X\phi_V\kappa)}{a\varrho\lambda\theta}(TP_3 - 1), \bar{D} = \frac{a\varrho\phi_Q + \phi_X\phi_V\kappa}{a\varrho\theta}(TP_3 - 1),$$

where

$$TP_3 = \frac{a\varrho\lambda\theta\left(\alpha + \phi_Z b\right)}{a\varrho\phi_Q + \phi_X\phi_V\kappa}\left(\frac{\eta}{\left(\phi_E\phi_D\left(\phi_Z + \alpha\right)\right.} + \frac{\kappa\xi\rho}{(\rho + \phi_N)\left[\phi_Y\lambda\theta\left(\alpha + \phi_Z b\right) + \mu\phi_E\phi_D\left(\phi_Z + \alpha\right)\right]}\right),$$

$$TP_4 = \frac{a\varrho\lambda\theta\xi\rho\left(\alpha + \phi_Z b\right)}{\phi_X\phi_V(\rho + \phi_N)\left[\phi_Y\lambda\theta\left(\alpha + \phi_Z b\right) + \mu\phi_E\phi_D\left(\phi_Z + \alpha\right)\right]}.$$

Thus, if $TP_3 > 1$ and $TP_4 > 1$, then FP_3 exists. Parameters TP_3 and TP_4 determine the situation of SARS-CoV-2/HIV co-dynamics. The above results can be summarized as:

Lemma 2. There exist four threshold numbers $TP_j > 0$, $j = 1, 2, 3, 4$

(a) if $TP_1 \leq 1$, then, the infection-free fixed point, $FP_0 = (X^0, 0, 0, 0, Q^0, 0, 0, 0)$ is the only fixed point,

(b) if $TP_1 > 1$, then, in addition to FP_0, there is a HIV mono infection fixed point, $FP_1 = (\hat{X}, 0, 0, 0, \hat{Q}, \hat{Z}, \hat{E}, \hat{D})$,

(c) if $TP_2 > 1$, then, in addition to FP_0, there is a SARS-CoV-2 mono infection fixed point, $FP_2 = (\tilde{X}, \tilde{N}, \tilde{Y}, \tilde{V}, \tilde{Q}, 0, 0, 0)$,

(d) if $TP_3 > 1$ and $TP_4 > 1$, then, in addition to FP_0, there is an SARS-CoV-2/HIV co-dynamics fixed point , $FP_3 = (\bar{X}, \bar{N}, \bar{Y}, \bar{V}, \bar{Q}, \bar{Z}, \bar{E}, \bar{D})$.

5. Global stability

In this section we demonstrate the global asymptotic stability of all fixed points using Lyapunov method. We use the function $G(x) \geq 0$ as $G(x) = x - 1 - \ln x$. We have

$$\ln x \leq x - 1. \tag{5.1}$$

Theorem 5.1. *If $TP_1 \leq 1$ and $TP_2 \leq 1$, then $FP_0 = (X^0, 0, 0, 0, Q^0, 0, 0, 0)$ is globally asymptotically stable (GAS).*

Proof. We formulate a discrete Lyapunov function $\Xi_n(X_n, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n)$ as

$$\begin{aligned}\Xi_n = \frac{1}{\Pi(d)}\Bigg[&X^0 G\left(\frac{X_n}{X^0}\right) + N_n + \frac{\rho + \phi_N}{\rho}Y_n + \frac{\varrho X^0}{\phi_V}\left(1 + \Pi(d)\phi_V\right)V_n \\ &+ \frac{\mu(\rho + \phi_N)}{\kappa\rho}Q^0 G\left(\frac{Q_n}{Q^0}\right) + \frac{\mu\alpha(\rho + \phi_N)}{\kappa\rho(\alpha + \phi_Z b)}Z_n + \frac{\mu(\rho + \phi_N)(\phi_Z + \alpha)}{\kappa\rho(\alpha + \phi_Z b)}E_n \\ &+ \frac{\mu\phi_E(\rho + \phi_N)(\phi_Z + \alpha)}{\lambda\kappa\rho(\alpha + \phi_Z b)}\left(1 + \Pi(d)\phi_D\right)D_n\Bigg].\end{aligned}$$

We have, $\Xi_n > 0$ for all $X_n > 0, N_n > 0, Y_n > 0, V_n > 0, Q_n > 0, Z_n > 0, E_n > 0, D_n > 0$. Further, $\Xi_n(X^0, 0, 0, 0, Q^0, 0, 0, 0) = 0$. Computing $\Delta\Xi_n = \Xi_{n+1} - \Xi_n$ as:

$$\Delta\Xi_n = \frac{1}{\Pi(d)}\Bigg[X^0 G\left(\frac{X_{n+1}}{X^0}\right) + N_{n+1} + \frac{\rho + \phi_N}{\rho}Y_{n+1} + \frac{\varrho X^0}{\phi_V}\left(1 + \Pi(d)\phi_V\right)V_{n+1}$$

$$+\frac{\mu(\rho+\phi_N)}{\kappa\rho}Q^0G\left(\frac{Q_{n+1}}{Q^0}\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}Z_{n+1}+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}E_{n+1}$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1+\Pi(d)\phi_D\right)D_{n+1}-X^0G\left(\frac{X_n}{X^0}\right)-N_n-\frac{\rho+\phi_N}{\rho}Y_n$$
$$-\frac{\varrho X^0}{\phi_V}\left(1+\Pi(d)\phi_V\right)V_n-\frac{\mu(\rho+\phi_N)}{\kappa\rho}Q^0G\left(\frac{Q_n}{Q^0}\right)-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}Z_n$$
$$-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}E_n-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1+\Pi(d)\phi_D\right)D_n\Bigg]$$
$$=\frac{1}{\Pi(d)}\left[X^0\left(\frac{X_{n+1}-X_n}{X^0}+\ln\left(\frac{X_n}{X_{n+1}}\right)\right)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)\right.$$
$$+\frac{\varrho X^0}{\phi_V}\left(1+\Pi(d)\phi_V\right)(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}Q^0\left(\frac{Q_{n+1}-Q_n}{Q^0}+\ln\left(\frac{Q_n}{Q_{n+1}}\right)\right)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(Z_{n+1}-Z_n)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}(E_{n+1}-E_n)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1+\Pi(d)\phi_D\right)(D_{n+1}-D_n)\right].$$

Using inequality (5.1) we obtain

$$\Delta\Xi_n\le\frac{1}{\Pi(d)}\left[\left(X_{n+1}-X_n+X^0\left(\frac{X_n}{X_{n+1}}-1\right)\right)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)\right.$$
$$+\frac{\varrho X^0}{\phi_V}\left(1+\Pi(d)\phi_V\right)(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(Q_{n+1}-Q_n+Q^0\left(\frac{Q_n}{Q_{n+1}}-1\right)\right)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(Z_{n+1}-Z_n)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}(E_{n+1}-E_n)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1+\Pi(d)\phi_D\right)(D_{n+1}-D_n)\right]. \tag{5.2}$$

Inequality (5.2), can be written as:

$$=\frac{1}{\Pi(d)}\left[\left(1-\frac{X^0}{X_{n+1}}\right)(X_{n+1}-X_n)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)\right.$$
$$+\frac{\varrho X^0}{\phi_V}\left(1+\Pi(d)\phi_V\right)(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{Q^0}{Q_{n+1}}\right)(Q_{n+1}-Q_n)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(Z_{n+1}-Z_n)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}(E_{n+1}-E_n)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1+\Pi(d)\phi_D\right)(D_{n+1}-D_n)\right].$$

From Eqs. (2.1)-(2.8), we have

$$\Delta\Xi_n\le\left(1-\frac{X^0}{X_{n+1}}\right)(\xi-\phi_X X_{n+1}-\varrho V_n X_{n+1})+(\varrho V_n X_{n+1}-(\rho+\phi_N)N_{n+1})$$
$$+\frac{\rho+\phi_N}{\rho}(\rho N_{n+1}-\phi_Y Y_{n+1}-\mu Y_{n+1}Q_{n+1})+\frac{\varrho X^0}{\phi_V}(aY_{n+1}-\phi_V V_{n+1})+\varrho X^0V_{n+1}$$

$$+ \varrho X^0 V_n + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{Q^0}{Q_{n+1}}\right)\left(\eta+\kappa Y_{n+1}Q_{n+1}-\phi_Q Q_{n+1}-\theta D_n Q_{n+1}\right)$$
$$+ \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left((1-b)\theta D_n Q_{n+1}-(\alpha+\phi_Z)Z_{n+1}\right)$$
$$+ \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(b\theta D_n Q_{n+1}+\alpha Z_{n+1}-\phi_E E_{n+1}\right)$$
$$+ \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(\lambda E_{n+1}-\phi_D D_{n+1}\right)+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}$$
$$- \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n.$$

Collecting terms yields

$$\Delta\Xi_n \le \left(1-\frac{X^0}{X_{n+1}}\right)(\xi-\phi_X X_{n+1})+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{Q^0}{Q_{n+1}}\right)\left(\eta-\phi_Q Q_{n+1}\right)$$
$$+\left(\frac{\varrho X^0}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}Q^0\right)Y_{n+1}+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta Q^0-\frac{\phi_D\phi_E(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n.$$

We have

$$\xi=\phi_X X^0 \text{ and } \eta=\phi_Q Q^0,$$

then we obtain

$$\Delta\Xi_n \le \left(1-\frac{X^0}{X_{n+1}}\right)\left(\phi_X X^0-\phi_X X_{n+1}\right)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{Q^0}{Q_{n+1}}\right)\left(\phi_Q Q^0-\phi_Q Q_{n+1}\right)$$
$$+\left(\frac{\varrho X^0}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}Q^0\right)Y_{n+1}+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta Q^0-\frac{\phi_D\phi_E(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n$$
$$=-\phi_X\frac{\left(X_{n+1}-X^0\right)^2}{X_{n+1}}-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{\left(Q_{n+1}-Q^0\right)^2}{Q_{n+1}}$$
$$+\frac{(\rho+\phi_N)(\phi_Y\phi_Q+\eta\mu)}{\rho\phi_Q}\left(\frac{a\xi\varrho\rho\phi_Q}{\phi_X\phi_V(\rho+\phi_N)(\phi_Y\phi_Q+\eta\mu)}-1\right)Y_{n+1}$$
$$+\frac{\mu\phi_D\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho\lambda(\alpha+\phi_Z b)}\left(\frac{\eta\theta\lambda(\alpha+\phi_z b)}{\phi_Q\phi_E\phi_D(\phi_z+\alpha)}-1\right)D_n$$
$$=-\phi_X\frac{\left(X_{n+1}-X^0\right)^2}{X_{n+1}}-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{\left(Q_{n+1}-Q^0\right)^2}{Q_{n+1}}+\frac{(\rho+\phi_N)(\phi_Y\phi_Q+\eta\mu)}{\rho\phi_Q}\left(TP_2-1\right)Y_{n+1}$$
$$+\frac{\mu\phi_D\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho\lambda(\alpha+\phi_Z b)}\left(TP_1-1\right)D_n.$$

Since $TP_1 \le 1$ and $TP_2 \le 1$, then Ξ_n is monotonically decreasing. Clearly $\Xi_n \ge 0$, and hence, there is a limit, $\lim_{n\to\infty}\Xi_n \ge 0$ and $\lim_{n\to\infty}\Delta\Xi_n = 0$, which provides $\lim_{n\to\infty} X_n = X^0$, $\lim_{n\to\infty} Q_n = Q^0$, $\lim_{n\to\infty}(TP_2-1)Y_{n+1}=0$ and $\lim_{n\to\infty}(TP_1-1)D_n=0$. We study four cases:

(i) $TP_1 = 1$ and $TP_2 = 1$, and then from Eq. (2.1),

$$0=\xi-\phi_X X^0-\varrho X^0\lim_{n\to\infty}V_n \Rightarrow \lim_{n\to\infty}V_n=0. \tag{5.3}$$

Moreover, from Eqs. (2.2), (2.4), (2.5) and (2.6) we get

$$0 = \varrho X^0 \lim_{n\to\infty} V_n - (\rho+\phi_N) \lim_{n\to\infty} N_{n+1} \Rightarrow \lim_{n\to\infty} N_n = 0, \tag{5.4}$$

$$0 = a \lim_{n\to\infty} Y_{n+1} - \phi_V \lim_{n\to\infty} V_{n+1} \Rightarrow \lim_{n\to\infty} Y_n = 0, \tag{5.5}$$

$$0 = \eta + \kappa \lim_{n\to\infty} Y_{n+1} Q^0 - \phi_Q Q^0 - \theta \lim_{n\to\infty} D_n Q^0 \Rightarrow \lim_{n\to\infty} D_n = 0, \tag{5.6}$$

$$0 = (1-b)\theta \lim_{n\to\infty} D_n Q^0 - (\alpha+\phi_Z) \lim_{n\to\infty} Z_{n+1} \Rightarrow \lim_{n\to\infty} Z_n = 0 \tag{5.7}$$

Then, from Eq. (2.8), we have

$$0 = \lambda \lim_{n\to\infty} E_{n+1} - \phi_D \lim_{n\to\infty} D_{n+1} \Rightarrow \lim_{n\to\infty} E_n = 0. \tag{5.8}$$

(ii) $TP_1 = 1$, $TP_2 < 1$ and $\lim_{n\to\infty} Y_n = 0$. Eqs. (5.3), (5.4) and (5.6)-(5.8) yield $\lim_{n\to\infty} N_n = 0$, $\lim_{n\to\infty} V_n = 0$, $\lim_{n\to\infty} Z_n = 0$, $\lim_{n\to\infty} E_n = 0$ and $\lim_{n\to\infty} D_n = 0$.

