Int. J. Anal. Appl. (2024), 22:71

International Journal of Analysis and Applications

Mathematical Analysis for a *Zika Virus* Dynamics in a Seasonal Environment

Fatema Ahmed Al Najim*

Department of Mathematics and Statistics, College of Science, King Faisal University, P. O. Box 400, Al Ahsa 31982, Saudi Arabia

**Corresponding author:* Falnajem@kfu.edu.sa

Abstract. We propose a mathematical model for the *Zika virus* (ZIKV) spread under the influence of a seasonal environment. The basic reproduction number $\mathcal{R}_0$ was calculated for both cases, the fixed and seasonal environment permitting the characterisation of the extinction and the persistence of the disease for both cases. We proved that the virus-free steady state is globally asymptotically stable if $\mathcal{R}_0 \leq 1$, while the disease will be persist if $\mathcal{R}_0 > 1$. Finally, extensive numerical simulations are given to confirm the theoretical findings.

1. Introduction

Infectious diseases are communicable diseases. They are transmitted either directly from one person to another as is the case for the flu, measles or diphtheria. among others; either through a vector such as the mosquito for chikungunya or malaria, the tick for Lyme disease, etc. ; either through the environment passing through for example through food or water for salmonellosis or cholera. Many pathogens are present and circulate simultaneously in populations and are therefore likely to interact both with their environment and with each other. These interactions occur at different levels, from the population to the host scale, and are likely to affect the epidemiological dynamics of the pathogens concerned or the diseases they cause. The interactions studied in this thesis work are intra-host interactions only [1]. The Zika virus is a Flavivirus, of the Flaviviridae family, like the West Nile virus, those of yellow fever and dengue. The Zika virus is transmitted primarily by mosquitoes of the Aedes genus. Aedes mosquitoes generally bite during the day. The main vector of the Zika virus, Aedes aegypti, is found mainly in tropical and subtropical regions. As for Aedes albopictus, it is present in more temperate zones, on all continents, with the exception of Antarctica. Primates (humans and non-humans) constitute the reservoir of the virus [2]. In 70

Received: Mar. 6, 2024.

2020 *Mathematics Subject Classification.* 34K13, 37N25, 92D30.

Key words and phrases. Zika virus; seasonal environment; periodic solution; uniform persistence; extinction; basic reproduction number.

https://doi.org/10.28924/2291-8639-22-2024-71

ISSN: 2291-8639

to 80% of cases, no symptoms are apparent. In others, symptoms are usually mild and last 2 to 7 days. They are mainly characterized by (low-grade) fever, joint pain, conjunctivitis and skin rashes. Other symptoms include fatigue, muscle pain and headaches [3]. Rare cases of encephalopathy, meningoencephalitis, myelitis, uveitis and severe thrombocytopenia have been reported. Neurological complications such as Guillain-Barré syndrome (GBS) have been documented in people infected with the Zika virus. GBS is often triggered by a bacterial or viral infection. In the case of the Zika virus, an unusual increase in GBS cases has been reported in countries affected by the virus. Data currently suggests a causal link between Zika virus infection and GBS, with the most likely hypothesis being that the virus is a trigger for GBS. A causal relationship has been established between prenatal Zika virus infections and cases of microcephaly and other congenital brain abnormalities [4]. Indeed, maternal-fetal transmission of the Zika virus is possible, and it can cause adverse events in the fetus: fetal death in utero, microcephaly, malformations of the brain, limbs and eyes. The World Health Organization (WHO) estimates that 5 to 15% of infants born to mothers infected with the Zika virus during pregnancy may have complications related to the virus. Birth defects can occur as a result of symptomatic or asymptomatic infection. The risk is greater if the mother contracts the virus during the first trimester, but all trimesters are at risk. Other complications include stillbirths and prematurity.

A mathematical model is the representation of an object, a system, a phenomenon, a theory through mathematical language. This representation is necessarily a simplification of reality, based on a set of hypotheses. Depending on the objective sought when constructing the model, the hypotheses will not be the same and the way of representing the same object will be different. For example, a plastic bottle will not be represented in the same way in models seeking to study its resistance to changes in temperature. temperature, to optimize the production of a mineral water company or to improve recycling processes. A model must be adapted to the question it seeks to answer, neither too complex nor too simple. The formulation of hypotheses is therefore essential in order to include all the aspects necessary for the study of the problem and to avoid any redundancy or superfluity [5]. The role of a mathematical model is to represent an aspect of reality in a correct manner as simply as possible, according to the principle of parsimony. In the case of the study of infectious diseases, the primary objective is often to describe the mode of transmission of the pathogen or disease considered, based on surveillance data to try to understand the biological mechanisms at play [5]. Once a model can adequately reproduce the data already available, it can potentially be used for prediction or evaluation of public health measures. In a compartmental model, the population is distributed into different compartments according to the infectious status of the individuals. This infectious status is the only characteristic allowing individuals to be discriminated: within a compartment they are indistinguishable from each other. The dynamics of this type of model is generally governed by a set of equations differentials. Several researchers worked on some mathematical models for several infectious diseases [6–10]. In particular, the modeling of the behavior of *Zika virus* was considered in several works [11, 12].

Note that seasonality in infectious is very repetitive [13]. In particular, each year with the return of cold weather, infectious diseases spread among the population. Although they are often temporary and harmless, they can nevertheless be much more serious, particularly in the weakest people. Several works modeling the infectious diseases that reflect the seasonality were considered [14–16]. The basic reproduction number is expressed either using the time-averaged system (autonomous) as in [17,18] or other definition as in [10,19,20]. In [21–26], several epidemic models were considered under the influence of seasonality. We aim in this paper to study the impact of seasonality on the dynamics of *ZIKV*. The basic reproduction number is expressed as the spectral radius of a linear integral operator. We analysed the global stability of the disease-free solution where we proved that it is globally asymptotically stable if $\mathcal{R}_0 < 1$. However, $\mathcal{R}_0 > 1$, we proved that the dynamics is persistent. We validate the theoretical results trough several numerical tests.

The rest of this article is organized as follows. In Section 2, we discuss the bilinear *ZIKV* model influenced by the seasonality. In Section 3, we consider on the case of fixed environment, and we calculated $\mathcal{R}_0$ and we studied the global analysis of both, the disease-free and the endemic equilibrium points. However, in section 6, we focus on the stability of virus-free periodic solution and the persistence of the disease for the case of seasonal environment. We give several numerical test confirming the theoretical results. Finally, in section 7, we give some concluding remarks.

2. Mathematical model for *Zika virus* dynamics in a seasonal environment

The virus is spread as follows. Mosquitoes contract the virus by biting human infected with it. They the transmission of the virus increases by biting an uninfected human [27]. The mathematical model for *Zika virus* spread that we proposed here is a compartmental one. Therefore, the model is given by the fifth dimensional system of differential equations hereafter.

$$\left\{\begin{array}{lcl} \dot{S}_h(t) & = & S_{in}^h(t) - \rho_h(t)I_v(t)S_h(t) - d_h(t)S_h(t), \\ \dot{I}_h(t) & = & \rho_h(t)I_v(t)S_h(t) - (r_h(t) + d_h(t))I_h(t), \\ \dot{R}_h(t) & = & r_h(t)I_h(t) - d_h(t)R_h(t), \\ \dot{S}_v(t) & = & S_{in}^v(t) - \rho_v(t)I_h(t)S_v(t) - d_v(t)S_v(t), \\ \dot{I}_v(t) & = & \rho_v(t)I_h(t)S_v(t) - d_v(t)I_v(t). \end{array}\right. \tag{2.1}$$

S_h, I_h and R_h describe the susceptible, infected and recovered in the human population, respectively. S_v and I_v describe, respectively, the susceptible and infected in the mosquito population. More details concerning the significance of the model parameters are given in Table 1.

Notation	Definition
S_{in}^h	Periodic recruitment rate of human population
S_{in}^v	Periodic recruitment rate of mosquito population
ρ_h	Periodic contact rate between I_v and S_h
ρ_v	Periodic contact rate between I_h and S_v
d_h	Periodic natural death rate of human population
d_v	Periodic natural death rate of mosquito population
r_h	Periodic natural recovery rate of symptomatic infective human population

Table 1. Parameters and variables of system (6.1).

The model parameters satisfy the following assumption:

Assumption 2.1. *The functions $S_{in}^h(t)$, $S_{in}^v(t)$, $\rho_h(t)$, $\rho_v(t)$, $d_h(t)$, $d_v(t)$, and $r_h(t)$ are non-negative continuous bounded and T-periodic.*

3. Case of fixed environment

By considering fixed parameters for the model (2.1), we obtain a simple mathematical model equivalent to the ones given in [28–30].

$$\begin{cases} \dot{S}_h &= S_{in}^h - \rho_h I_v S_h - d_h S_h, \\ \dot{I}_h &= \rho_h I_v S_h - (r_h + d_h) I_h, \\ \dot{R}_h &= r_h I_h - d_h R_h, \\ \dot{S}_v &= S_{in}^v - \rho_v I_h S_v - d_v S_v, \\ \dot{I}_v &= \rho_v I_h S_v - d_v I_v. \end{cases} \tag{3.1}$$

We aim in section to study firstly the behavior of the model (3.1). We start by giving some basic classic properties.

3.1. **Basic properties.** The variables of the model (3.1) remain positive and bounded and the system is conservative.

Lemma 3.1.

$$\Gamma = \left\{ (S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^5 : \ N_h = \frac{S_{in}^h}{d_h}, \ N_v = \frac{S_{in}^v}{d_v} \right\}$$

is a positively invariant attractor of all solutions of system (3.1) where $N_h = S_h + I_h + R_h$ and $N_v = S_v + I_v$ are the total population sizes of human and mosquitoes, respectively.

Proof. We have

$$\begin{array}{rcl} \dot{S}_h \mid_{S_h=0} & = & S_{in}^h > 0, \\ \dot{I}_h \mid_{I_h=0} & = & \rho_h I_v S_h \geq 0, \\ \dot{R}_h \mid_{R_h=0} & = & r_h I_h \geq 0, \\ \dot{S}_v \mid_{S_v=0} & = & S_{in}^v > 0, \\ \dot{I}_v \mid_{I_v=0} & = & \rho_v I_h S_v \geq 0. \end{array}$$

Thus one can deduce that the closed non-negative cone $\mathbb{R}_+^5$ is positively invariant by system (3.1). From Eqs. (3.1) we get $\dot{N}_h = S_{in}^h - d_h N_h$. Hence $N_h = \frac{S_{in}^h}{d_h}$ if $N_h(0) = \frac{S_{in}^h}{d_h}$. Similarly, $\dot{N}_v = S_{in}^v - d_v N_v$. Hence $N_v = \frac{S_{in}^v}{d_v}$ if $N_v(0) = \frac{S_{in}^v}{d_v}$. □

3.2. **Steady states: Existence and uniqueness.** The next-generation matrix method introduced by Diekmann [31, 32]. In compartmental models describing the infectious diseases transmission, this method is used to derive the basic reproduction number. In our case, by considering $F = \begin{pmatrix} 0 & \rho_h \frac{S_{in}^h}{d_h} \\ \rho_v \frac{S_{in}^v}{d_v} & 0 \end{pmatrix}$ and $V = \begin{pmatrix} r_h + d_h & 0 \\ 0 & d_v \end{pmatrix}$, we obtain the next generation matrix $FV^{-1} = \begin{pmatrix} 0 & \frac{\rho_h S_{in}^h}{d_v d_h} \\ \frac{\rho_v S_{in}^v}{d_v(r_h + d_h)} & 0 \end{pmatrix}$ and then the basic reproduction number (spectral radius of FV^{-1}) for (3.1) is given by.

$$\mathcal{R}_0 = r(FV^{-1}) = \sqrt{\frac{\rho_h \rho_v S_{in}^h S_{in}^v}{d_v^2 d_h (r_h + d_h)}}.$$

Lemma 3.2.
- *If $\mathcal{R}_0 \leq 1$, then model (3.1) has a trivial steady state E_0.*
- *If $\mathcal{R}_0 > 1$, then model (3.1) has two steady states E_0 and E^*.*

Proof. Let $E(S_h, I_h, R_h, S_v, I_v)$ be any steady state satisfying

$$\begin{cases} 0 = S_{in}^h - \rho_h I_v S_h - d_h S_h, \\ 0 = \rho_h I_v S_h - (r_h + d_h) I_h, \\ 0 = r_h I_h - d_h R_h, \\ 0 = S_{in}^v - \rho_v I_h S_v - d_v S_v, \\ 0 = \rho_v I_h S_v - d_v I_v. \end{cases} \tag{3.2}$$

which is equivalent to

$$\begin{cases} S_h &= \dfrac{S_{in}^h-(r_h+d_h)I_h}{d_h}=\dfrac{S_{in}^h}{d_h}-\dfrac{(r_h+d_h)}{d_h}I_h, \\ R_h &= \dfrac{r_hI_h}{d_h}, \\ I_v &= \dfrac{\rho_vI_hS_v}{d_v}=\dfrac{S_{in}^v}{d_v}\dfrac{\rho_vI_h}{\rho_vI_h+d_v}, \\ S_v &= \dfrac{S_{in}^v}{d_v+\rho_vI_h}. \end{cases} \tag{3.3}$$

From the second equation of system (3.2), we have

$$\begin{aligned} 0 &= \rho_hI_vS_h-(r_h+d_h)I_h \\ &= \frac{S_{in}^v}{d_v}\frac{\rho_h\rho_vI_h}{\rho_vI_h+d_v}\Big(\frac{S_{in}^h}{d_h}-\frac{(r_h+d_h)}{d_h}I_h\Big)-(r_h+d_h)I_h \end{aligned} \tag{3.4}$$

If $I_h=0$, then we get a steady state given by the ZIKV-free steady state $E_0=\left(\dfrac{S_{in}^h}{d_h},0,0,\dfrac{S_{in}^v}{d_v},0\right)$. If $I_h\neq 0$, define the function

$$f(I_h)=\frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v}{\rho_vI_h+d_v}\Big(\frac{S_{in}^h}{d_h}-\frac{(r_h+d_h)}{d_h}I_h\Big)-(r_h+d_h).$$

Then we obtain

$$\begin{aligned} \lim_{I_h\to 0^+}f(I_h) &= \frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v}{d_v}\frac{S_{in}^h}{d_h}-(r_h+d_h) \\ &= (r_h+d_h)(\mathcal{R}_0^2-1)>0 \text{ if } \mathcal{R}_0>1. \end{aligned}$$

Now, we have

$$\begin{aligned} f\Big(\frac{S_{in}^h}{d_h}\Big) &= \frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v}{\rho_v\dfrac{S_{in}^h}{d_h}+d_v}\Big(\frac{S_{in}^h}{d_h}-\frac{(r_h+d_h)}{d_h}\frac{S_{in}^h}{d_h}\Big)-(r_h+d_h) \\ &= -\frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v}{\rho_v\dfrac{S_{in}^h}{d_h}+d_v}\frac{r_h}{d_h}\frac{S_{in}^h}{d_h}-(r_h+d_h)<0. \end{aligned}$$

The derivative of the function f on $(0,\dfrac{S_{in}^h}{d_h})$ is given by

$$\begin{aligned} f'(I_h) &= -\frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v^2}{(\rho_vI_h+d_v)^2}\Big(\frac{S_{in}^h}{d_h}-\frac{(r_h+d_h)}{d_h}I_h\Big)-\frac{S_{in}^v}{d_v}\frac{\rho_h\rho_v}{\rho_vI_h+d_v}\frac{(r_h+d_h)}{d_h} \\ &\leq 0. \end{aligned}$$

Thus f is an decreasing function. Then the equation $f(I_h) = 0$ admits a unique solution $\bar{I}_h \in (0, \frac{S^h_{in}}{d_h})$. Therefore,

$$\begin{cases} \bar{S}_h &= \dfrac{S^h_{in}}{d_h} - \dfrac{(r_h + d_h)}{d_h}\bar{I}_h, \\ \bar{R}_h &= \dfrac{r_h \bar{I}_h}{d_h}, \\ \bar{I}_v &= \dfrac{S^v_{in}}{d_v}\dfrac{\rho_v \bar{I}_h}{\rho_v \bar{I}_h + d_v}, \\ \bar{S}_v &= \dfrac{S^v_{in}}{d_v + \rho_v \bar{I}_h}, \end{cases} \tag{3.5}$$

and the infected steady state $E^* = (\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v)$ exists only if $\mathcal{R}_0 > 1$. □

4. Local stability of steady states

The $\mathcal{R}_0$ rate is important to determine the risk of a new pathogen causing an epidemic (impossible if $\mathcal{R}_0 < 1$, possible if $\mathcal{R}_0 > 1$) and to estimate the final size of the epidemic, with or without control measures.

Theorem 4.1. *If $\mathcal{R}_0 < 1$, then the trivial equilibrium point E_0 is locally asymptotically stable.*

Proof. The Jacobian matrix of system (3.1) at a point $(S_h, I_h, R_h, S_v, I_v)$ is given by:

$$J = \begin{pmatrix} -(\rho_h I_v + d_h) & 0 & 0 & 0 & -\rho_h S_h \\ \rho_h I_v & -(r_h + d_h) & 0 & 0 & \rho_h S_h \\ 0 & r_h & -d_h & 0 & 0 \\ 0 & -\rho_v S_v & 0 & -(\rho_v I_h + d_v) & 0 \\ 0 & \rho_v S_v & 0 & \rho_v I_h & -d_v \end{pmatrix}$$

The Jacobian matrix evaluated at E_0 is then given by:

$$J_0 = \begin{pmatrix} -d_h & 0 & 0 & 0 & -\rho_h \dfrac{S^h_{in}}{d_h} \\ 0 & -(r_h + d_h) & 0 & 0 & \rho_h \dfrac{S^h_{in}}{d_h} \\ 0 & r_h & -d_h & 0 & 0 \\ 0 & -\rho_v \dfrac{S^v_{in}}{d_v} & 0 & -d_v & 0 \\ 0 & \rho_v \dfrac{S^v_{in}}{d_v} & 0 & 0 & -d_v \end{pmatrix}$$

J_0 admits five eigenvalues. The first three eigenvalues are given by $\lambda_1 = \lambda_2 = -d_h < 0$ and $\lambda_3 = -d_v < 0$. The other two eigenvalues are those of the following sub-matrix

$$S_0 := \begin{pmatrix} -(r_h + d_h) & \rho_h \dfrac{S^h_{in}}{d_h} \\ \rho_v \dfrac{S^v_{in}}{d_v} & -d_v \end{pmatrix}$$

where the trace is given by

$$\text{Trace } (S_0) = -(r_h + d_h + d_v) < 0$$

and the determinant is given by

$$\text{Det } (S_0) = d_v(r_h + d_h) - \rho_h\rho_v \frac{S^h_{in}}{d_h}\frac{S^v_{in}}{d_v} = d_v(r_h + d_h)(1 - \mathcal{R}_0^2).$$

Then all eigenvalues must have negative real parts once $\mathcal{R}_0 < 1$ and the trivial equilibrium point E_0 is then locally asymptotically stable once $\mathcal{R}_0 < 1$. □

Theorem 4.2. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium point $E^* = (\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v)$ is locally asymptotically stable.*