(iii) $TP_1 < 1$, $TP_2 = 1$ and $\lim_{n\to\infty} D_n = 0$. Eqs. (5.3)-(5.5), (5.7) and (5.8) give $\lim_{n\to\infty} N_n = 0$, $\lim_{n\to\infty} V_n = 0$, $\lim_{n\to\infty} Y_n = 0$, $\lim_{n\to\infty} Z_n = 0$, and $\lim_{n\to\infty} E_n = 0$.

(iv) $TP_1 < 1$, $TP_2 < 1$, $\lim_{n\to\infty} Y_n = 0$ and $\lim_{n\to\infty} D_n = 0$. From Eqs.(5.3), (5.4), (5.7) and (5.8) we get $\lim_{n\to\infty} N_n = 0$, $\lim_{n\to\infty} V_n = 0$ $\lim_{n\to\infty} Z_n = 0$, and $\lim_{n\to\infty} E_n = 0$.

As a result, if $TP_1 \leq 1$ and $TP_2 \leq 1$, then $\lim_{n\to\infty}(X_n, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n) = (X^0, 0, 0, 0, Q^0, 0, 0, 0)$. This provides that, FP_0 is GAS.□

Theorem 5.2. *If $TP_1 > 1$ and $TP_4 \leq 1$ then $FP_1 = (\hat{X}, 0, 0, 0, \hat{Q}, \hat{Z}, \hat{E}, \hat{D})$ is GAS.*

Proof. Define

$$\begin{aligned} \Upsilon_n = \frac{1}{\Pi(d)} \Bigg[& \hat{X} G\left(\frac{X_n}{\hat{X}}\right) + N_n + \frac{\rho+\phi_N}{\rho} Y_n + \frac{\varrho \hat{X}}{\phi_V} \left(1 + \Pi(d)\phi_V\right) V_n \\ & + \frac{\mu(\rho+\phi_N)}{\kappa\rho} \hat{Q} G\left(\frac{Q_n}{\hat{Q}}\right) + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)} \hat{Z} G\left(\frac{Z_n}{\hat{Z}}\right) + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)} \hat{E} G\left(\frac{E_n}{\hat{E}}\right) \\ & + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)} \left(1 + \Pi(d)\phi_D\right) \hat{D} G\left(\frac{D_n}{\hat{D}}\right) \Bigg]. \end{aligned}$$

Obviously $\Upsilon_n > 0$ for all $X_n > 0, N_n > 0, Y_n > 0, V_n > 0, Q_n > 0, Z_n > 0, E_n > 0, D_n > 0$. Moreover, $\Upsilon_n(\hat{X}, 0, 0, 0, \hat{Q}, \hat{Z}, \hat{E}, \hat{D}) = 0$. Computing $\Delta\Upsilon_n = \Upsilon_{n+1} - \Upsilon_n$ as:

$$\begin{aligned} \Upsilon_n = \frac{1}{\Pi(d)} \Bigg[& \hat{X} G\left(\frac{X_{n+1}}{\hat{X}}\right) + N_{n+1} + \frac{\rho+\phi_N}{\rho} Y_{n+1} + \frac{\varrho \hat{X}}{\phi_V} \left(1 + \Pi(d)\phi_V\right) V_{n+1} \\ & + \frac{\mu(\rho+\phi_N)}{\kappa\rho} \hat{Q} G\left(\frac{Q_{n+1}}{\hat{Q}}\right) + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)} \hat{Z} G\left(\frac{Z_{n+1}}{\hat{Z}}\right) + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)} \hat{E} G\left(\frac{E_{n+1}}{\hat{E}}\right) \\ & + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)} \left(1 + \Pi(d)\phi_D\right) \hat{D} G\left(\frac{D_{n+1}}{\hat{D}}\right) - \hat{X} G\left(\frac{X_n}{\hat{X}}\right) - N_n - \frac{\rho+\phi_N}{\rho} Y_n \\ & - \frac{\varrho \hat{X}}{\phi_V} \left(1 + \Pi(d)\phi_V\right) V_n - \frac{\mu(\rho+\phi_N)}{\kappa\rho} \hat{Q} G\left(\frac{Q_n}{\hat{Q}}\right) - \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)} \hat{Z} G\left(\frac{Z_n}{\hat{Z}}\right) \\ & - \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)} \hat{E} G\left(\frac{E_n}{\hat{E}}\right) - \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)} \left(1 + \Pi(d)\phi_D\right) \hat{D} G\left(\frac{D_n}{\hat{D}}\right) \Bigg] \end{aligned}$$

$$= \frac{1}{\Pi(d)}\left[\hat{X}\left(\frac{X_{n+1}-X_n}{\hat{X}}+\ln\left(\frac{X_n}{X_{n+1}}\right)\right)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)\right.$$
$$+\frac{\varrho\hat{X}}{\phi_V}(1+\Pi(d)\phi_V)(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\hat{Q}\left(\frac{Q_{n+1}-Q_n}{\hat{Q}}+\ln\left(\frac{Q_n}{Q_{n+1}}\right)\right)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\hat{Z}\left(\frac{Z_{n+1}-Z_n}{\hat{Z}}+\ln\left(\frac{Z_n}{Z_{n+1}}\right)\right)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\hat{E}\left(\frac{E_{n+1}-E_n}{\hat{E}}+\ln\left(\frac{E_n}{E_{n+1}}\right)\right)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}(1+\Pi(d)\phi_D)\hat{D}\left(\frac{D_{n+1}-D_n}{\hat{D}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right)\right].$$

Using inequality (5.1) we get

$$\Delta Y_n \leq \frac{1}{\Pi(d)}\left[X_{n+1}-X_n+\hat{X}\left(\frac{X_n}{X_{n+1}}-1\right)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)+\frac{\varrho\hat{X}}{\phi_V}(V_{n+1}-V_n)\right.$$
$$+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(Q_{n+1}-Q_n+\hat{Q}\left(\frac{Q_n}{Q_{n+1}}-1\right)\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(Z_{n+1}-Z_n+\hat{Z}\left(\frac{Z_n}{Z_{n+1}}-1\right)\right)$$
$$+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(E_{n+1}-E_n+\hat{E}\left(\frac{E_n}{E_{n+1}}-1\right)\right)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(D_{n+1}-D_n+\hat{D}\left(\frac{D_n}{D_{n+1}}-1\right)\right)\right]$$
$$+\varrho\hat{X}V_{n+1}-\varrho\hat{X}V_n+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}\ln\left(\frac{D_n}{D_{n+1}}\right). \tag{5.9}$$

Inequality (5.9), can be written as:

$$\Delta Y_n \leq \frac{1}{\Pi(d)}\left[\left(1-\frac{\hat{X}}{X_{n+1}}\right)(X_{n+1}-X_n)+(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}(Y_{n+1}-Y_n)\right.$$
$$+\frac{\varrho\hat{X}}{\phi_V}(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\hat{Q}}{Q_{n+1}}\right)(Q_{n+1}-Q_n)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{Z}}{Z_{n+1}}\right)(Z_{n+1}-Z_n)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{E}}{E_{n+1}}\right)(E_{n+1}-E_n)$$
$$\left.+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{D}}{D_{n+1}}\right)(D_{n+1}-D_n)\right]+\varrho\hat{X}V_{n+1}-\varrho\hat{X}V_n$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}\ln\left(\frac{D_n}{D_{n+1}}\right).$$

From Eqs. (2.1)-(2.8) we have

$$\Delta Y_n \leq \left(1-\frac{\hat{X}}{X_{n+1}}\right)(\xi-\phi_X X_{n+1}-\varrho V_n X_{n+1})+(\varrho V_n X_{n+1}-(\rho+\phi_N)N_{n+1})$$
$$+\frac{\rho+\phi_N}{\rho}(\rho N_{n+1}-\phi_Y Y_{n+1}-\mu Y_{n+1}Q_{n+1})$$

$$+\frac{\varrho\hat{X}}{\phi_V}(aY_{n+1}-\phi_V V_{n+1})+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\hat{Q}}{Q_{n+1}}\right)\Big(\eta+\kappa Y_{n+1}Q_{n+1}-\phi_Q Q_{n+1}-\theta D_n Q_{n+1}\Big)$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{Z}}{Z_{n+1}}\right)((1-b)\theta D_n Q_{n+1}-(\alpha+\phi_Z)Z_{n+1})$$
$$+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{E}}{E_{n+1}}\right)(b\theta D_n Q_{n+1}+\alpha Z_{n+1}-\phi_E E_{n+1})$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\hat{D}}{D_{n+1}}\right)(\lambda E_{n+1}-\phi_D D_{n+1})+\varrho\hat{X}V_{n+1}-\varrho\hat{X}V_n$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}\ln\left(\frac{D_n}{D_{n+1}}\right).$$

Collecting terms we get

$$\Delta Y_n\le\left(1-\frac{\hat{X}}{X_{n+1}}\right)(\xi-\phi_X X_{n+1})-\frac{\rho+\phi_N}{\rho}\phi_Y Y_{n+1}+\frac{\varrho\hat{X}}{\phi_V}aY_{n+1}$$
$$+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\hat{Q}}{Q_{n+1}}\right)\Big(\eta-\phi_Q Q_{n+1}\Big)-\frac{\mu(\rho+\phi_N)}{\rho}Y_{n+1}\hat{Q}$$
$$-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta D_n Q_{n+1}\frac{\hat{Z}}{Z_{n+1}}+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(\alpha+\phi_Z)\hat{Z}$$
$$-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta D_n Q_{n+1}\frac{\hat{E}}{E_{n+1}}-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\alpha Z_{n+1}\frac{\hat{E}}{E_{n+1}}$$
$$+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\phi_E\hat{E}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}E_{n+1}\frac{\hat{D}}{D_{n+1}}$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}\ln\left(\frac{D_n}{D_{n+1}}\right).$$

Utilizing the following conditions for FP_1:

$$\xi=\phi_X\hat{X},\qquad \eta=\phi_Q\hat{Q}+\theta\hat{D}\hat{Q},$$
$$(1-b)\theta\hat{D}\hat{Q}=(\alpha+\phi_Z)\hat{Z},\quad b\theta\hat{D}\hat{Q}=\phi_E\hat{E}-\alpha\hat{Z},\quad \lambda\hat{E}=\phi_D\hat{D},$$

we get

$$\Delta Y_n\le\left(1-\frac{\hat{X}}{X_{n+1}}\right)\Big(\phi_X\hat{X}-\phi_X X_{n+1}\Big)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\hat{Q}}{Q_{n+1}}\right)\Big(\phi_Q\hat{Q}-\phi_Q Q_{n+1}\Big)$$
$$+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\theta\hat{D}\hat{Q}\left(1-\frac{\hat{Q}}{Q_{n+1}}\right)-\frac{\rho+\phi_N}{\rho}\phi_Y Y_{n+1}+\frac{\varrho\hat{X}}{\phi_V}aY_{n+1}-\frac{\mu(\rho+\phi_N)}{\rho}Y_{n+1}\hat{Q}$$
$$-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta D_n Q_{n+1}\frac{\hat{Z}}{Z_{n+1}}+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(\alpha+\phi_Z)\hat{Z}$$
$$-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta D_n Q_{n+1}\frac{\hat{E}}{E_{n+1}}-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\alpha Z_{n+1}\frac{\hat{E}}{E_{n+1}}$$

$$+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\phi_E\hat{E}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}E_{n+1}\frac{\hat{D}}{D_{n+1}}$$
$$+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\hat{D}\ln\left(\frac{D_n}{D_{n+1}}\right).$$
$$=-\phi_X\frac{\left(X_{n+1}-\hat{X}\right)^2}{X_{n+1}}-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{\left(Q_{n+1}-\hat{Q}\right)^2}{Q_{n+1}}+\left(\frac{\varrho\hat{X}}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}\hat{Q}\right)Y_{n+1}$$
$$+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\hat{D}\hat{Q}\left(4-\frac{\hat{Q}}{Q_{n+1}}-\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}-\frac{Z_{n+1}\hat{E}}{\hat{Z}E_{n+1}}-\frac{D_nQ_{n+1}\hat{Z}}{\hat{D}\hat{Q}Z_{n+1}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right)$$
$$+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\hat{D}\hat{Q}\left(3-\frac{\hat{Q}}{Q_{n+1}}-\frac{Q_{n+1}\hat{E}D_n}{\hat{D}\hat{Q}E_{n+1}}-\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right).$$