Proof. The Jacobian matrix at the endemic equilibrium $E^* = (\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v)$ is given by:

$$J_1 = \begin{pmatrix} -(\rho_h\bar{I}_v + d_h) & 0 & 0 & 0 & -\rho_h\bar{S}_h \\ \rho_h\bar{I}_v & -(r_h + d_h) & 0 & 0 & \rho_h\bar{S}_h \\ 0 & r_h & -d_h & 0 & 0 \\ 0 & -\rho_v\bar{S}_v & 0 & -(\rho_v\bar{I}_h + d_v) & 0 \\ 0 & \rho_v\bar{S}_v & 0 & \rho_v\bar{I}_h & -d_v \end{pmatrix}.$$

The characteristic polynomial is then given by:

$$\begin{aligned} P(X) &= \begin{vmatrix} -(X + \rho_h\bar{I}_v + d_h) & 0 & 0 & 0 & -\rho_h\bar{S}_h \\ \rho_h\bar{I}_v & -(X + r_h + d_h) & 0 & 0 & \rho_h\bar{S}_h \\ 0 & r_h & -(X + d_h) & 0 & 0 \\ 0 & -\rho_v\bar{S}_v & 0 & -(X + \rho_v\bar{I}_h + d_v) & 0 \\ 0 & \rho_v\bar{S}_v & 0 & \rho_v\bar{I}_h & -(X + d_v) \end{vmatrix} \\ &= \begin{vmatrix} -(X + \rho_h\bar{I}_v + d_h) & 0 & 0 & 0 & -\rho_h\bar{S}_h \\ \rho_h\bar{I}_v & -(X + r_h + d_h) & 0 & 0 & \rho_h\bar{S}_h \\ 0 & r_h & -(X + d_h) & 0 & 0 \\ 0 & -\rho_v\bar{S}_v & 0 & -(X + \rho_v\bar{I}_h + d_v) & 0 \\ 0 & 0 & 0 & -(X + d_v) & -(X + d_v) \end{vmatrix} \\ &= -(X + d_v)\begin{vmatrix} -(X + \rho_h\bar{I}_v + d_h) & 0 & 0 & 0 \\ \rho_h\bar{I}_v & -(X + r_h + d_h) & 0 & 0 \\ 0 & r_h & -(X + d_h) & 0 \\ 0 & -\rho_v\bar{S}_v & 0 & -(X + \rho_v\bar{I}_h + d_v) \end{vmatrix} \\ &\quad +(X + d_v)\begin{vmatrix} -(X + \rho_h\bar{I}_v + d_h) & 0 & 0 & -\rho_h\bar{S}_h \\ \rho_h\bar{I}_v & -(X + r_h + d_h) & 0 & \rho_h\bar{S}_h \\ 0 & r_h & -(X + d_h) & 0 \\ 0 & -\rho_v\bar{S}_v & 0 & 0 \end{vmatrix} \\ &= -(X + d_v)(X + d_h)(X + \rho_v\bar{I}_h + d_v)(X + \rho_h\bar{I}_v + d_h)(X + r_h + d_h) \end{aligned}$$

$$
\begin{aligned}
& -\rho_v\bar{S}_v(X+d_v)(X+d_h)\begin{vmatrix} -(X+\rho_h\bar{I}_v+d_h) & -\rho_h\bar{S}_h \\ \rho_h\bar{I}_v & \rho_h\bar{S}_h \end{vmatrix} \\
= \quad & -(X+d_v)(X+d_h)\Big[(X+\rho_v\bar{I}_h+d_v)(X+\rho_h\bar{I}_v+d_h)(X+R_h+d_h)-\rho_v\rho_h\bar{S}_h\bar{S}_v(X+d_h)\Big].
\end{aligned}
$$

Then $\lambda_1 = -d_v < 0$ and $\lambda_2 = -d_h < 0$ are two eigenvalues. The other three eigenvalues are the roots of

$$
\begin{aligned}
Q(X) &= (X+\rho_v\bar{I}_h+d_v)(X+\rho_h\bar{I}_v+d_h)(X+r_h+d_h)-\rho_v\rho_h\bar{S}_h\bar{S}_v(X+d_h) \\
&= X^3+(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h)\,X^2 \\
&\quad +((\rho_v\bar{I}_h+d_v+r_h+d_h)(\rho_h\bar{I}_v+d_h)+(\rho_v\bar{I}_h+d_v)(r_h+d_h)-\rho_v\rho_h\bar{S}_h\bar{S}_v)\,X \\
&\quad (\rho_v\bar{I}_h+d_v)\,(\rho_h\bar{I}_v+d_h)\,(r_h+d_h)-d_h\rho_v\rho_h\bar{S}_h\bar{S}_v \\
&= X^3+a_2X^2+a_1X+a_0
\end{aligned}
$$

where

$$
\begin{aligned}
& a_2 = \rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h > 0, \\
& a_1 = (\rho_v\bar{I}_h+d_v+r_h+d_h)(\rho_h\bar{I}_v+d_h)+(\rho_v\bar{I}_h+d_v)(r_h+d_h)-\rho_v\rho_h\bar{S}_h\bar{S}_v, \\
& a_0 = (\rho_v\bar{I}_h+d_v)(\rho_h\bar{I}_v+d_h)(r_h+d_h)-d_h\rho_v\rho_h\bar{S}_h\bar{S}_v.
\end{aligned}
$$

Using the fact that $r_h+d_h = \dfrac{\rho_h\bar{I}_v\bar{S}_h}{\bar{I}_h}$, and $d_v = \dfrac{\rho_v\bar{I}_h\bar{S}_v}{\bar{I}_v}$, we obtains

$$
\begin{aligned}
a_1 &\geq (\rho_v\bar{I}_h+d_v)(\rho_h\bar{I}_v+d_h)+(r_h+d_h)(\rho_h\bar{I}_v+d_h)+\rho_h\rho_v\bar{I}_v\bar{S}_h > 0, \\
a_0 &= \rho_v\rho_h^2\bar{I}_v\bar{S}_h\bar{S}_v+\rho_h\rho_v\bar{I}_v\bar{S}_h(\rho_h\bar{I}_v+d_h) > 0, \\
a_2a_1-a_0 &= \Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h\Big)(\rho_v\bar{I}_h+d_v)(\rho_h\bar{I}_v+d_h) \\
&\quad +\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h\Big)(r_h+d_h)(\rho_h\bar{I}_v+d_h) \\
&\quad +\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h\Big)(\rho_v\bar{I}_h+d_v)(r_h+d_h) \\
&\quad +\rho_h\rho_v\bar{I}_v\bar{S}_h(\rho_v\bar{I}_h+d_v+r_h+d_h) \\
&\quad -\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h\Big)\rho_v\rho_h\bar{S}_h\bar{S}_v \\
&\geq \Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h\Big)(\rho_v\bar{I}_h+d_v)(\rho_h\bar{I}_v+d_h) \\
&\quad +\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h\Big)(r_h+d_h)(\rho_h\bar{I}_v+d_h) \\
&\quad +\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h\Big)\rho_v\rho_h\bar{S}_h\bar{S}_v \\
&\quad -\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h\Big)\rho_v\rho_h\bar{S}_h\bar{S}_v \\
&\quad +\rho_h\rho_v\bar{I}_v\bar{S}_h(\rho_v\bar{I}_h+d_v+r_h+d_h) \\
&= \Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h\Big)(\rho_v\bar{I}_h+d_v)(\rho_h\bar{I}_v+d_h) \\
&\quad +\Big(\rho_v\bar{I}_h+d_v+\rho_h\bar{I}_v+d_h+r_h+d_h\Big)(r_h+d_h)(\rho_h\bar{I}_v+d_h)+d_h\rho_v\rho_h\bar{S}_h\bar{S}_v \\
&\quad +\rho_h\rho_v\bar{I}_v\bar{S}_h(\rho_v\bar{I}_h+d_v+r_h+d_h) \\
&> 0.
\end{aligned}
$$

Then the eigenvalues have negative real parts. Therefore, the equilibrium E^* (exists if $\mathcal{R}_0 > 1$) is locally asymptotically stable. □

5. Global properties of steady states

Theorem 5.1. *By considering the dynamics (3.1), if $\mathcal{R}_0 \leq 1$, then E_0 is globally asymptotically stable.*

Proof. Assume that $\mathcal{R}_0 \leq 1$ and define the following Lyapunov function $U_0(S_h, I_h, R_h, S_v, I_v)$:

$$U_0(S_h, I_h, R_h, S_v, I_v) = \frac{d_v}{\rho_h} I_h + \frac{S_{in}^h}{d_h} I_v.$$

Clearly, $U_0(S_h, I_h, R_h, S_v, I_v) > 0$ for all $S_h, I_h, R_h, S_v, I_v > 0$ and $U_0\left(\frac{S_{in}^h}{d_h}, 0, 0, \frac{S_{in}^v}{d_v}, 0\right) = 0$. The derivative of U_0 with respect to time along system (3.1)is given by:

$$\begin{aligned}
\frac{dU_0}{dt} &= \frac{d_v}{\rho_h}\Big(\rho_h I_v S_h - (r_h + d_h) I_h\Big) + \frac{S_{in}^h}{d_h}\Big(\rho_v I_h S_v - d_v I_v\Big) \\
&\leq \frac{d_v}{\rho_h}\Big(\rho_h I_v \frac{S_{in}^h}{d_h} - (r_h + d_h) I_h\Big) + \frac{S_{in}^h}{d_h}\Big(\rho_v I_h \frac{S_{in}^v}{d_v} - d_v I_v\Big) \\
&\leq \frac{d_v}{\rho_h}\Big(\rho_h I_v \frac{S_{in}^h}{d_h} - (r_h + d_h) I_h\Big) + \frac{S_{in}^h}{d_h}\Big(\rho_v I_h \frac{S_{in}^v}{d_v} - d_v I_v\Big) \\
&\leq \left(\frac{S_{in}^h}{d_h}\frac{S_{in}^v}{d_v}\rho_v - \frac{d_v}{\rho_h}(r_h + d_h)\right) I_h \\
&\leq \frac{d_v (r_h + d_h)}{\rho_h}\left(\frac{S_{in}^h S_{in}^v \rho_h \rho_v}{d_h d_v^2 (r_h + d_h)} - 1\right) I_h \\
&= \frac{d_v (r_h + d_h)}{\rho_h}(\mathcal{R}_0^2 - 1) I_h.
\end{aligned} \tag{5.1}$$

If $\mathcal{R}_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $S_h, I_h, R_h, S_v, I_v > 0$. Let $W_0 = \left\{(S_h, I_h, R_h, S_v, I_v) : \frac{dU_0}{dt} = 0\right\}$. It can be easily shown that $W_0 = \{E_0\}$. Applying LaSalle's invariance principle [33] (see [34] for other applications), we deduce that E_0 is GAS when $\mathcal{R}_0 \leq 1$. □

We use the same process as in [6] and we define the set

$$\Omega = \left\{(S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^5 : \ 0 < S_h \leq \bar{S}_h, 0 < I_h \leq \bar{I}_h, 0 < R_h \leq \bar{R}_h, 0 < S_v \leq \bar{S}_v, 0 < I_v \leq \bar{I}_v\right\}.$$

Theorem 5.2. *If $\mathcal{R}_0 > 1$, then $\bar{E} = (\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v)$ is globally asymptotically stable in Ω.*

Proof. Consider the positive function defined by $G(s) = s - 1 - \ln s$. Let the Lyapunov function $\bar{U}(S_h, I_h, R_h, S_v, I_v)$ given by

$$\bar{U}(S_h, I_h, R_h, S_v, I_v) = \frac{1}{\rho_h \bar{I}_v} G\left(\frac{S_h}{\bar{S}_h}\right) + \frac{\bar{I}_h}{\rho_h \bar{I}_v \bar{S}_h} G\left(\frac{I_h}{\bar{I}_h}\right) + \frac{1}{2}\left(R_h - \bar{R}_h\right)^2 + \frac{1}{\rho_v \bar{I}_h} G\left(\frac{S_v}{\bar{S}_v}\right) + \frac{1}{\rho_v \bar{S}_v} G\left(\frac{I_v}{\bar{I}_v}\right).$$

It is easy to see that $\bar{U}(S_h, I_h, R_h, S_v, I_v) > 0$ for all $S_h, I_h, R_h, S_v, I_v > 0$ and $\bar{U}(\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v) = 0$. Calculating $\frac{d\bar{U}}{dt}$ along the solutions of (3.1), we obtain

$$\begin{aligned}\frac{d\bar{U}}{dt} \;=\;& \frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{S}_h}{S_h}\right)\left(S_{in}^h - \rho_h I_v S_h - d_h S_h\right) + \frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\rho_h I_v S_h - (r_h+d_h) I_h\right)\\ &+(R_h-\bar{R}_h)\left(r_h I_h - d_h R_h\right) + \frac{1}{\rho_v \bar{I}_h \bar{S}_v}\left(1-\frac{\bar{S}_v}{S_v}\right)\left(S_{in}^v - \rho_v I_h S_v - d_v S_v\right)\\ &+\frac{1}{\rho_v \bar{S}_v \bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\rho_v I_h S_v - d_v I_v\right)\end{aligned}$$

Applying the steady state conditions for $\bar{E}$

$$S_{in}^h = \rho_h \bar{I}_v \bar{S}_h + d_h \bar{S}_h, (r_h+d_h)\bar{I}_h = \rho_h \bar{I}_v \bar{S}_h, S_{in}^v = \rho_v \bar{I}_h \bar{S}_v + d_v \bar{S}_v, d_v \bar{I}_v = \rho_v \bar{I}_h \bar{S}_v,$$

we get

$$\begin{aligned}\frac{d\bar{U}}{dt} \;=\;& \frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{S}_h}{S_h}\right)\left(\rho_h \bar{I}_v \bar{S}_h + d_h \bar{S}_h - \rho_h I_v S_h - d_h S_h\right)\\ &+\frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\rho_h I_v S_h - (r_h+d_h) I_h\right) + (R_h-\bar{R}_h)\left(r_h I_h - d_h R_h\right)\\ &+\frac{1}{\rho_v \bar{I}_h \bar{S}_v}\left(1-\frac{\bar{S}_v}{S_v}\right)\left(\rho_v \bar{I}_h \bar{S}_v + d_v \bar{S}_v - \rho_v I_h S_v - d_v S_v\right)\\ &+\frac{1}{\rho_v \bar{S}_v \bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\rho_v I_h S_v - d_v I_v\right)\\ =\;& -\frac{d_h}{\rho_h \bar{I}_v \bar{S}_h}\frac{(S_h-\bar{S}_h)^2}{S_h} + \frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{S}_h}{S_h}\right)\left(\rho_h \bar{I}_v \bar{S}_h - \rho_h I_v S_h\right)\\ &+\frac{1}{\rho_h \bar{I}_v \bar{S}_h}\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\rho_h I_v S_h - (r_h+d_h) I_h\right) + (R_h-\bar{R}_h)\left(r_h I_h - d_h R_h\right)\\ &-\frac{d_v}{\rho_v \bar{I}_h \bar{S}_v}\frac{(S_v-\bar{S}_v)^2}{S_v} + \frac{1}{\rho_v \bar{I}_h \bar{S}_v}\left(1-\frac{\bar{S}_v}{S_v}\right)\left(\rho_v \bar{I}_h \bar{S}_v - \rho_v I_h S_v\right)\\ &+\frac{\bar{I}_h}{\rho_v \bar{I}_h \bar{S}_v \bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\rho_v I_h S_v - d_v I_v\right)\\ =\;& -\frac{d_h}{\rho_h \bar{I}_v \bar{S}_h}\frac{(S_h-\bar{S}_h)^2}{S_h} - \frac{d_v}{\rho_v \bar{I}_h \bar{S}_v}\frac{(S_v-\bar{S}_v)^2}{S_v} + \frac{1}{\bar{I}_v \bar{S}_h}(1-\frac{\bar{S}_h}{S_h})\left(\bar{I}_v \bar{S}_h - I_v S_h\right)\\ &+\frac{1}{\bar{I}_v \bar{S}_h}\left(1-\frac{\bar{I}_h}{I_h}\right)\left(I_v S_h - \bar{I}_v \bar{S}_h \frac{I_h}{\bar{I}_h}\right) + (R_h-\bar{R}_h)\left(r_h I_h - d_h R_h\right)\\ &+\frac{1}{\bar{I}_h \bar{S}_v}\left(1-\frac{\bar{S}_v}{S_v}\right)\left(\bar{I}_h \bar{S}_v - I_h S_v\right) + \frac{\bar{I}_h}{\bar{I}_h \bar{S}_v \bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(I_h S_v - \bar{I}_h \bar{S}_v \frac{I_v}{\bar{I}_v}\right)\\ =\;& -\frac{d_h}{\rho_h \bar{I}_v \bar{S}_h}\frac{(S_h-\bar{S}_h)^2}{S_h} - \frac{d_v}{\rho_v \bar{I}_h \bar{S}_v}\frac{(S_v-\bar{S}_v)^2}{S_v} + \left(1-\frac{\bar{S}_h}{S_h}\right)\left(1-\frac{I_v S_h}{\bar{I}_v \bar{S}_h}\right)\\ &+\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\frac{I_v S_h}{\bar{I}_v \bar{S}_h} - \frac{I_h}{\bar{I}_h}\right) + (R_h-\bar{R}_h)\left(r_h(\frac{S_{in}^h}{d_h} - S_h - R_h) - d_h R_h\right)\\ &+\left(1-\frac{\bar{S}_v}{S_v}\right)\left(1-\frac{I_h S_v}{\bar{I}_h \bar{S}_v}\right) + \frac{\bar{I}_h}{\bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\frac{I_h S_v}{\bar{I}_h \bar{S}_v} - \frac{I_v}{\bar{I}_v}\right)\\ =\;& -\frac{d_h}{\rho_h \bar{I}_v \bar{S}_h}\frac{(S_h-\bar{S}_h)^2}{S_h} - \frac{d_v}{\rho_v \bar{I}_h \bar{S}_v}\frac{(S_v-\bar{S}_v)^2}{S_v} + \left(1-\frac{\bar{S}_h}{S_h}\right)\left(1-\frac{I_v S_h}{\bar{I}_v \bar{S}_h}\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\frac{I_vS_h}{\bar{I}_v\bar{S}_h}-\frac{I_h}{\bar{I}_h}\right)+(R_h-\bar{R}_h)\left(r_h(\bar{S}_h+\frac{(r_h+d_h)}{r_h}\bar{R}_h-S_h-R_h)-d_hR_h\right)\\
&+\left(1-\frac{\bar{S}_v}{S_v}\right)\left(1-\frac{I_hS_v}{\bar{I}_h\bar{S}_v}\right)+\frac{\bar{I}_h}{\bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\frac{I_hS_v}{\bar{I}_h\bar{S}_v}-\frac{I_v}{\bar{I}_v}\right)\\
= &-\frac{d_h}{\rho_h\bar{I}_v\bar{S}_h}\frac{(S_h-\bar{S}_h)^2}{S_h}-\frac{d_v}{\rho_v\bar{I}_h\bar{S}_v}\frac{(S_v-\bar{S}_v)^2}{S_v}+\left(1-\frac{\bar{S}_h}{S_h}\right)\left(1-\frac{I_vS_h}{\bar{I}_v\bar{S}_h}\right)\\
&+\left(1-\frac{\bar{I}_h}{I_h}\right)\left(\frac{I_vS_h}{\bar{I}_v\bar{S}_h}-\frac{I_h}{\bar{I}_h}\right)+(R_h-\bar{R}_h)\left(r_h(\bar{S}_h-S_h)+(r_h+d_h)(\bar{R}_h-R_h)\right)\\
&+\left(1-\frac{\bar{S}_v}{S_v}\right)\left(1-\frac{I_hS_v}{\bar{I}_h\bar{S}_v}\right)+\frac{\bar{I}_h}{\bar{I}_v}\left(1-\frac{\bar{I}_v}{I_v}\right)\left(\frac{I_hS_v}{\bar{I}_h\bar{S}_v}-\frac{I_v}{\bar{I}_v}\right).
\end{aligned}
$$

Therefore, $\frac{d\bar{U}}{dt} \leq 0$ for all $S_h, I_h, R_h, S_v, I_v \in \Omega$ and $\frac{d\bar{U}}{dt} = 0$ if and only if $(S_h, I_h, R_h, S_v, I_v) = (\bar{S}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v) = 0$. By applying the LaSalle's invariance principle [33], one can easily deduce that $\bar{E}$ is globally stable (see [35–37] for other applications). □

5.1. **Numerical examples.** In this section, we adopted the numerical simulations validating analytical findings for the case of fixed environment. The numerical values of ρ_h, ρ_v, d_h, d_v and r_h are considered in Table 2. In Figures 1-2, the calculated trajectories of dynamics (3.1) converge

Table 2. Used values for ρ_h, ρ_v, d_h, d_v and r_h.