Using the following equalities:

$$\ln\left(\frac{D_n}{D_{n+1}}\right)=\ln\left(\frac{\hat{Q}}{Q_{n+1}}\right)+\ln\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right)+\ln\left(\frac{Z_{n+1}\hat{E}}{\hat{Z}E_{n+1}}\right)+\ln\left(\frac{D_nQ_{n+1}\hat{Z}}{\hat{D}\hat{Q}Z_{n+1}}\right), \tag{5.10}$$
$$\ln\left(\frac{D_n}{D_{n+1}}\right)=\ln\left(\frac{\hat{Q}}{Q_{n+1}}\right)+\ln\left(\frac{Q_{n+1}\hat{E}D_n}{\hat{D}\hat{Q}E_{n+1}}\right)+\ln\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right). \tag{5.11}$$

We get

$$\Delta\Upsilon_n\leq-\phi_X\frac{\left(X_{n+1}-\hat{X}\right)^2}{X_{n+1}}-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{\left(Q_{n+1}-\hat{Q}\right)^2}{Q_{n+1}}$$
$$+\frac{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}{\lambda\rho\theta\left(\alpha+\phi_Z b\right)}\times$$
$$\left(\frac{a\varrho\lambda\theta\xi\rho\left(\alpha+\phi_Z b\right)}{\phi_X\phi_V(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}-1\right)Y_{n+1}$$
$$-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\hat{D}\hat{Q}\left[G\left(\frac{\hat{Q}}{Q_{n+1}}\right)+G\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right)+G\left(\frac{Z_{n+1}\hat{E}}{\hat{Z}E_{n+1}}\right)+G\left(\frac{D_nQ_{n+1}\hat{Z}}{\hat{D}\hat{Q}Z_{n+1}}\right)\right]$$
$$-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\hat{D}\hat{Q}\left[G\left(\frac{\hat{Q}}{Q_{n+1}}\right)+G\left(\frac{Q_{n+1}\hat{E}D_n}{\hat{D}\hat{Q}E_{n+1}}\right)+G\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right)\right]$$
$$=-\phi_X\frac{\left(X_{n+1}-\hat{X}\right)^2}{X_{n+1}}-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{\left(Q_{n+1}-\hat{Q}\right)^2}{Q_{n+1}}$$
$$+\frac{(\rho+\phi_N)\left[\phi_Y\lambda\theta\left(\alpha+\phi_Z b\right)+\mu\phi_E\phi_D\left(\phi_Z+\alpha\right)\right]}{\lambda\rho\theta\left(\alpha+\phi_Z b\right)}\left(TP_4-1\right)Y_{n+1}$$
$$-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\hat{D}\hat{Q}\left[G\left(\frac{\hat{Q}}{Q_{n+1}}\right)+G\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right)+G\left(\frac{Z_{n+1}\hat{E}}{\hat{Z}E_{n+1}}\right)+G\left(\frac{D_nQ_{n+1}\hat{Z}}{\hat{D}\hat{Q}Z_{n+1}}\right)\right]$$
$$-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\hat{D}\hat{Q}\left[G\left(\frac{\hat{Q}}{Q_{n+1}}\right)+G\left(\frac{Q_{n+1}\hat{E}D_n}{\hat{D}\hat{Q}E_{n+1}}\right)+G\left(\frac{E_{n+1}\hat{D}}{\hat{E}D_{n+1}}\right)\right].$$

Since $TP_1>1$ and if $TP_4\leq 1$, then Υ_n is monotonically decreasing. We have $\Upsilon_n\geq 0$, and hence there is a limit, $\lim_{n\to\infty}\Upsilon_n\geq 0$ and thus $\lim_{n\to\infty}\Delta\Upsilon_n=0$, which gives $\lim_{n\to\infty}X_n=\hat{X}$, $\lim_{n\to\infty}Q_n=\hat{Q}$,

$\lim_{n\to\infty} Z_n = \hat{Z}$, $\lim_{n\to\infty} E_n = \hat{E}$, $\lim_{n\to\infty} D_n = \hat{D}$ and $\lim_{n\to\infty} (TP_4 - 1) Y_{n+1} = 0$. Let us address the following cases:

(i) $TP_4 = 1$. From Eq. (2.1), we get

$$0 = \xi - \phi_X \hat{X} - \varrho \hat{X} \lim_{n\to\infty} V_n \Rightarrow \lim_{n\to\infty} V_n = 0. \tag{5.12}$$

Moreover, from Eqs. (2.2) and (2.4) we get

$$0 = \varrho \hat{X} \lim_{n\to\infty} V_n - (\rho + \phi_N) \lim_{n\to\infty} N_{n+1} \Rightarrow \lim_{n\to\infty} N_n = 0. \tag{5.13}$$

$$0 = \mu \lim_{n\to\infty} Y_{n+1} - \phi_V \lim_{n\to\infty} V_{n+1} \Rightarrow \lim_{n\to\infty} Y_n = 0, \tag{5.14}$$

(ii) $TP_4 < 1$ and then $\lim_{n\to\infty} Y_n = 0$. Eqs. (5.12) and (5.14) we obtain $\lim_{n\to\infty} N_n = 0$ *and* $\lim_{n\to\infty} V_n = 0$. Hence, FP_1 is GAS.□

Theorem 5.3. *If $TP_2 > 1$ and $TP_3 \leq 1$, then $FP_2 = (\tilde{X}, \tilde{N}, \tilde{Y}, \tilde{V}, \tilde{Q}, 0, 0, 0)$ is GAS.*

Proof. Consider

$$\begin{aligned}\Omega_n = \frac{1}{\Pi(d)} \Bigg[&\tilde{X} G\left(\frac{X_n}{\tilde{X}}\right) + \tilde{N} G\left(\frac{N_n}{\tilde{N}}\right) + \frac{\rho + \phi_N}{\rho} \tilde{Y} G\left(\frac{Y_n}{\tilde{Y}}\right) + \frac{\varrho \tilde{X}}{\phi_V} (1 + \Pi(d)\phi_V) \tilde{V} G\left(\frac{V_n}{\tilde{V}}\right) \\ &+ \frac{\mu(\rho + \phi_N)}{\kappa \rho} \tilde{Q} G\left(\frac{Q_n}{\tilde{Q}}\right) + \frac{\mu \alpha (\rho + \phi_N)}{\kappa \rho (\alpha + \phi_Z b)} Z_n + \frac{\mu (\rho + \phi_N)(\phi_Z + \alpha)}{\kappa \rho (\alpha + \phi_Z b)} E_n \\ &+ \frac{\mu \phi_E (\rho + \phi_N)(\phi_Z + \alpha)}{\lambda \kappa \rho (\alpha + \phi_Z b)} (1 + \Pi(d)\phi_D) D_n \Bigg].\end{aligned}$$

Computing $\Delta\Omega_n = \Omega_{n+1} - \Omega_n$ as:

$$\begin{aligned}\Delta\Omega_n = \frac{1}{\Pi(d)} \Bigg[&\tilde{X} G\left(\frac{X_{n+1}}{\tilde{X}}\right) + \tilde{N} G\left(\frac{N_{n+1}}{\tilde{N}}\right) + \frac{\rho + \phi_N}{\rho} \tilde{Y} G\left(\frac{Y_{n+1}}{\tilde{Y}}\right) + \frac{\varrho \tilde{X}}{\phi_V} (1 + \Pi(d)\phi_V) \tilde{V} G\left(\frac{V_{n+1}}{\tilde{V}}\right) \\ &+ \frac{\mu(\rho + \phi_N)}{\kappa \rho} \tilde{Q} G\left(\frac{Q_{n+1}}{\tilde{Q}}\right) + \frac{\mu \alpha (\rho + \phi_N)}{\kappa \rho (\alpha + \phi_Z b)} Z_{n+1} + \frac{\mu (\rho + \phi_N)(\phi_Z + \alpha)}{\kappa \rho (\alpha + \phi_Z b)} E_{n+1} \\ &+ \frac{\mu \phi_E (\rho + \phi_N)(\phi_Z + \alpha)}{\lambda \kappa \rho (\alpha + \phi_Z b)} (1 + \Pi(d)\phi_D) D_{n+1} - \tilde{X} G\left(\frac{X_n}{\tilde{X}}\right) - \tilde{N} G\left(\frac{N_n}{\tilde{N}}\right) - \frac{\rho + \phi_N}{\rho} \tilde{Y} G\left(\frac{Y_n}{\tilde{Y}}\right) \\ &- \frac{\varrho \tilde{X}}{\phi_V} (1 + \Pi(d)\phi_V) \tilde{V} G\left(\frac{V_n}{\tilde{V}}\right) - \frac{\mu(\rho + \phi_N)}{\kappa \rho} \tilde{Q} G\left(\frac{Q_n}{\tilde{Q}}\right) - \frac{\mu \alpha (\rho + \phi_N)}{\kappa \rho (\alpha + \phi_Z b)} Z_n \\ &- \frac{\mu (\rho + \phi_N)(\phi_Z + \alpha)}{\kappa \rho (\alpha + \phi_Z b)} E_n - \frac{\mu \phi_E (\rho + \phi_N)(\phi_Z + \alpha)}{\lambda \kappa \rho (\alpha + \phi_Z b)} (1 + \Pi(d)\phi_D) D_n \Bigg] \\ = \frac{1}{\Pi(d)} \Bigg[&\tilde{X} \left(\frac{X_{n+1} - X_n}{\tilde{X}} + \ln\left(\frac{X_n}{X_{n+1}}\right)\right) + \tilde{N} \left(\frac{N_{n+1} - N_n}{\tilde{N}} + \ln\left(\frac{N_n}{N_{n+1}}\right)\right) \\ &+ \frac{\rho + \phi_N}{\rho} \tilde{Y} \left(\frac{Y_{n+1} - Y_n}{\tilde{Y}} + \ln\left(\frac{Y_n}{Y_{n+1}}\right)\right) + \frac{\varrho \tilde{X}}{\phi_V} (1 + \Pi(d)\phi_V) \tilde{V} \left(\frac{V_{n+1} - V_n}{\tilde{V}} + \ln\left(\frac{V_n}{V_{n+1}}\right)\right) \\ &+ \frac{\mu(\rho + \phi_N)}{\kappa \rho} \tilde{Q} \left(\frac{Q_{n+1} - Q_n}{\tilde{Q}} + \ln\left(\frac{Q_n}{Q_{n+1}}\right)\right) + \frac{\mu \alpha (\rho + \phi_N)}{\kappa \rho (\alpha + \phi_Z b)} (Z_{n+1} - Z_n) \\ &+ \frac{\mu (\rho + \phi_N)(\phi_Z + \alpha)}{\kappa \rho (\alpha + \phi_Z b)} (E_{n+1} - E_n) + \frac{\mu \phi_E (\rho + \phi_N)(\phi_Z + \alpha)}{\lambda \kappa \rho (\alpha + \phi_Z b)} (1 + \Pi(d)\phi_D) (D_{n+1} - D_n) \Bigg].\end{aligned}$$