Parameter	ρ_h	ρ_v	d_h	d_v	r_h
Value	1.2	0.9	0.9	0.7	1.6

asymptotically to $\mathcal{E}^*$ if $\mathcal{R}_0 > 1$. However, in Figures 3-4, the calculated trajectories of the dynamics (3.1) converge to the disease-free steady state $\mathcal{E}_0$, then confirming the global asymptotic stability of $\mathcal{E}_0$ if $\mathcal{R}_0 \leq 1$. In Figures 2 and 4, different initial conditions were considered and for each one of them, the solution converge to the same steady state.

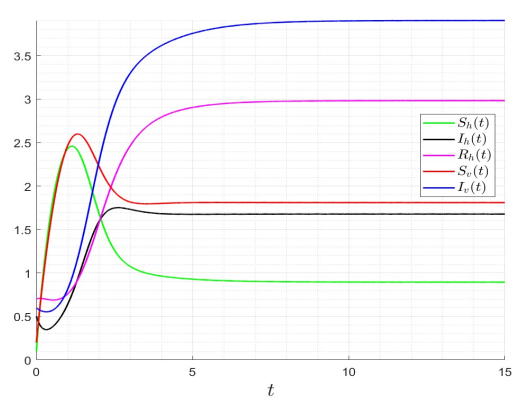

Figure 1. Behavior of the dynamics (6.1) for $S^h_{in} = 5$ and $S^v_{in} = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

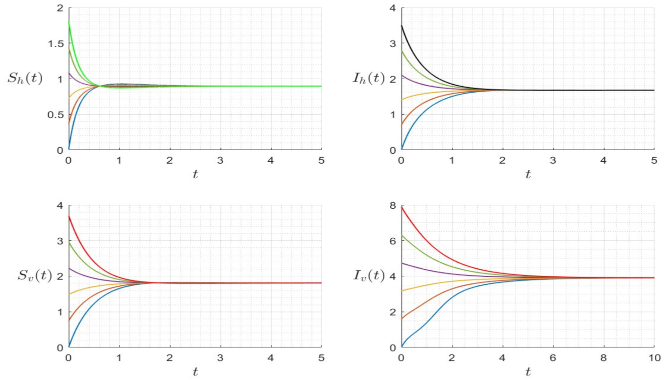

FIGURE 2. Behavior of the dynamics (6.1) for $S_{in}^h = 5$ and $S_{in}^v = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

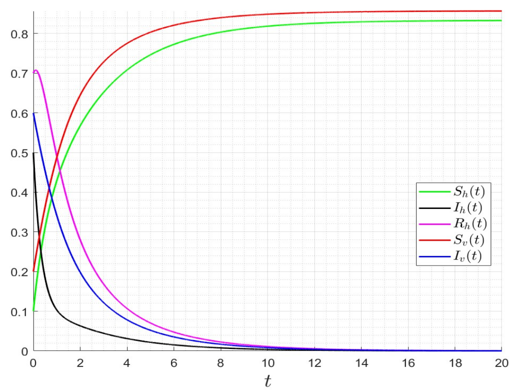

FIGURE 3. Behavior of the dynamics (6.1) for $S_{in}^h = 0.75$ and $S_{in}^v = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

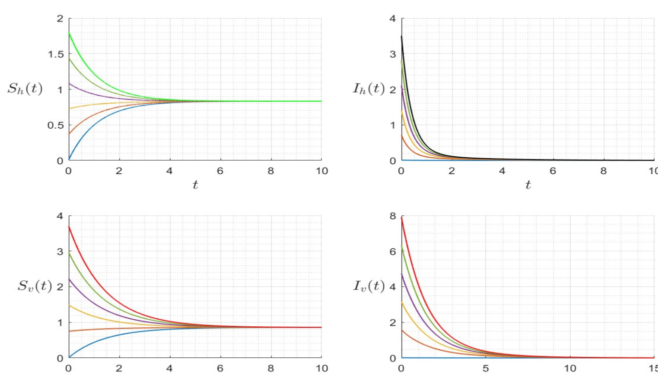

FIGURE 4. Behavior of the dynamics (6.1) for $S_{in}^h = 0.75$ and $S_{in}^v = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

6. Case of seasonal environment

Let return to the main dynamics (2.1) for a seasonal environment. Since recovered compartment doesn't affect any of the other compartments, we consider only the following reduced dynamics

$$\begin{cases} \dot{S}_h(t) = S_{in}^h(t) - \rho_h(t)I_v(t)S_h(t) - d_h(t)S_h(t), \\ \dot{I}_h(t) = \rho_h(t)I_v(t)S_h(t) - (r_h(t) + d_h(t))I_h(t), \\ \dot{R}_h(t) = r_h(t)I_h(t) - d_h(t)R_h(t), \\ \dot{S}_v(t) = S_{in}^v(t) - \rho_v(t)I_h(t)S_v(t) - d_v(t)S_v(t), \\ \dot{I}_v(t) = \rho_v(t)I_h(t)S_v(t) - d_v(t)I_v(t). \end{cases} \tag{6.1}$$

For any continuous, positive T-periodic function $g(t)$, we define $g^u = \max\limits_{t\in[0,T)} g(t)$ and $g^l = \min\limits_{t\in[0,T)} g(t)$.

6.1. **Preliminary.** Let $A(t)$ to be a T-periodic $m \times m$ matrix continuous function that it is irreducible and cooperative. Let $\beta_A(t)$ to be the fundamental matrix with positive entries, solution of

$$\dot{w}(t) = A(t)w(t). \tag{6.2}$$

Let us denote the spectral radius of the matrix $\beta_A(T)$ by $r(\beta_A(T))$. By using the Perron-Frobenius theorem, one can define $r(\beta_A(T))$ to be the principal eigenvalue of $\beta_A(T)$. According to [38], we have:

Lemma 6.1. *[38]. (6.2) admits a positive T-periodic function $x(t)$ such that $w(t) = x(t)e^{at}$ with $a = \dfrac{1}{T}\ln(r(\beta_A(T)))$.*

In order to define the disease-free periodic trajectory of model (6.1), let us consider the subsystem

$$\begin{cases} \dot{S}_h(t) = S_{in}^h(t) - d_h(t)S_h(t), \\ \dot{S}_v(t) = S_{in}^v(t) - d_v(t)S_v(t). \end{cases} \tag{6.3}$$

with the initial condition $(S_h(0), S_v(0)) \in \mathbb{R}_+^2$. The dynamics (6.3) has a unique T-periodic trajectory $(\bar{S}_h(t), \bar{S}_v(t))$ such that $\bar{S}_h(t) > 0$ and $\bar{S}_v(t) > 0$. This solution is globally attractive in $\mathbb{R}_+^2$; therefore, the dynamics (6.1) admits a unique disease-free periodic trajectory $(\bar{S}_h(t), 0, 0, \bar{S}_v(t), 0)$.

Let us define $\sigma(t) = \min\limits_{t\geq 0}(\mu(t), m(t))$ and then we have

Proposition 6.1. *The compact set*

$$\Sigma^u = \left\{ (S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^5 \ / \ S_h + I_h + R_h \leq \frac{S_{in}^{h^u}}{d_h^l}; S_v + I_v \leq \frac{S_{in}^{v^u}}{d_v^l} \right\}$$

is a positively invariant and attractor of trajectories of dynamics (6.1) with

$$\begin{aligned} &\lim_{t\to\infty} S_h(t) + I_h(t) + R_h(t) - \bar{S}_h(t) = 0, \\ &\lim_{t\to\infty} S_v(t) + I_v(t) - \bar{S}_v(t) = 0. \end{aligned} \tag{6.4}$$

Proof. Using the dynamics (6.1), we obtain

$$\begin{array}{rcl}\dot{S}_h(t)+\dot{I}_h(t)+\dot{R}_h(t) & = & S_{in}^h(t)-d_h(t)(S_h(t)+I_h(t)+R_h(t)) \\ & \leq & S_{in}^{h^u}-d_h^l(S_h(t)+I_h(t)+R_h(t)) \\ & \leq & 0, \text{ if } S_h(t)+I_h(t)+R_h(t)\geq \dfrac{S_{in}^{h^u}}{d_h^l},\end{array} \tag{6.5}$$

and

$$\begin{array}{rcl}\dot{S}_v(t)+\dot{I}_v(t) & = & S_{in}^v(t)-d_v(t)(S_v(t)+I_v(t)) \\ & \leq & S_{in}^{v^u}-d_v^l(S_v(t)+I_v(t)) \\ & \leq & 0, \text{ if } S_v(t)+I_v(t)\geq \dfrac{S_{in}^{v^u}}{d_v^l},\end{array} \tag{6.6}$$

Let $Y_1(t)=S_h(t)+I_h(t)+R_h(t)$ and $Y_2(t)=S_v(t)+I_v(t)$. For $x_1(t)=Y_1(t)-\bar{S}_h(t), t\geq 0$, it follows that $\dot{x}_1(t)=-d_h(t)x_1(t)$, and thus $\lim\limits_{t\to\infty}x_1(t)=\lim\limits_{t\to\infty}(Y_1(t)-\bar{S}_h(t))=0$. By the same way, let $x_2(t)=Y_2(t)-\bar{S}_v(t), t\geq 0$, then $\dot{x}_2(t)\leq -d_v(t)x_2(t)$, and thus $\lim\limits_{t\to\infty}x_2(t)=\lim\limits_{t\to\infty}(Y_2(t)-\bar{S}_v(t))=0$. □

In section 6.2, we aim to define the basic reproduction number; $\mathcal{R}_0$, the disease-free and then its global stability for $\mathcal{R}_0\leq 1$. Later, in section 6.3, we aim to prove that compartments $I_h(t)$ and $I_v(t)$ persists if $\mathcal{R}_0>1$.

6.2. **Disease-free trajectory.** By using the definition of $\mathcal{R}_0$ given by the theory in [20]. For $Y=(I_h,I_v,R_h,S_h,S_v)$, let

$$\mathcal{F}(t,Y)=\begin{pmatrix}\rho_h(t)I_v(t)S_h(t)\\ \rho_v(t)I_h(t)S_v(t)\\ 0\\ 0\\ 0\end{pmatrix},\quad \mathcal{V}^-(t,Y)=\begin{pmatrix}(r_h(t)+d_h(t))I_h(t)\\ d_v(t)I_v(t)\\ d_h(t)R_h(t)\\ \rho_h(t)I_v(t)S_h(t)+d_h(t)S_h(t)\\ \rho_v(t)I_h(t)S_v(t)+d_v(t)S_v(t)\end{pmatrix},\text{ and }\mathcal{V}^+(t,Y)=\begin{pmatrix}0\\ 0\\ r_h(t)I_h(t)\\ S_{in}^h(t)\\ S_{in}^v(t)\end{pmatrix}.$$

Our aim is to satisfy conditions (A1)–(A7) in [20, Section 1]. The dynamics (6.1) can take the form hereafter:

$$\dot{Y}=\mathcal{F}(t,Y)-\mathcal{V}(t,Y)=\mathcal{F}(t,Y)-\mathcal{V}^-(t,Y)+\mathcal{V}^+(t,Y). \tag{6.7}$$

Thus, the first five conditions (A1)–(A5) are satisfied.

The dynamics (6.7) has a disease-free periodic solution $\bar{Y}(t)=(0,0,0,\bar{S}_h(t),\bar{S}_v(t))$. Let $f(t,Y(t))=\mathcal{F}(t,Y)-\mathcal{V}^-(t,Y)+\mathcal{V}^+(t,Y)$ and $M(t)=\left(\dfrac{\partial f_i(t,\bar{Y}(t))}{\partial Y_j}\right)_{3\leq i,j\leq 5}$ where $f_i(t,Y(t))$ and Y_i are the i-th components of $f(t,Y(t))$ and Y, respectively. A simple calculation give us

$M(t) = \begin{pmatrix} -d_h(t) & 0 & 0 \\ 0 & -d_h(t) & 0 \\ 0 & 0 & -d_v(t) \end{pmatrix}$ and thus $r(\beta_M(T)) < 1$. Therefore, the trajectory $\bar{Y}(t)$ is linearly asymptotically stable in $\Omega_s = \left\{(0,0,0,S_h,S_v) \in R_+^5\right\}$. Therefore, the condition (A6) in [20, Section 1] is also fulfilled.

Let us define $\mathbf{F}(t)$ and $\mathbf{V}(t)$ to be two matrices defined by $\mathbf{F}(t) = \left(\dfrac{\partial \mathcal{F}_i(t,\bar{Y}(t))}{\partial Y_j}\right)_{1\le i,j\le 3}$ and $\mathbf{V}(t) = \left(\dfrac{\partial \mathcal{V}_i(t,\bar{Y}(t))}{\partial Y_j}\right)_{1\le i,j\le 3}$ where $\mathcal{F}_i(t,Y)$ and $\mathcal{V}_i(t,Y)$ are the i-th components of $\mathcal{F}(t,Y)$ and $\mathcal{V}(t,Y)$, respectively. A simple calculation by using (6.7) give us the expressions of matrices $\mathbf{F}(t)$ and $\mathbf{V}(t)$ as the following:

$$\mathbf{F}(t) == \begin{pmatrix} 0 & \rho_h(t)\bar{S}_h(t) \\ \rho_v(t)\bar{S}_v(t) & 0 \end{pmatrix}, \mathbf{V}(t) = \begin{pmatrix} (r_h(t)+d_h(t)) & 0 \\ 0 & d_v(t) \end{pmatrix}.$$

Consider $Z(t_1,t_2)$ to be the three by three matrix solution of the system $\dfrac{d}{dt}Z(t_1,t_2) = -\mathbf{V}(t_1)Z(t_1,t_2)$ for any $t_1 \ge t_2$, with $Z(t_1,t_1) = I_3$, i.e., the 3×3 identity matrix. Therefore, condition (A7) is also fulfilled.

Denote by C_T the ordered Banach space of T-periodic functions that are defined on $\mathbb{R} \mapsto \mathbb{R}^2$, with the maximum norm $\|.\|_\infty$ and the positive cone $C_T^+ = \{\psi \in C_T : \psi(s) \ge 0, \text{ for any } s \in \mathbb{R}\}$. Consider the linear operator $K : C_T \to C_T$ given by

$$(K\phi)(\omega) = \int_0^\infty Z(\omega,\omega-z)\mathbf{F}(\omega-z)\phi(\omega-z)dz, \quad \forall \omega \in \mathbb{R}, \phi \in C_T \tag{6.8}$$

Therefore, the basic reproduction number, $\mathcal{R}_0$, of dynamics (6.1) is given by $\mathcal{R}_0 = r(K)$.

Thus, the local stability of the disease-free periodic trajectory, $\mathcal{E}_0(t) = (\bar{S}_h(t),0,0,\bar{S}_h(t),0)$, of the dynamics (6.1) with respect to $\mathcal{R}_0$ is given hereafter.

Theorem 6.1. *[20, Theorem 2.2]*

- $\mathcal{R}_0 < 1 \;\Leftrightarrow\; r(\beta_{F-V}(T)) < 1$.
- $\mathcal{R}_0 = 1 \;\Leftrightarrow\; r(\beta_{F-V}(T)) = 1$.
- $\mathcal{R}_0 > 1 \;\Leftrightarrow\; r(\beta_{F-V}(T)) > 1$.

Then, $\mathcal{E}_0(t)$ is asymptotically stable if $\mathcal{R}_0 < 1$, however, it is unstable if $\mathcal{R}_0 > 1$.

Theorem 6.2. *$\mathcal{E}_0(t)$ is globally asymptotically stable if $\mathcal{R}_0 < 1$. It is unstable if $\mathcal{R}_0 > 1$.*

Proof. By Theorem 6.1, one has $\mathcal{E}_0(t)$ is locally stable if $\mathcal{R}_0 < 1$ however it is unstable if $\mathcal{R}_0 > 1$. Therefore, it remains to satisfy the global attractivity of $\mathcal{E}_0(t)$ once $\mathcal{R}_0 < 1$. Using (6.4) in Proposition 6.1, for any $m_1 > 0$, $\exists\ T_1 > 0$ such that $S_h(t)+I_h(t)+R_h(t) \le \bar{S}_h(t)+m_1$ and $S_v(t)+I_v(t) \le \bar{S}_v(t)+m_1$ for $t > T_1$. Therefore, $S_h(t) \le \bar{S}_h(t)+m_1$ and $S_v(t) \le \bar{S}_v(t)+m_1$; and

$$\begin{cases} \dot{I}_h(t) & \le & \rho_h(t)I_v(t)(\bar{S}_h(t)+m_1) - (r_h(t)+d_h(t))I_h(t), \\ \dot{I}_v(t) & \le & \rho_v(t)I_h(t)(\bar{S}_v(t)+m_1) - d_v(t)I_v(t), \end{cases} \tag{6.9}$$

for $t > T_1$. Let $M_2(t)$ be the two by two matrix function given hereafter

$$M_2(t) = \begin{pmatrix} 0 & \rho_h(t) \\ \rho_v(t) & 0 \end{pmatrix}. \tag{6.10}$$

using the equivalences in Theorem 6.1, one has $r(\varphi_{F-V}(T)) < 1$. By choosing $m_1 > 0$ satisfying $r(\varphi_{F-V+m_1M_2}(T)) < 1$ and we consider the dynamics hereafter,

$$\begin{cases} \dot{\bar{I}}_h(t) &= \rho_h(t)\bar{I}_v(t)(\bar{S}_h(t) + m_1) - (r_h(t) + d_h(t))\bar{I}_h(t), \\ \dot{\bar{I}}_v(t) &= \rho_v(t)\bar{I}_h(t)(\bar{S}_v(t) + m_1) - d_v(t)\bar{I}_v(t). \end{cases} \tag{6.11}$$