Using inequality (5.1) we obtain

$$
\begin{aligned}
\Delta\Omega_n \leq \frac{1}{\Pi(d)} \Bigg[& \left(X_{n+1} - X_n + \tilde{X}\left(\frac{X_n}{X_{n+1}} - 1\right)\right) + \left(N_{n+1} - N_n + \tilde{N}\left(\frac{N_n}{N_{n+1}} - 1\right)\right) \\
& + \frac{\rho + \phi_N}{\rho}\left(Y_{n+1} - Y_n + \tilde{Y}\left(\frac{Y_n}{Y_{n+1}} - 1\right)\right) + \frac{\varrho \tilde{X}}{\phi_V}\left(V_{n+1} - V_n + \tilde{V}\left(\frac{V_n}{V_{n+1}} - 1\right)\right) \\
& + \frac{\mu(\rho + \phi_N)}{\kappa\rho}\left(Q_{n+1} - Q_n + \tilde{Q}\left(\frac{Q_n}{Q_{n+1}} - 1\right)\right) + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(Z_{n+1} - Z_n\right) \\
& + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(E_{n+1} - E_n\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(D_{n+1} - D_n\right)\Bigg] \\
& + \varrho\tilde{X}V_{n+1} - \varrho\tilde{X}V_n + \varrho\tilde{X}\tilde{V}\ln\left(\frac{V_n}{V_{n+1}}\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1} \\
& - \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n. \qquad (5.15)
\end{aligned}
$$

Inequality (5.15), can be written as:

$$
\begin{aligned}
\Delta\Omega_n \leq \frac{1}{\Pi(d)} \Bigg[& \left(1 - \frac{\tilde{X}}{X_{n+1}}\right)(X_{n+1} - X_n) + \left(1 - \frac{\tilde{N}}{N_{n+1}}\right)(N_{n+1} - N_n) + \frac{\rho+\phi_N}{\rho}\left(1 - \frac{\tilde{Y}}{Y_{n+1}}\right)(Y_{n+1} - Y_n) \\
& + \frac{\varrho\tilde{X}}{\phi_V}\left(1 - \frac{\tilde{V}}{V_{n+1}}\right)(V_{n+1} - V_n) + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1 - \frac{\tilde{Q}}{Q_{n+1}}\right)(Q_{n+1} - Q_n) + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(Z_{n+1} - Z_n\right) \\
& + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(E_{n+1} - E_n\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(D_{n+1} - D_n\right)\Bigg] \\
& + \varrho\tilde{X}V_{n+1} - \varrho\tilde{X}V_n + \varrho\tilde{X}\tilde{V}\ln\left(\frac{V_n}{V_{n+1}}\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1} \\
& - \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n.
\end{aligned}
$$

From Eqs. (2.1)-(2.8) we have

$$
\begin{aligned}
\Delta\Omega_n \leq & \left(1 - \frac{\tilde{X}}{X_{n+1}}\right)(\xi - \phi_X X_{n+1} - \varrho V_n X_{n+1}) + \left(1 - \frac{\tilde{N}}{N_{n+1}}\right)(\varrho V_n X_{n+1} - (\rho+\phi_N)N_{n+1}) \\
& + \frac{\rho+\phi_N}{\rho}\left(1 - \frac{\tilde{Y}}{Y_{n+1}}\right)(\rho N_{n+1} - \phi_Y Y_{n+1} - \mu Y_{n+1}Q_{n+1}) + \frac{\varrho\tilde{X}}{\phi_V}\left(1 - \frac{\tilde{V}}{V_{n+1}}\right)(a Y_{n+1} - \phi_V V_{n+1}) \\
& + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1 - \frac{\tilde{Q}}{Q_{n+1}}\right)(\eta + \kappa Y_{n+1}Q_{n+1} - \phi_Q Q_{n+1} - \theta D_n Q_{n+1}) \\
& + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left((1-b)\theta D_n Q_{n+1} - (\alpha+\phi_Z)Z_{n+1}\right) \\
& + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(b\theta D_n Q_{n+1} + \alpha Z_{n+1} - \phi_E E_{n+1}\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(\lambda E_{n+1} - \phi_D D_{n+1}\right) \\
& + \varrho\tilde{X}V_{n+1} - \varrho\tilde{X}V_n + \varrho\tilde{X}\tilde{V}\ln\left(\frac{V_n}{V_{n+1}}\right) + \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1} - \frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n.
\end{aligned}
$$

By collecting terms we get

$$\begin{aligned}\Delta\Omega_n \leq & \left(1-\frac{\tilde{X}}{X_{n+1}}\right)(\xi-\phi_X X_{n+1})-\varrho V_n X_{n+1}\frac{\tilde{N}}{N_{n+1}}+(\rho+\phi_N)\tilde{N}-(\rho+\phi_N)N_{n+1}\frac{\tilde{Y}}{Y_{n+1}}\\ & +\frac{\rho+\phi_N}{\rho}\phi_Y\tilde{Y}+\frac{\rho+\phi_N}{\rho}\mu\tilde{Y}Q_{n+1}-\frac{\varrho\tilde{X}}{\phi_V}aY_{n+1}\frac{\tilde{V}}{V_{n+1}}+\varrho\tilde{X}\tilde{V}\ln\left(\frac{V_n}{V_{n+1}}\right)\\ & +\varrho\tilde{X}\tilde{V}+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\tilde{Q}}{Q_{n+1}}\right)(\eta-\phi_Q Q_{n+1})\\ & +\left(\frac{\varrho\tilde{X}}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Q}\right)Y_{n+1}\\ & +\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n.\end{aligned}$$

Using the conditions of fixed point FP_2 :

$$\begin{aligned}\xi &= \phi_X\tilde{X}+\varrho\tilde{V}\tilde{X},\\ \varrho\tilde{V}\tilde{X} &= (\rho+\phi_N)\tilde{N},\\ \rho\tilde{N} &= \phi_Y\tilde{Y}+\mu\tilde{Y}\tilde{Q},\\ a\tilde{Y} &= \phi_V\tilde{V},\\ \eta &= \phi_Q\tilde{Q}-\kappa\tilde{Y}\tilde{Q}.\end{aligned}$$

We obtain

$$\begin{aligned}\Delta\Omega_n \leq & \left(1-\frac{\tilde{X}}{X_{n+1}}\right)(\phi_X\tilde{X}-\phi_X X_{n+1})+\varrho\tilde{V}\tilde{X}-\varrho\tilde{V}\tilde{X}\frac{\tilde{X}}{X_{n+1}}-\varrho\tilde{V}\tilde{X}\frac{V_n X_{n+1}\tilde{N}}{\tilde{V}\tilde{X}N_{n+1}}\\ & +\varrho\tilde{V}\tilde{X}-\varrho\tilde{V}\tilde{X}\frac{N_{n+1}\tilde{Y}}{\tilde{N}Y_{n+1}}+\varrho\tilde{V}\tilde{X}-\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}+\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}\frac{Q_{n+1}}{\tilde{Q}}\\ & -\varrho\tilde{X}\tilde{V}\frac{Y_{n+1}\tilde{V}}{\tilde{Y}V_{n+1}}+\varrho\tilde{X}\tilde{V}+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\tilde{Q}}{Q_{n+1}}\right)(\phi_Q\tilde{Q}-\phi_Q Q_{n+1})\\ & -\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}+\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}\frac{\tilde{Q}}{Q_{n+1}}+\varrho\tilde{X}\tilde{V}\ln\left(\frac{V_n}{V_{n+1}}\right)\\ & +\left(\frac{\varrho\tilde{X}}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Q}\right)Y_{n+1}\\ & +\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n.\\ & =-\phi_X\frac{\left(X_{n+1}-\tilde{X}\right)^2}{X_{n+1}}-\frac{\mu\phi_Q(\rho+\phi_N)}{\kappa\rho Q_{n+1}}\left(Q_{n+1}-\tilde{Q}\right)^2\\ & +\varrho\tilde{V}\tilde{X}\left(4-\frac{\tilde{X}}{X_{n+1}}-\frac{V_n X_{n+1}\tilde{N}}{\tilde{V}\tilde{X}N_{n+1}}-\frac{N_{n+1}\tilde{Y}}{\tilde{N}Y_{n+1}}-\frac{Y_{n+1}\tilde{V}}{\tilde{Y}V_{n+1}}+\ln\left(\frac{V_n}{V_{n+1}}\right)\right)\\ & -\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}\left(2-\frac{Q_{n+1}}{\tilde{Q}}-\frac{\tilde{Q}}{Q_{n+1}}\right)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n.\end{aligned}$$

Since we have

$$
\begin{aligned}
&-\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\tilde{Q}\left(2-\frac{Q_{n+1}}{\tilde{Q}}-\frac{\tilde{Q}}{Q_{n+1}}\right)-\phi_Q\frac{\mu(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}}\\
&=\frac{\mu(\rho+\phi_N)}{\rho}\tilde{Y}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}}-\phi_Q\frac{\mu(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}}\\
&=\frac{\mu(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}}\left(\kappa\tilde{Y}-\phi_Q\right)\\
&=-\frac{\mu\eta(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}\tilde{Q}}
\end{aligned}
$$

We get

$$
\begin{aligned}
\Delta\Omega_n \le& -\phi_X\frac{\left(X_{n+1}-\tilde{X}\right)^2}{X_{n+1}}-\frac{\mu\eta(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}\tilde{Q}}\\
&+\varrho\tilde{V}\tilde{X}\left(4-\frac{\tilde{X}}{X_{n+1}}-\frac{V_nX_{n+1}\tilde{N}}{\tilde{V}\tilde{X}N_{n+1}}-\frac{N_{n+1}\tilde{Y}}{\tilde{N}Y_{n+1}}-\frac{Y_{n+1}\tilde{V}}{\tilde{Y}V_{n+1}}+\ln\left(\frac{V_n}{V_{n+1}}\right)\right)\\
&+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Zb)}\right)D_n. \qquad (5.16)
\end{aligned}
$$

By using the following equality:

$$
\ln\left(\frac{V_n}{V_{n+1}}\right)=\ln\left(\frac{\tilde{X}}{X_{n+1}}\right)+\ln\left(\frac{V_nX_{n+1}\tilde{N}}{\tilde{V}\tilde{X}N_{n+1}}\right)+\ln\left(\frac{Y_{n+1}\tilde{V}}{\tilde{Y}V_{n+1}}\right)+\ln\left(\frac{N_{n+1}\tilde{Y}}{\tilde{N}Y_{n+1}}\right). \qquad (5.17)
$$

Then Eq. (5.16) becomes

$$
\begin{aligned}
\Delta\Omega_n \le& -\phi_X\frac{\left(X_{n+1}-\tilde{X}\right)^2}{X_{n+1}}-\frac{\mu\eta(\rho+\phi_N)}{\kappa\rho}\frac{\left(Q_{n+1}-\tilde{Q}\right)^2}{Q_{n+1}\tilde{Q}}\\
&-\varrho\tilde{V}\tilde{X}\left(G\left(\frac{\tilde{X}}{X_{n+1}}\right)+G\left(\frac{V_nX_{n+1}\tilde{N}}{\tilde{V}\tilde{X}N_{n+1}}\right)+G\left(\frac{Y_{n+1}\tilde{V}}{\tilde{Y}V_{n+1}}\right)+G\left(\frac{N_{n+1}\tilde{Y}}{\tilde{N}Y_{n+1}}\right)\right)\\
&+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Zb)}\right)D_n.
\end{aligned}
$$

Hence , if $TP_3 \le 1$, then FP_3 does not exist since $\bar{D} \le 0$, $\bar{E} \le 0$ and $\bar{Z} \le 0$. This gives

$$
\begin{aligned}
\frac{Z_{n+1}-Z_n}{\Pi(d)} &= (1-b)\theta D_nQ_{n+1}-(\alpha+\phi_Z)Z_{n+1} \le 0,\\
\frac{E_{n+1}-E_n}{\Pi(d)} &= b\theta D_nQ_{n+1}+\alpha Z_{n+1}-\phi_E E_{n+1} \le 0,\\
\frac{D_{n+1}-D_n}{\Pi(d)} &= \lambda E_{n+1}-\phi_D D_{n+1} \le 0.
\end{aligned}
$$