Using Lemma 6.1, there exists a positive T-periodic function $x_1(t)$ such that $w(t) \le x_1(t)e^{a_1 t}$ with $w(t) = \begin{pmatrix} I_h(t) \\ I_v(t) \end{pmatrix}$ and $a_1 = \dfrac{1}{T}\ln\left(r(\varphi_{F-V+m_1M_2}(T)\right) < 0$. Thus, $\lim\limits_{t\to\infty} I_h(t) = 0$ and $\lim\limits_{t\to\infty} I_v(t) = 0$. Furthermore, we have that $\lim\limits_{t\to\infty} S_h(t) - \bar{S}_h(t) = \lim\limits_{t\to\infty} Z_1(t) - I_h(t) - R_h(t) - \bar{S}_h(t) = 0$ and $\lim\limits_{t\to\infty} S_v(t) - \bar{S}_v(t) = \lim\limits_{t\to\infty} Z_2(t) - I_v(t) - \bar{S}_v(t) = 0$. Then, we deduce that the disease-free periodic trajectory $\mathcal{E}_0(t)$ is globally attractive. □

6.3. **Endemic trajectory .** Note that the dynamics (6.1) admits Σ^u as an invariant compact set. Let $Y_0 = (S_h(0), I_h(0), R_h(0), S_v(0), I_v(0))$ and $Y_1 = (\bar{S}_h(0), 0, 0, \bar{S}_v(0), 0)$. Define $\mathcal{P} : \mathbb{R}^5_+ \to \mathbb{R}^5_+$ to be the Poincaré map related to the dynamics (6.1) with $Y_0 \mapsto u(T, Y^0)$, where $u(t, Y^0)$ is the unique solution of dynamics (6.1) and initial condition $u(0, Y^0) = Y^0 \in \mathbb{R}^5_+$. Let us define

$$\Omega = \left\{(S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}^5_+\right\},\ \Omega_0 = Int(\mathbb{R}^5_+) \text{ and } \partial\Omega_0 = \Omega \setminus \Omega_0.$$

Ω and Ω_0 are both positively invariant. $\mathcal{P}$ is point dissipative. Define

$$M_\partial = \left\{(Y_0) \in \partial\Omega_0 : \mathcal{P}^k(Y_0) \in \partial\Omega_0,\ \text{for any } k \ge 0\right\}.$$

By using the persistence theory given in [39] (also in [38, Theorem 2.3]), we have

$$M_\partial = \{(S_h, 0, 0, S_v, 0),\ S_h \ge 0, S_v \ge 0\}. \tag{6.12}$$

It is easy to see that $M_\partial \supseteq \{(S_h, 0, 0, S_v, 0),\ S_h \ge 0, S_v \ge 0\}$. To prove that $M_\partial \setminus \{(S_h, 0, 0, S_v, 0),\ S_h \ge 0, S_v \ge 0\} = \emptyset$, consider $(Y_0) \in M_\partial \setminus \{(S_h, 0, 0, S_v, 0),\ S_h \ge 0, S_v \ge 0\}$. If $I_v(0) = 0$ and $0 < I_h(0)$, then $I_h(t) > 0$ for all $t > 0$. Then $\dot{I}_v(t)_{|t=0} = \rho_v(0)I_h(0)\bar{S}_v(0) > 0$.
If $I_v(0) > 0$ and $I_h(0) = 0$, then $I_v(t) > 0$ and $S_h(t) > 0$ for all $t > 0$. Thus, for all $t > 0$, we obtain

$$I_h(t) = \left[I_h(0) + \int_0^t (\rho_h(\omega)I_v(\omega)S_h(\omega))e^{\int_0^\omega (r_h(z) + d_h(z))dz} d\omega\right]e^{-\int_0^t (r_h(z) + d_h(z))dz} > 0$$

for all $t > 0$. This means that $Y(t) \notin \partial\Omega_0$ for $0 < t \ll 1$. Therefore, Ω_0 is positively invariant from which we deduce (6.12). Using the previous discussion, we deduce that there exists one fixed point Y_1 of $\mathcal{P}$ in M_∂. We deduce, therefore, the uniform persistence of the disease as follows.

Theorem 6.3. *Assume that $\mathcal{R}_0 > 1$. The dynamics (6.1) admits at least one periodic solution such that there exists $\varepsilon > 0$ that satisfies $\forall\ Y_0 \in \mathbb{R}_+ \times Int(\mathbb{R}^2_+) \times \mathbb{R}_+ \times Int(\mathbb{R}_+)$ and*

$$\liminf_{t\to\infty} I_h(t) \geq \varepsilon > 0, \liminf_{t\to\infty} I_v(t) \geq \varepsilon > 0.$$

Proof. We aim to prove that $\mathcal{P}$ is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$ which permits to prove that the solution of the dynamics (6.1) is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$ by using [39, Theorem 3.1.1]. From Theorem 6.1, we have $r(\varphi_{F-V}(T)) > 1$. Therefore, there exists $\xi > 0$ such that $r(\varphi_{F-V-\xi M_2}(T)) > 1$. Define the system of equations:

$$\begin{cases} \dot{S}^h_\alpha(t) &=& S^h_{in}(t) - \rho_h(t)\alpha S^h_\alpha(t) - d_h(t)S^h_\alpha(t), \\ \dot{S}^v_\alpha(t) &=& S^v_{in}(t) - \rho_v(t)\alpha S^v_\alpha(t) - d_v(t)S^v_\alpha(t). \end{cases} \tag{6.13}$$

$\mathcal{P}$ associated with the dynamics (6.13) admits a unique fixed point $(\bar{S}^h_\alpha, \bar{S}^v_\alpha)$ which is globally attractive in $\mathbb{R}^2_+$. By using the implicit function theorem, $\alpha \mapsto (\bar{S}^h_\alpha, \bar{S}^v_\alpha)$ is continuous. Thus, $\alpha > 0$ can be chosen small enough such that $\bar{S}^h_\alpha(t) > \bar{S}_h(t) - \xi,$ and $\bar{S}^v_\alpha(t) > \bar{S}_v(t) - \xi,\ \forall\ t > 0$. Using the continuity property of the solution with respect to the initial condition, $\exists\alpha^*$ such that $Y_0 \in \Omega_0$ with $\|Y_0 - u(t, Y_1)\| \leq \alpha^*$; then

$$\|u(t, Y_0) - u(t, Y_1)\| < \alpha \text{ for } 0 \leq t \leq T.$$

We prove by contradiction that

$$\limsup_{k\to\infty}\ d(\mathcal{P}^k(Y_0), Y_1) \geq \alpha^*\ \forall\ Y_0 \in \Omega_0. \tag{6.14}$$

Suppose that $\limsup_{k\to\infty} d(\mathcal{P}^k(Y_0), Y_1) < \alpha^*$ for some $Y_0 \in \Omega_0$. We can assume that $d(\mathcal{P}^k(Y_0), Y_1) < \alpha^*$ for all $k > 0$. Therefore

$$\|u(t, \mathcal{P}^k(Y_0)) - u(t, Y_1)\| < \alpha\ \forall\ k > 0 \text{ and } 0 \leq t \leq T.$$

For $t \geq 0$, let $t = kT + t_1$, where $t_1 \in [0, T)$ and $k = \lfloor \frac{t}{T} \rfloor$. Therefore

$$\|u(t, Y_0) - u(t, Y_1)\| = \|u(t_1, \mathcal{P}^k(Y_0)) - u(t_1, Y_1)\| < \alpha \text{ for all } t \geq 0.$$

Set $(S_h(t), I_h(t), R_h(t), S_v(t), I_v(t)) = u(t, Y_0)$. Therefore $0 \leq I_h(t), I_v(t) \leq \alpha, t \geq 0$ and

$$\begin{cases} \dot{S}_h(t) &\geq& S^h_{in}(t) - \rho_h(t)\alpha S_h(t) - d_h(t)S_h(t), \\ \dot{S}_v(t) &\geq& S^v_{in}(t) - \rho_v(t)\alpha S_v(t) - d_v(t)S_v(t). \end{cases} \tag{6.15}$$

$\mathcal{P}$ applied to the dynamics (6.13) admits a fixed point $(\bar{S}^h_\alpha, \bar{S}^v_\alpha)$ that it is globally attractive with $\bar{S}^h_\alpha(t) > \bar{S}_h - \xi,$ and $\bar{S}^v_\alpha(t) > \bar{S}_v(t) - \xi$; then, $\exists\ T_2 > 0$ such that $S_h(t) > \bar{S}_h(t) - \xi$ and $S_v(t) > \bar{S}_v(t) - \xi$ for $t > T_2$. Then, for $t > T_2$, we have

$$\begin{cases} \dot{I}_h(t) &\geq& \rho_h(t)I_v(t)(\bar{S}_h(t) - \xi) - (r_h(t) + d_h(t))I_h(t), \\ \dot{I}_v(t) &\geq& \rho_v(t)I_h(t)(\bar{S}_v(t) - \xi) - d_v(t)I_v(t). \end{cases} \tag{6.16}$$

Since $r(\varphi_{F-V-\xi M_2}(T)) > 1$, then by using Lemma 6.1, there exists a positive T-periodic function $x_2(t)$ such that $J(t) \geq e^{a_2 t}x_2(t)$ where $a_2 = \dfrac{1}{T}\ln\ r\left(\varphi_{F-V-\xi M_2}(T)\right) > 0$, then $\lim_{t\to\infty} I_h(t)(t) = \infty$ which contradicts the boundedness of the solution. Therefore, (6.14) is satisfied and $\mathcal{P}$ is weakly uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. By applying Proposition 6.1, $\mathcal{P}$ has a global attractor. We deduce that Y_1 is an isolated invariant set inside Ω and that $W^s(Y_1) \cap \Omega_0 = \emptyset$. All trajectories inside M_∂ converges to Y_1 which is acyclic in M_∂. Applying [39, Theorem 1.3.1 and Remark 1.3.1], we deduce that $\mathcal{P}$ is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. Moreover, by using [39, Theorem 1.3.6], $\mathcal{P}$ has a fixed point $\tilde{Y}_0 = (\tilde{S}_h, \tilde{I}_h, \tilde{R}_h, \tilde{S}_v, \tilde{I}_v) \in \Omega_0$ with $\tilde{Y}_0 \in R_+ \times Int(R_+^2) \times R_+ \times Int(R_+)$. Suppose that $\tilde{S}_h = 0$. From the first equation of the dynamics (6.1), $\tilde{S}_h(t)$ satisfies

$$\dot{\tilde{S}}_h(t) \quad = \quad S_{in}^h(t) - \rho_h(t)\tilde{I}_v(t)\tilde{S}_h(t) - d_h(t)\tilde{S}_h(t), \tag{6.17}$$

where $\tilde{S}_h = \tilde{S}_h(nT) = 0, n = 1, 2, 3, \cdots$. By using Proposition 6.1, $\forall\ m_3 > 0$, $\exists\ T_3 > 0$ such that $\tilde{I}_v(t) \leq S_{in}^{v^u} + m_3$ for $t > T_3$. Then, we obtain

$$\dot{\tilde{S}}_h(t) \geq S_{in}^h(t) - \rho_h(t)(S_{in}^{v^u} + m_3)\tilde{S}_h(t) - d_h(t)\tilde{S}_h(t),\ \text{for } t \geq T_3. \tag{6.18}$$