It follows that $\left(\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Zb)}\right)D_n \le 0$ for all $D_n > 0$.Thus, $\theta\tilde{Q}-\frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Zb)} \le 0$ and since, $TP_3 \le 1$, then $\Delta\Omega_n \le 0$, for all $n \ge 0$. Therefore, Ω_n is monotonically decreasing. Since $\Omega_n \ge 0$, then $\lim_{n\to\infty}\Omega_n \geq 0$ and thus $\lim_{n\to\infty}\Delta\Omega_n = 0$. It follows that, $\lim_{n\to\infty} X_n = \tilde{X}$, $\lim_{n\to\infty} N_n = \tilde{N}$, $\lim_{n\to\infty} Y_n = \tilde{Y}$, $\lim_{n\to\infty} V_n = \tilde{V}$, $\lim_{n\to\infty} Q_n = \tilde{Q}$ and $\lim_{n\to\infty}(TP_3-1)D_n = 0$. The following cases will be considered:

(i) $TP_3 = 1$, and from Eq. (2.8)

$$0 = \lambda \lim_{n\to\infty} E_{n+1} - \phi_D \lim_{n\to\infty} D_{n+1} \Longrightarrow \lim_{n\to\infty} E_n = \lim_{n\to\infty} D_n = 0. \tag{5.18}$$

Then, from Eq. (2.6), we have

$$0 = (1-b)\theta \lim_{n\to\infty} D_{n+1}\tilde{Q} - (\alpha+\phi_Z)\lim_{n\to\infty} Z_{n+1} \Longrightarrow \lim_{n\to\infty} Z_n = 0 \tag{5.19}$$

(ii) $TP_3 < 1$ and $\lim_{n\to\infty} D_n = 0$. Eq. (5.18) and (5.19) imply that $\lim_{n\to\infty} Z_n = 0$ and $\lim_{n\to\infty} E_n = 0$. This proves that, FP_2 is GAS.□

Define a parameter K as:

$$K = \frac{a\varrho\theta\lambda\eta(\alpha+\phi_Z b)}{\phi_E\phi_D(\phi_Z+\alpha)(a\varrho\phi_Q+\phi_X\phi_V\kappa)}.$$

Theorem 5.4. *If $TP_4 > 1$ and $1 < TP_3 \leq 1+K$, then $FP_3 = (\bar{X},\bar{N},\bar{Y},\bar{V},\bar{Q},\bar{Z},\bar{E},\bar{D})$ is GAS.*

Proof. Consider

$$\begin{aligned}\Delta F_n = \frac{1}{\Pi(d)}\Bigg[&\bar{X}G\left(\frac{X_n}{\bar{X}}\right)+\bar{N}G\left(\frac{N_n}{\bar{N}}\right)+\frac{\rho+\phi_N}{\rho}\bar{Y}G\left(\frac{Y_n}{\bar{Y}}\right)+\frac{\varrho\bar{X}}{\phi_V}(1+\Pi(d)\phi_V)\bar{V}G\left(\frac{V_n}{\bar{V}}\right)\\&+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\bar{Q}G\left(\frac{Q_n}{\bar{Q}}\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\bar{Z}G\left(\frac{Z_n}{\bar{Z}}\right)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\bar{E}G\left(\frac{E_n}{\bar{E}}\right)\\&+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}(1+\Pi(d)\phi_D)\bar{D}G\left(\frac{D_n}{\bar{D}}\right)\Bigg].\end{aligned}$$

Computing $\Delta F_n = F_{n+1} - F_n$ as:

$$\begin{aligned}\Delta F_n = \frac{1}{\Pi(d)}\Bigg[&\bar{X}G\left(\frac{X_{n+1}}{\bar{X}}\right)+\bar{N}G\left(\frac{N_{n+1}}{\bar{N}}\right)+\frac{\rho+\phi_N}{\rho}\bar{Y}G\left(\frac{Y_{n+1}}{\bar{Y}}\right)\\&+\frac{\varrho\bar{X}}{\phi_V}(1+\Pi(d)\phi_V)\bar{V}G\left(\frac{V_{n+1}}{\bar{V}}\right)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\bar{Q}G\left(\frac{Q_{n+1}}{\bar{Q}}\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\bar{Z}G\left(\frac{Z_{n+1}}{\bar{Z}}\right)\\&+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\bar{E}G\left(\frac{E_{n+1}}{\bar{E}}\right)+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}(1+\Pi(d)\phi_D)\bar{D}G\left(\frac{D_{n+1}}{\bar{D}}\right)\\&-\bar{X}G\left(\frac{X_n}{\bar{X}}\right)-\bar{N}G\left(\frac{N_n}{\bar{N}}\right)-\frac{\rho+\phi_N}{\rho}\bar{Y}G\left(\frac{Y_n}{\bar{Y}}\right)-\frac{\varrho\bar{X}}{\phi_V}(1+\Pi(d)\phi_V)\bar{V}G\left(\frac{V_n}{\bar{V}}\right)\\&-\frac{\mu(\rho+\phi_N)}{\kappa\rho}\bar{Q}G\left(\frac{Q_n}{\bar{Q}}\right)-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\bar{Z}G\left(\frac{Z_n}{\bar{Z}}\right)-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\bar{E}G\left(\frac{E_n}{\bar{E}}\right)\\&-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}(1+\Pi(d)\phi_D)\bar{D}G\left(\frac{D_n}{\bar{D}}\right)\Bigg]\\=\frac{1}{\Pi(d)}\Bigg[&\bar{X}\left(\frac{X_{n+1}-X_n}{\bar{X}}+\ln\left(\frac{X_n}{X_{n+1}}\right)\right)+\bar{N}\left(\frac{N_{n+1}-N_n}{\bar{N}}+\ln\left(\frac{N_n}{N_{n+1}}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\rho+\phi_N}{\rho}\bar{Y}\left(\frac{Y_{n+1}-Y_n}{\bar{Y}}+\ln\left(\frac{Y_n}{Y_{n+1}}\right)\right)+\frac{\varrho\bar{X}}{\phi_V}\left(1+\Pi(d)\phi_V\right)\bar{V}\left(\frac{V_{n+1}-V_n}{\bar{V}}+\ln\left(\frac{V_n}{V_{n+1}}\right)\right)\\
&+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\bar{Q}\left(\frac{Q_{n+1}-Q_n}{\bar{Q}}+\ln\left(\frac{Q_n}{Q_{n+1}}\right)\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\bar{Z}\left(\frac{Z_{n+1}-Z_n}{\bar{Z}}+\ln\left(\frac{Z_n}{Z_{n+1}}\right)\right)\\
&+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\bar{E}\left(\frac{E_{n+1}-E_n}{\bar{E}}+\ln\left(\frac{E_n}{E_{n+1}}\right)\right)+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\times\\
&\left(1+\Pi(d)\phi_D\right)\bar{D}\left(\frac{D_{n+1}-D_n}{\bar{D}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right)\Bigg].
\end{aligned}
$$

Using inequality (5.1), we obtain

$$
\begin{aligned}
\Delta F_n \le \frac{1}{\Pi(d)}\Bigg[&\left(X_{n+1}-X_n+\bar{X}\left(\frac{X_n}{X_{n+1}}-1\right)\right)+\left(N_{n+1}-N_n+\bar{N}\left(\frac{N_n}{N_{n+1}}-1\right)\right)\\
&+\frac{\rho+\phi_N}{\rho}\left(Y_{n+1}-Y_n+\bar{Y}\left(\frac{Y_n}{Y_{n+1}}-1\right)\right)+\frac{\varrho\bar{X}}{\phi_V}\left(V_{n+1}-V_n+\bar{V}\left(\frac{V_n}{V_{n+1}}-1\right)\right)\\
&+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(Q_{n+1}-Q_n+\bar{Q}\left(\frac{Q_n}{Q_{n+1}}-1\right)\right)+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(Z_{n+1}-Z_n+\bar{Z}\left(\frac{Z_n}{Z_{n+1}}-1\right)\right)\\
&+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(E_{n+1}-E_n+\bar{E}\left(\frac{E_n}{E_{n+1}}-1\right)\right)+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\times\\
&\left(D_{n+1}-D_n+\bar{D}\left(\frac{D_n}{D_{n+1}}-1\right)\right)\Bigg]+\varrho\bar{X}V_{n+1}-\varrho\bar{X}V_n+\varrho\bar{X}\bar{V}\ln\left(\frac{V_n}{V_{n+1}}\right)\\
&+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n\\
&+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\bar{D}\ln\left(\frac{D_n}{D_{n+1}}\right).
\end{aligned} \tag{5.20}
$$

Inequality (5.20), can be written as:

$$
\begin{aligned}
\Delta F_n \le \frac{1}{\Pi(d)}\Bigg[&\left(1-\frac{\bar{X}}{X_{n+1}}\right)(X_{n+1}-X_n)+\left(1-\frac{\bar{N}}{N_{n+1}}\right)(N_{n+1}-N_n)+\frac{\rho+\phi_N}{\rho}\left(1-\frac{\bar{Y}}{Y_{n+1}}\right)(Y_{n+1}-Y_n)\\
&+\frac{\varrho\bar{X}}{\phi_V}\left(1-\frac{\bar{V}}{V_{n+1}}\right)(V_{n+1}-V_n)+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\bar{Q}}{Q_{n+1}}\right)(Q_{n+1}-Q_n)\\
&+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{Z}}{Z_{n+1}}\right)(Z_{n+1}-Z_n)+\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{E}}{E_{n+1}}\right)(E_{n+1}-E_n)\\
&+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{D}}{D_{n+1}}\right)(D_{n+1}-D_n)\Bigg]\\
&+\varrho\bar{X}V_{n+1}-\varrho\bar{X}V_n+\varrho\bar{X}\bar{V}\ln\left(\frac{V_n}{V_{n+1}}\right)+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}\\
&-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\bar{D}\ln\left(\frac{D_n}{D_{n+1}}\right).
\end{aligned}
$$

From Eqs. (2.1)-(2.8) we have

$$\begin{aligned}
\Delta F_n \leq & \left(1-\frac{\bar{X}}{X_{n+1}}\right)(\xi-\phi_X X_{n+1}-\varrho V_n X_{n+1})+\left(1-\frac{\bar{N}}{N_{n+1}}\right)(\varrho V_n X_{n+1}-(\rho+\phi_N)N_{n+1}) \\
& +\frac{\rho+\phi_N}{\rho}\left(1-\frac{\bar{Y}}{Y_{n+1}}\right)(\rho N_{n+1}-\phi_Y Y_{n+1}-\mu Y_{n+1}Q_{n+1})+\frac{\varrho\bar{X}}{\phi_V}\left(1-\frac{\bar{V}}{V_{n+1}}\right)(aY_{n+1}-\phi_V V_{n+1}) \\
& +\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\bar{Q}}{Q_{n+1}}\right)(\eta+\kappa Y_{n+1}Q_{n+1}-\phi_Q Q_{n+1}-\theta D_n Q_{n+1}) \\
& +\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{Z}}{Z_{n+1}}\right)((1-b)\theta D_n Q_{n+1}-(\alpha+\phi_Z)Z_{n+1}) \\
& +\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{E}}{E_{n+1}}\right)(b\theta D_n Q_{n+1}+\alpha Z_{n+1}-\phi_E E_{n+1}) \\
& +\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\left(1-\frac{\bar{D}}{D_{n+1}}\right)(\lambda E_{n+1}-\phi_D D_{n+1})+\varrho\bar{X}V_{n+1}-\varrho\bar{X}V_n+\varrho\bar{X}\bar{V}\ln\left(\frac{V_n}{V_{n+1}}\right) \\
& +\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_{n+1}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n \\
& +\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\bar{D}\ln\left(\frac{D_n}{D_{n+1}}\right).
\end{aligned}$$

By collecting terms we get

$$\begin{aligned}
\Delta F_n \leq & \left(1-\frac{\bar{X}}{X_{n+1}}\right)(\xi-\phi_X X_{n+1})+\left(\frac{\varrho\bar{X}}{\phi_V}a-\frac{\rho+\phi_N}{\rho}\phi_Y-\frac{\mu(\rho+\phi_N)}{\rho}\bar{Q}\right)Y_{n+1} \\
& -\varrho V_n X_{n+1}\frac{\bar{N}}{N_{n+1}}+(\rho+\phi_N)\bar{N}-(\rho+\phi_N)N_{n+1}\frac{\bar{Y}}{Y_{n+1}}+\frac{\rho+\phi_N}{\rho}\phi_Y\bar{Y}+\frac{\rho+\phi_N}{\rho}\mu\bar{Y}Q_{n+1} \\
& -\frac{\varrho\bar{X}}{\phi_V}aY_{n+1}\frac{\bar{V}}{V_{n+1}}+\varrho\bar{X}\bar{V}+\frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\bar{Q}}{Q_{n+1}}\right)(\eta-\phi_Q Q_{n+1}) \\
& +\frac{\mu(\rho+\phi_N)}{\kappa\rho}\theta D_n\bar{Q}-\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta D_n Q_{n+1}\frac{\bar{Z}}{Z_{n+1}}+\frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(\alpha+\phi_Z)\bar{Z} \\
& -\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta D_n Q_{n+1}\frac{\bar{E}}{E_{n+1}}-\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\alpha Z_{n+1}\frac{\bar{E}}{E_{n+1}} \\
& +\frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}\phi_E\bar{E}-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}E_{n+1}\frac{\bar{D}}{D_{n+1}}+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\bar{D} \\
& +\varrho\bar{X}\bar{V}\ln\left(\frac{V_n}{V_{n+1}}\right)-\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D D_n+\frac{\mu\phi_E(\rho+\phi_N)(\phi_Z+\alpha)}{\lambda\kappa\rho(\alpha+\phi_Z b)}\phi_D\bar{D}\ln\left(\frac{D_n}{D_{n+1}}\right).
\end{aligned}$$

Using the conditions of fixed point FP_3 :

$$\begin{aligned}
\xi &= \phi_X\bar{X}+\varrho\bar{V}\bar{X}, \\
\varrho\bar{V}\bar{X} &= (\rho+\phi_N)\bar{N}, \\
\rho\bar{N} &= \phi_Y\bar{Y}+\mu\bar{Y}\bar{Q}, \\
a\bar{Y} &= \phi_V\bar{V},
\end{aligned}$$

$$\eta = \phi_Q\bar{Q} - \kappa\bar{Y}\bar{Q} + \theta\bar{D}\bar{Q},$$
$$(1-b)\theta\bar{D}\bar{Q} = (\alpha+\phi_Z)\bar{Z},$$
$$b\theta\bar{D}\bar{Q} = \phi_E\bar{E} - \alpha\bar{Z},$$
$$\lambda\bar{E} = \phi_D\bar{D},$$

we get

$$\begin{aligned}
\Delta F_n \le & \left(1-\frac{\bar{X}}{X_{n+1}}\right)(\phi_X\bar{X} - \phi_X X_{n+1}) - \varrho\bar{V}\bar{X}\frac{\bar{X}}{X_{n+1}} + \varrho\bar{X}\bar{V} + -\varrho\bar{V}\bar{X}\frac{V_nX_{n+1}\bar{N}}{\bar{V}\bar{X}N_{n+1}} + \varrho\bar{X}\bar{V} \\
& - \varrho\bar{X}\bar{V}\frac{N_{n+1}\bar{Y}}{\bar{N}Y_{n+1}} + \varrho\bar{X}\bar{V} - \frac{\rho+\phi_N}{\rho}\mu\bar{Y}\bar{Q} + \frac{\rho+\phi_N}{\rho}\mu\bar{Y}\bar{Q}\frac{Q_{n+1}}{\bar{Q}} - \varrho\bar{X}\bar{V}\frac{Y_{n+1}\bar{V}}{\bar{Y}V_{n+1}} + \varrho\bar{X}\bar{V} \\
& + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(1-\frac{\bar{Q}}{Q_{n+1}}\right)(\phi_Q\bar{Q} - \phi_Q Q_{n+1}) - \frac{\mu(\rho+\phi_N)}{\rho}\bar{Y}\bar{Q} + \frac{\mu(\rho+\phi_N)}{\rho}\bar{Y}\bar{Q}\frac{\bar{Q}}{Q_{n+1}} \\
& + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\theta\bar{D}\bar{Q} - \frac{\mu(\rho+\phi_N)}{\kappa\rho}\theta\bar{D}\bar{Q}\frac{\bar{Q}}{Q_{n+1}} - \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q}\frac{D_nQ_{n+1}\bar{Z}}{\bar{D}\bar{Q}Z_{n+1}} \\
& + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q} - \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q}\frac{D_nQ_{n+1}\bar{E}}{\bar{D}\bar{Q}E_{n+1}} \\
& - \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q}\frac{Z_{n+1}\bar{E}}{\bar{Z}E_{n+1}} + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q} + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q} \\
& - \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q}\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}} - \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q}\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}} \\
& + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q} + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q} + \varrho\bar{X}\bar{V}\ln\left(\frac{V_n}{V_{n+1}}\right) \\
& + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\left(\theta\bar{Q} - \frac{\phi_E\phi_D(\phi_Z+\alpha)}{\lambda(\alpha+\phi_Z b)}\right)D_n + \frac{\mu(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q}\ln\left(\frac{D_n}{D_{n+1}}\right) \\
& + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q}\ln\left(\frac{D_n}{D_{n+1}}\right). \\
= & -\phi_X\frac{(X_{n+1}-\bar{X})^2}{X_{n+1}} - \frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}} - \frac{\rho+\phi_N}{\rho}\mu\bar{Y}\bar{Q}\left(2 - \frac{Q_{n+1}}{\bar{Q}} - \frac{\bar{Q}}{Q_{n+1}}\right) \\
& + \varrho\bar{V}\bar{X}\left(4 - \frac{\bar{X}}{X_{n+1}} - \frac{V_nX_{n+1}\bar{N}}{\bar{V}\bar{X}N_{n+1}} - \frac{N_{n+1}\bar{Y}}{\bar{N}Y_{n+1}} - \frac{Y_{n+1}\bar{V}}{\bar{Y}V_{n+1}} + \ln\left(\frac{V_n}{V_{n+1}}\right)\right) \\
& + \frac{\mu(\rho+\phi_N)}{\kappa\rho}\theta\bar{D}\bar{Q}\left(1-\frac{\bar{Q}}{Q_{n+1}}\right) \\
& + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Z b)}(1-b)\theta\bar{D}\bar{Q}\left(3 - \frac{D_nQ_{n+1}\bar{Z}}{\bar{D}\bar{Q}Z_{n+1}} - \frac{Z_{n+1}\bar{E}}{\bar{Z}E_{n+1}} - \frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}} + \ln\left(\frac{D_n}{D_{n+1}}\right)\right) \\
& + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Z b)}b\theta\bar{D}\bar{Q}\left(2 - \frac{D_nQ_{n+1}\bar{E}}{\bar{D}\bar{Q}E_{n+1}} - \frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}} + \ln\left(\frac{D_n}{D_{n+1}}\right)\right).
\end{aligned} \tag{5.21}$$

Since we have

$$-\frac{\rho+\phi_N}{\rho}\mu\bar{Y}\bar{Q}\left(2-\frac{Q_{n+1}}{\bar{Q}} - \frac{\bar{Q}}{Q_{n+1}}\right) - \frac{\mu(\rho+\phi_N)}{\kappa\rho}\phi_Q\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}$$

$$= \frac{\mu(\rho+\phi_N)}{\rho}\bar{Y}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}} - \frac{\mu\phi_Q(\rho+\phi_N)}{\kappa\rho}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}$$
$$= \frac{\mu(\rho+\phi_N)}{\rho}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}\left(\bar{Y}-\frac{\phi_Q}{\kappa}\right)$$
$$= \frac{\mu(\rho+\phi_N)(\kappa\phi_X\phi_V+a\varrho\phi_Q)}{a\varrho\kappa\rho}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}\left(TP_3-K-1\right).$$

Collecting the terms of Eq. (5.21), we get

$$\begin{aligned}\Delta F_n \leq & -\phi_X\frac{(X_{n+1}-\bar{X})^2}{X_{n+1}} + \frac{\mu(\rho+\phi_N)(\kappa\phi_X\phi_V+a\varrho\phi_Q)}{a\varrho\kappa\rho}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}\left(TP_3-K-1\right) \\ & + \varrho\bar{V}\bar{X}\left(4-\frac{\bar{X}}{X_{n+1}}-\frac{V_nX_{n+1}\bar{N}}{\bar{V}\bar{X}N_{n+1}}-\frac{N_{n+1}\bar{Y}}{\bar{N}Y_{n+1}}-\frac{Y_{n+1}\bar{V}}{\bar{Y}V_{n+1}}+\ln\left(\frac{V_n}{V_{n+1}}\right)\right) \\ & + \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Zb)}(1-b)\theta\bar{D}\bar{Q}\left(4-\frac{\bar{Q}}{Q_{n+1}}-\frac{D_nQ_{n+1}\bar{Z}}{\bar{D}\bar{Q}Z_{n+1}}-\frac{Z_{n+1}\bar{E}}{\bar{Z}E_{n+1}}-\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right) \\ & + \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Zb)}b\theta\bar{D}\bar{Q}\left(3-\frac{\bar{Q}}{Q_{n+1}}-\frac{D_nQ_{n+1}\bar{E}}{\bar{D}\bar{Q}E_{n+1}}-\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}+\ln\left(\frac{D_n}{D_{n+1}}\right)\right). \end{aligned} \tag{5.22}$$

By using the following equality:

$$\ln\left(\frac{V_n}{V_{n+1}}\right) = \ln\left(\frac{\bar{X}}{X_{n+1}}\right)+\ln\left(\frac{V_nX_{n+1}\bar{N}}{\bar{V}\bar{X}N_{n+1}}\right)+\ln\left(\frac{N_{n+1}\bar{Y}}{\bar{N}Y_{n+1}}\right)+\ln\left(\frac{Y_{n+1}\bar{V}}{\bar{Y}V_{n+1}}\right),$$
$$\ln\left(\frac{D_n}{D_{n+1}}\right) = \ln\left(\frac{\bar{Q}}{Q_{n+1}}\right)+\ln\left(\frac{D_nQ_{n+1}\bar{Z}}{\bar{D}\bar{Q}Z_{n+1}}\right)+\ln\left(\frac{Z_{n+1}\bar{E}}{\bar{Z}E_{n+1}}\right)+\ln\left(\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}\right),$$
$$\ln\left(\frac{D_n}{D_{n+1}}\right) = \ln\left(\frac{\bar{Q}}{Q_{n+1}}\right)+\ln\left(\frac{D_nQ_{n+1}\bar{E}}{\bar{D}\bar{Q}E_{n+1}}\right)+\ln\left(\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}\right).$$

Then Eq. (5.22) becomes

$$\begin{aligned}\Delta F_n \leq & -\phi_X\frac{(X_{n+1}-\bar{X})^2}{X_{n+1}} + \frac{\mu(\rho+\phi_N)(\kappa\phi_X\phi_V+a\varrho\phi_Q)}{a\varrho\kappa\rho}\frac{(Q_{n+1}-\bar{Q})^2}{Q_{n+1}}\left(TP_3-K-1\right) \\ & - \varrho\bar{V}\bar{X}\left[G\left(\frac{\bar{X}}{X_{n+1}}\right)+G\left(\frac{V_nX_{n+1}\bar{N}}{\bar{V}\bar{X}N_{n+1}}\right)+G\left(\frac{N_{n+1}\bar{Y}}{\bar{N}Y_{n+1}}\right)+G\left(\frac{Y_{n+1}\bar{V}}{\bar{Y}V_{n+1}}\right)\right] \\ & - \frac{\mu\alpha(\rho+\phi_N)}{\kappa\rho(\alpha+\phi_Zb)}(1-b)\theta\bar{D}\bar{Q}\left[G\left(\frac{\bar{Q}}{Q_{n+1}}\right)+G\left(\frac{D_nQ_{n+1}\bar{Z}}{\bar{D}\bar{Q}Z_{n+1}}\right)+G\left(\frac{Z_{n+1}\bar{E}}{\bar{Z}E_{n+1}}\right)+G\left(\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}\right)\right] \\ & - \frac{\mu(\rho+\phi_N)(\phi_Z+\alpha)}{\kappa\rho(\alpha+\phi_Zb)}b\theta\bar{D}\bar{Q}\left[G\left(\frac{\bar{Q}}{Q_{n+1}}\right)+G\left(\frac{D_nQ_{n+1}\bar{E}}{\bar{D}\bar{Q}E_{n+1}}\right)+G\left(\frac{E_{n+1}\bar{D}}{\bar{E}D_{n+1}}\right)\right]. \end{aligned}$$

We note that $\Delta F_n \leq 0$. since $1 < TP_3 \leq 1+K$. Therefore F_n is monotonically decreasing. Since $F_n \geq 0$, Then $\lim_{n\to\infty} F_n \geq 0$ and $\lim_{n\to\infty} F_n = 0$. This gives, $\lim_{n\to\infty}(X_n, N_n, Y_n, V_n, Q_n, Z_n, E_n, D_n) = (\bar{X}, \bar{N}, \bar{Y}, \bar{V}, \bar{Q}, \bar{Z}, \bar{E}, \bar{D})$. Hence, FP_3 is GAS.□

6. Numerical simulations

To conduct numerical simulations for the discrete-time model (2.1)-(2.8), we use the following values: $\xi = 0.0224$, $\phi_X = 0.001$, $\rho = 4.08$, $\phi_N = 0.001$, $\phi_Y = 0.11, a = 0.24$, $\eta = 10$, $\kappa = 0.1$,

$\phi_Q = 0.01$, $\alpha = 0.2$, $\phi_Z = 0.02$, $b = 0.7$, $\phi_E = 0.5$, $\lambda = 5$, $\phi_D = 2$ and $d = 0.1$. The other parameters will be chosen below. We take the following initial values:

$$\begin{aligned}
\text{IV1}: &\quad (X_0, N_0, Y_0, V_0, U_0, L_0, A_0) = (10, 0.0002, 0.0003, 0.0004, 150, 10, 15, 20),\\
\text{IV2}: &\quad (X_0, N_0, Y_0, V_0, U_0, L_0, A_0) = (15, 0.002, 0.003, 0.004, 250, 15, 20, 25),\\
\text{IV3}: &\quad (X_0, N_0, Y_0, V_0, U_0, L_0, A_0) = (20, 0.003, 0.004, 0.005, 350, 20, 25, 30).
\end{aligned}$$

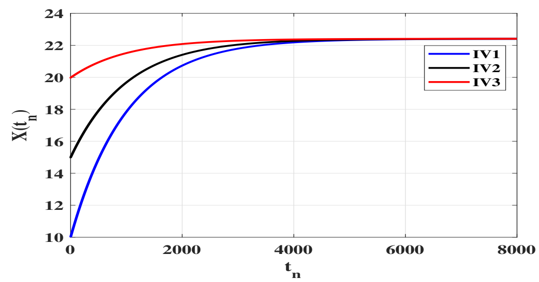
We choose ϱ, κ, ϕ_V, and λ as:

Case (C1) $\varrho = 0.8$, $\mu = 1.2$, $\phi_V = 5.40$, and $\theta = 0.0002$. This gives $TP_1 = 0.9727 < 1$ and $TP_2 = 0.0007 < 1$. Figure (1) illustrates that, the concentrations of uninfected epithelial cells and uninfected CD4$^+$T cells increase and tend to the healthy values $X^0 = 22.41$ and $U^0 = 1000$, while the concentrations of other compartments decrease and converge to zero. Therefore, FP_0 is GAS and this agrees the result of Theorem 5.1. In this case, both SARS-CoV-2 and HIV are cleared.

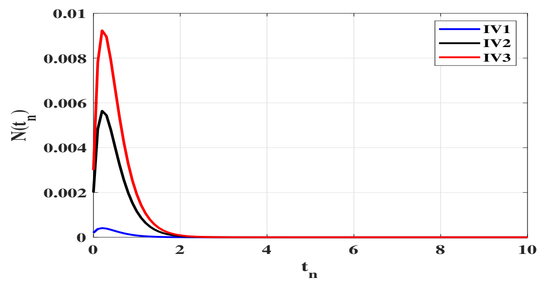
Case (C2) $\varrho = 0.66$, $\mu = 1.2$, $\phi_V = 5.40$, and $\theta = 0.0017$. These values give $TP_1 = 8.2682 > 1$ and $TP_4 = 0.0045 < 1$. Figure (2) shows that, the solutions of the discrete-time model tend to the fixed point $FP_1 = (22.41, 0, 0, 0, 120.946, 11.9871, 17.1016, 42.754)$. Hence, FP_1 exists and based on Theorem 5.2, it is GAS. This describes the situation of a patient who infected by HIV, while the SARS-CoV-2 infected is cleared.

Case (C3) $\varrho = 2.8$, $\mu = 0.03$, $\phi_V = 0.04$, and $\theta = 0.0002$, and then $TP_2 = 12.5007 > 1$ and $TP_3 = 0.9794 < 1$. Figure (3) clarifies that the solutions of the discrete-time model reach the fixed point $FP_2 = (1.8079, 0.005, 0.001, 0.0034, 1006.8, 0, 0, 0)$ for arbitrary initial conditions. Lemma 2 and Theorem 5.3 state that FP_2 exists and it is GAS. This describes the status of a patient who infected by SARS-CoV-2, while the HIV infection is removed.

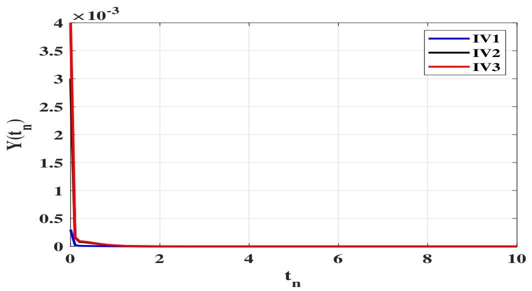
Case (C4) $\varrho = 2.8$, $\mu = 0.03$, $\phi_V = 0.2$, and $\theta = 0.0017$, and thus, $TP_4 = 20.1369 > 1$, $TP_3 = 8.3034 > 1$, and $TP_3 < 1 + K = 8.2437$. Figure (4) shows that the solutions of the discrete-time model starting arbitrary initial converge to a fixed point $FP_3 = (1.113, 0.005, 0.006, 0.007, 120.946, 12.081, 17.236, 43.089)$. Lemma 2 and Theorem 5.4 state that, FP_3 exists and it is GAS.



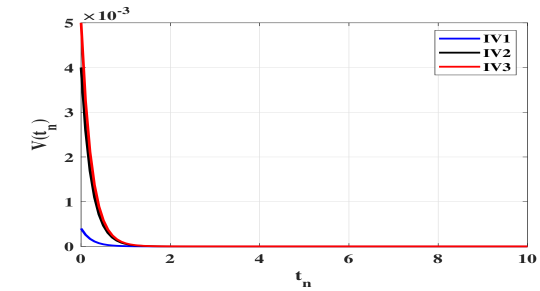
(A) Uninfected ECs



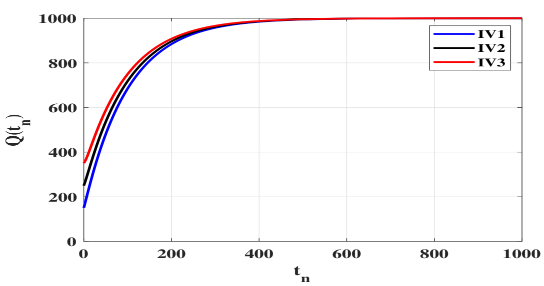
(B) Latently SARS-CoV-2-infected ECs



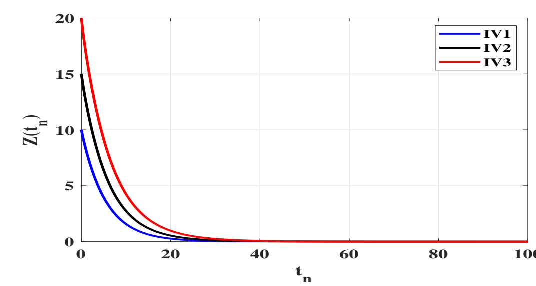
(C) Actively SARS-CoV-2- infected ECs



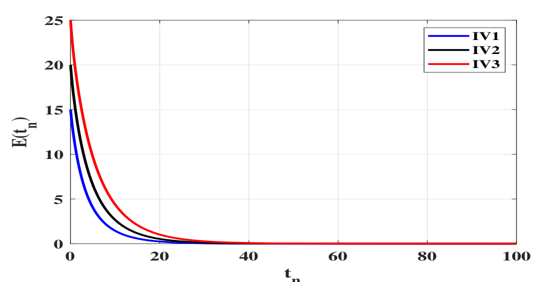
(D) SARS-CoV-2 particles



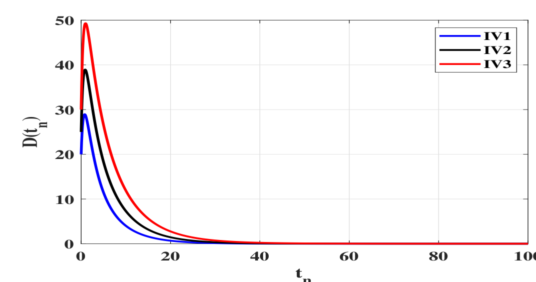
(E) Uninfected CD4$^+$T cells



(F) Latenlty HIV infected CD4$^+$T cells

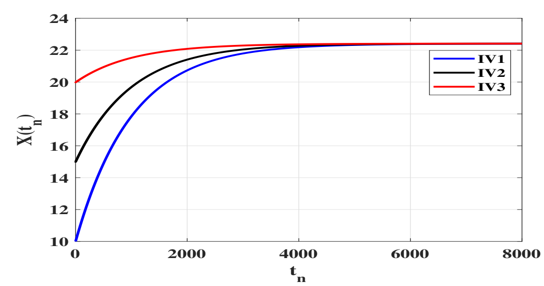


(G) Actively HIV-infected CD4$^+$T cells

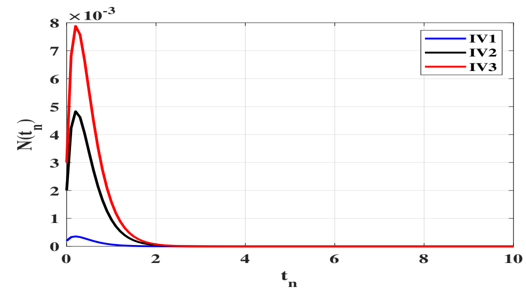


(H) HIV particles

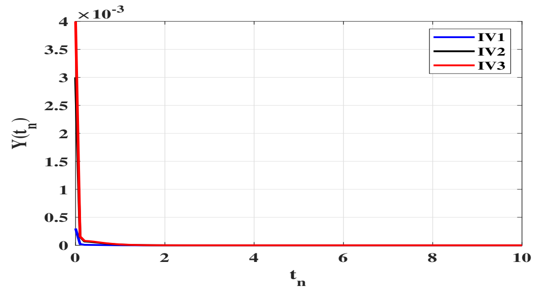
FIGURE 1. Solutions of system (2.1)-(2.8) with initial conditions IV1-IV3 in case of $TP_1 < 1$ and $TP_2 < 1$.



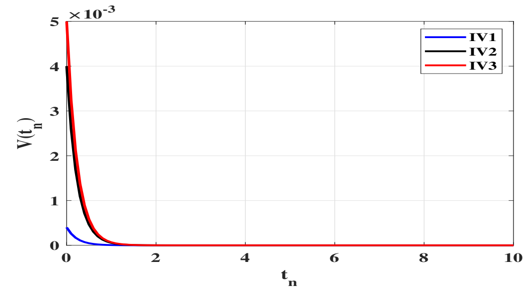
(A) Uninfected ECs



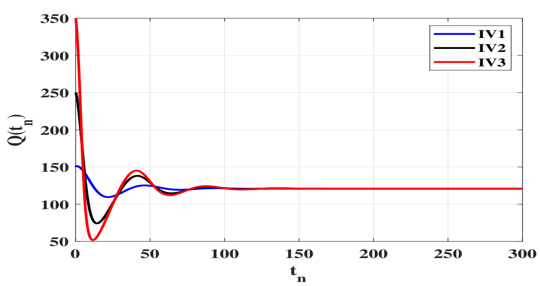
(B) Latently SARS-CoV-2-infected ECs



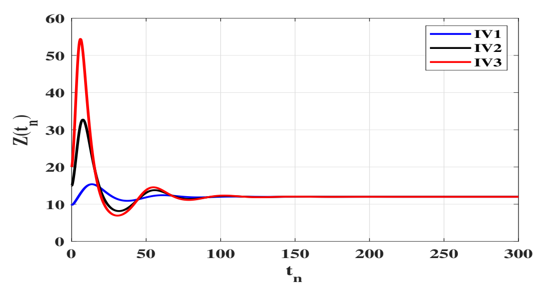
(C) Actively SARS-CoV-2- infected ECs



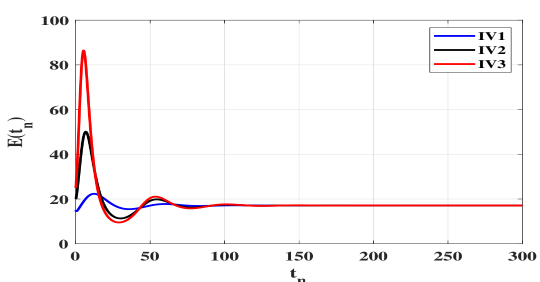
(D) SARS-CoV-2 particles



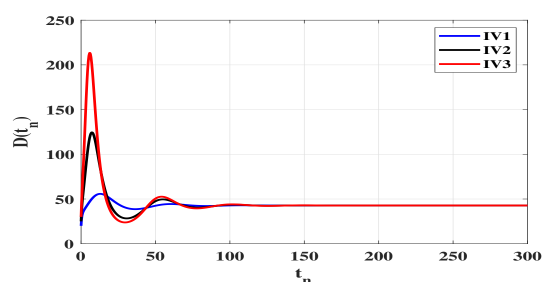
(E) Uninfected $CD4^{+}T$ cells



(F) Latenlty HIV infected $CD4^{+}T$ cells

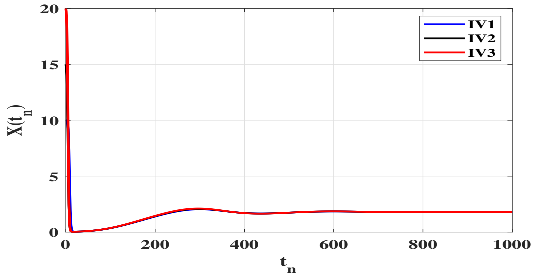


(G) Actively HIV-infected $CD4^{+}T$ cells

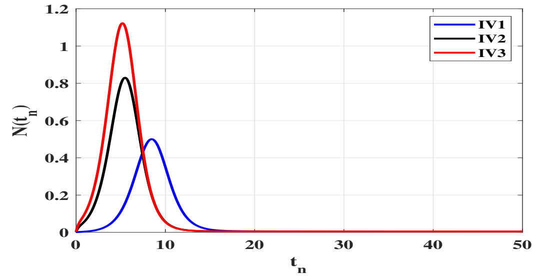


(H) HIV particles

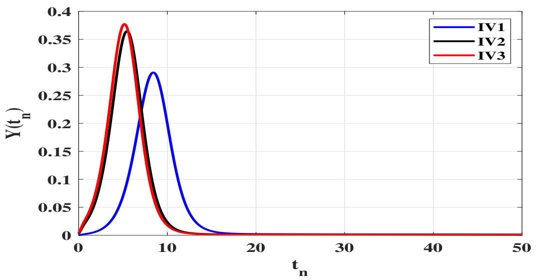
FIGURE 2. Solutions of system (2.1)-(2.8) with initial conditions IV1-IV3 in case of $TP_1 > 1$ and $TP_4 < 1$.



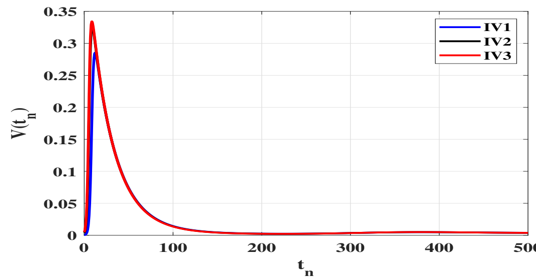
(A) Uninfected ECs



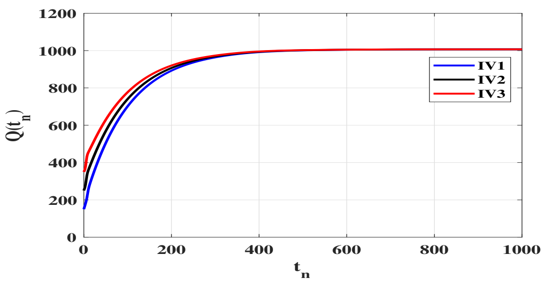
(B) Latently SARS-CoV-2-infected ECs



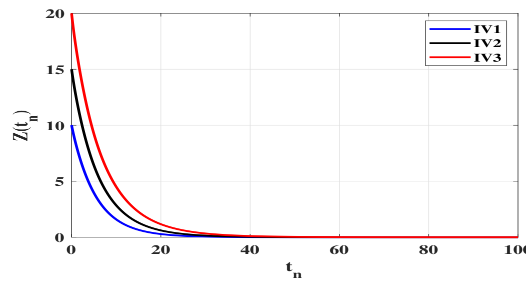
(C) Actively SARS-CoV-2- infected ECs



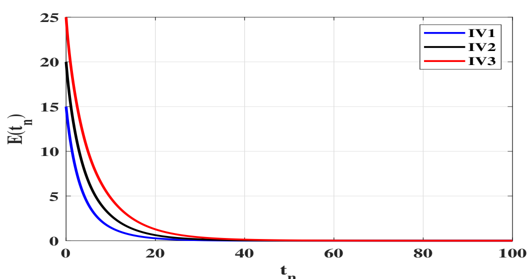
(D) SARS-CoV-2 particles



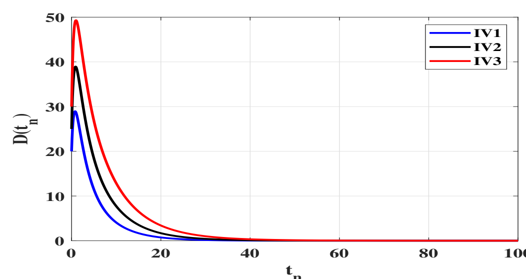
(E) Uninfected $CD4^{+}T$ cells



(F) Latenlty HIV infected $CD4^{+}T$ cells

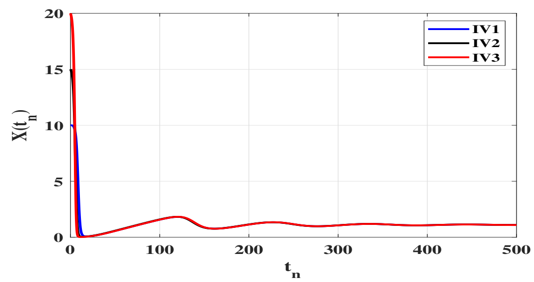


(G) Actively HIV-infected $CD4^{+}T$ cells

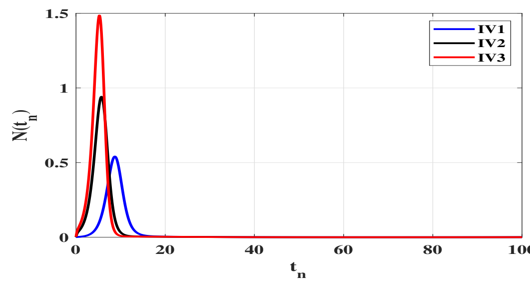


(H) HIV particles

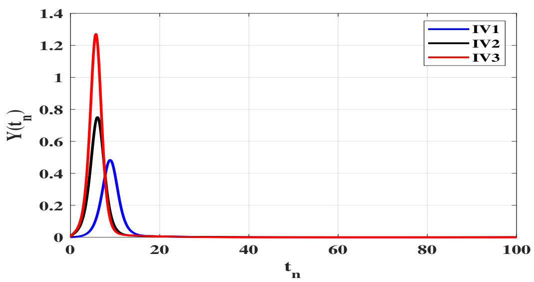
FIGURE 3. Solutions of system (2.1)-(2.8) with initial conditions IV1-IV3 in case of $TP_2 > 1$ and $TP_3 < 1$.



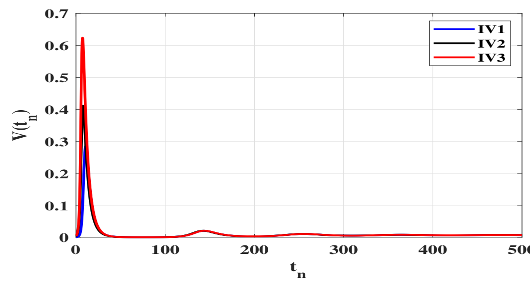
(A) Uninfected ECs



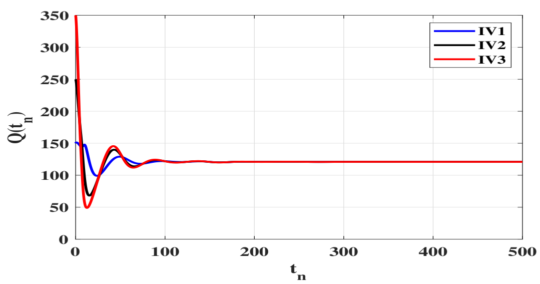
(B) Latently SARS-CoV-2-infected ECs



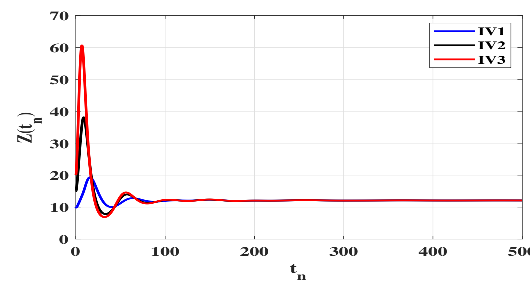
(C) Actively SARS-CoV-2- infected ECs



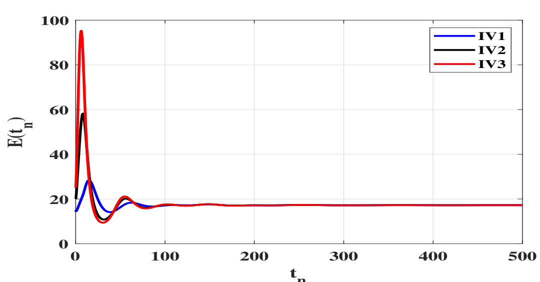
(D) SARS-CoV-2 particles



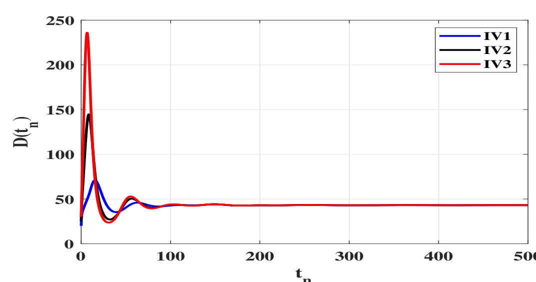
(E) Uninfected $CD4^{+}T$ cells



(F) Latenlty HIV infected $CD4^{+}T$ cells



(G) Actively HIV-infected $CD4^{+}T$ cells



(H) HIV particles

FIGURE 4. Solutions of system (2.1)-(2.8) with initial conditions IV1-IV3 in case of $TP_4 > 1$ and $TP_3 > 1$.

7. Conclusion

In this paper, we constructed a discrete SARS-CoV-2/HIV co-dynamics model that explores the interplay among eight components: uninfected ECs, latently infected ECs, actively infected ECs, free SARS-CoV-2 particles, uninfected $CD4^{+}T$ cells, latently infected $CD4^{+}T$ cells, actively infected $CD4^{+}T$ cells and free HIV particles. This model has four fixed points depending on the four threshold parameters (TP_i, $i = 1, 2, 3, 4$) as the following:
(a) Infection-free fixed point FP_0 always exists. It is GAS when $TP_1 \leq 1$ and $TP_2 \leq 1$. This represents the healthy state when the person is free from both SARS-CoV-2 and HIV infections.
(b) The HIV mono-infection fixed point FP_1 is defined if $TP_1 > 1$, and it is GAS if $TP_4 \leq 1$. This represents the case of a patient who has contracted HIV infection only.
(c) The SARS-CoV-2 mono-infection fixed point FP_2 is defined if $TP_2 > 1$ and it is GAS if $TP_3 \leq 1$. This represents the case of a patient who has only SARS-CoV-2 infection.
(d) The SARS-CoV-2/HIV co-dynamics fixed point FP_3 is defined and GAS if $TP_4 > 1$ and $1 < TP_3 \leq 1 + K$. This case simulates the occurrence of SARS-CoV-2 infection in individuals who are already infected with HIV. The numerical results are entirely consistent with the theoretical results. The model examined in this paper could be enhanced by taking into account time delays associated with various biological processes. Additionally, incorporating the impact of treatments might yield significant insights that can contribute to the development of therapies for this specific group of patients.

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