$\exists\ \bar{n}$ such that $nT > T_3$ for all $n > \bar{n}$. Therefore

$$\tilde{S}_h(nT) \geq \left[\tilde{S}_h(0) + \int_0^{nT} S_{in}^h(\omega) e^{\int_0^\omega \left(\rho_h(s)(S_{in}^{v^u} + m_3) + d_h(s)\right)ds} d\omega\right] e^{-\int_0^{nT} \left(\rho_h(s)(S_{in}^{v^u} + m_3) + d_h(s)\right)ds}$$

for all $n > \bar{n}$ which contradicts the fact that $\tilde{S}_h(nT) = 0$. Then, $\tilde{S}_h(0) > 0$ and $\tilde{Y}_0$ is a positive T-periodic solution of the dynamics (6.1). □

6.4. **Numerical examples.** In this subsection, we used some numerical example to validate theoretical findings. The periodic functions are given as follows

$$\begin{cases} S_{in}^h(t) &=& S_{in}^{h0}(1 + S_{in}^{h1}\cos(2\pi(t+\phi))), & S_{in}^v(t) &=& S_{in}^{v0}(1 + S_{in}^{v1}\cos(2\pi(t+\phi))), \\ \rho_h(t) &=& \rho_h^0(1 + \rho_v^1\cos(2\pi(t+\phi))), & \rho_v(t) &=& \rho_v^0(1 + \rho_v^1\cos(2\pi(t+\phi))), \\ d_h(t) &=& d_h^0(1 + d_h^1\cos(2\pi(t+\phi))), & d_v(t) &=& d_v^0(1 + d_v^1\cos(2\pi(t+\phi))), \\ r_h(t) &=& r_h^0(1 + r_h^1\cos(2\pi(t+\phi))), & & & , \end{cases} \tag{6.19}$$

with $|S_{in}^{h1}|$, $|S_{in}^{v1}|$, $|\rho_h^1|$, $|\rho_v^1|$, $|d_h^1|$, $|d_v^1|$, $|r_h^1|$ and $|u^1|$ describe the seasonal cycles frequencies, however, ϕ describes the phase shift. The numerical values of S_{in}^{h0}, S_{in}^{v0}, ρ_h^0, ρ_v^0, d_h^0, d_v^0, r_h^0, u^0, S_{in}^{h1}, S_{in}^{v1}, ρ_h^1, ρ_v^1, d_h^1, d_v^1, r_h^1 and u^1 are considered in Table 3.

TABLE 3. Used values for ϕ, ρ_h^0, ρ_v^0, d_h^0, d_v^0, r_h^0, S_{in}^{h1}, S_{in}^{v1}, ρ_h^1, ρ_v^1, d_h^1, d_v^1 and r_h^1.

Parameter	ϕ	ρ_h^0	ρ_v^0	d_h^0	d_v^0	r_h^0	S_{in}^{h1}	S_{in}^{v1}	ρ_h^1	ρ_v^1	d_h^1	d_v^1	r_h^1
Value	0	1.2	0.9	0.9	0.7	1.6	0.13	0.19	0.11	0.12	0.3	0.4	0.8

Two scenarios were consider here. The first one was allocated to the case where only the contact rates are seasonal. The second case was allocated to the case where all parameters are periodic. Let us start be the case where only the contact rates, ρ_h and ρ_v are seasonal functions reflecting periodic contact between human and mosquito. We give the results of some numerical simulations confirming the stability results for system (6.1). The approximation of the basic reproduction number $\mathcal{R}_0$ was performed using the time-averaged system. In Figures 6-5, the calculated trajectories of the dynamics (6.1) converge asymptotically to the periodic solution corresponding to the disease persistence if $\mathcal{R}_0 > 1$. In Figures 5 and 7, different initial conditions were considered and for each one of them, the solution converge to the same periodic solution. In Figures 8 and 7, the calculated trajectories of the dynamics (6.1) converge to the disease-free periodic solution $\mathcal{E}_0(t) = (S^*(t), 0, 0, P^*(t))$ for the case where $\mathcal{R}_0 \leq 1$.

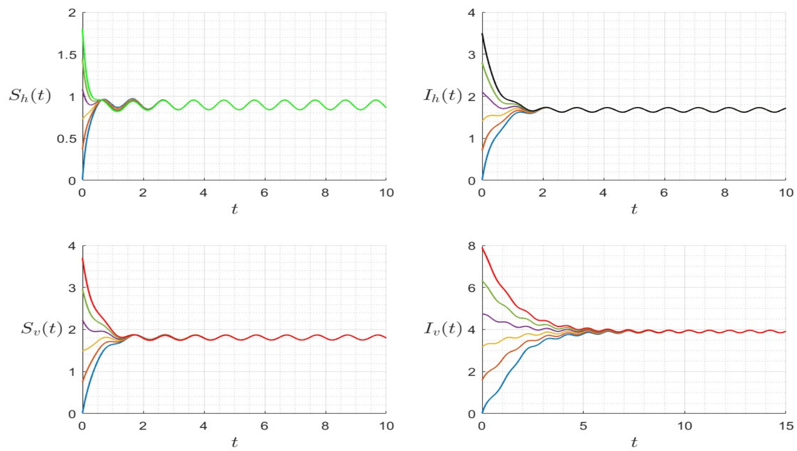

FIGURE 5. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 5$ and $S_{in}^{v0} = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

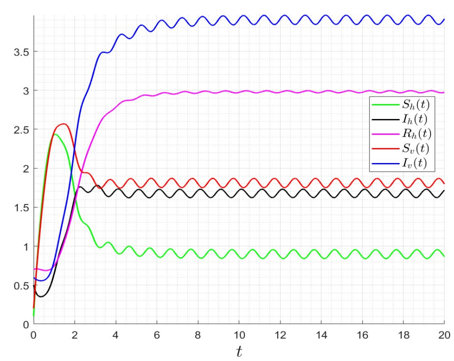

FIGURE 6. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 5$ and $S_{in}^{v0} = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

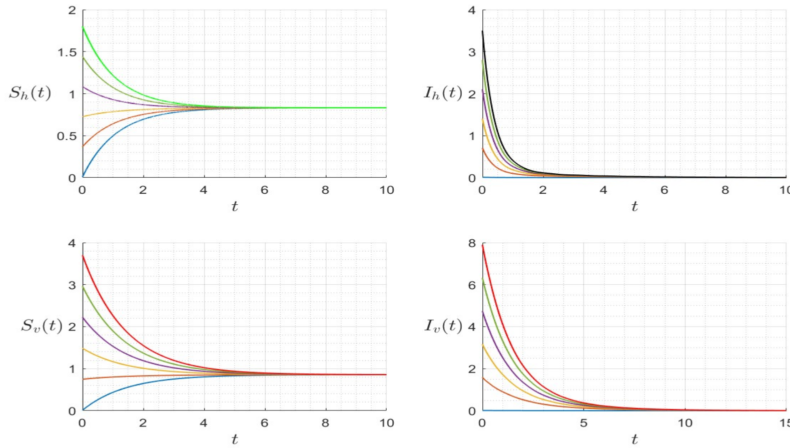

FIGURE 7. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 0.75$ and $S_{in}^{v0} = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

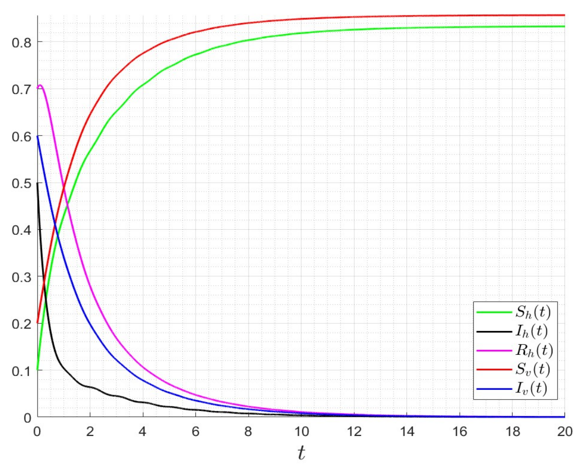

FIGURE 8. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 0.75$ and $S_{in}^{v0} = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

In the second part, we performed numerical simulations for the system (6.1) where all parameters were set as T-periodic functions. The basic reproduction number $\mathcal{R}_0$ was agian approximated by using the time-averaged system. In Figures 10-9, the calculated trajectories of the dynamics (6.1) converge asymptotically to the periodic solution corresponding to the disease persistence if $\mathcal{R}_0 > 1$. In Figures 9 and 11, different initial conditions were considered and for each one of them, the solution converge to the same periodic solution. In Figures 12 and 11, the calculated trajectories of the dynamics (6.1) converge to the disease-free periodic solution $\mathcal{E}_0(t) = (S^*(t), 0, 0, P^*(t))$ for the case where $\mathcal{R}_0 \leq 1$.

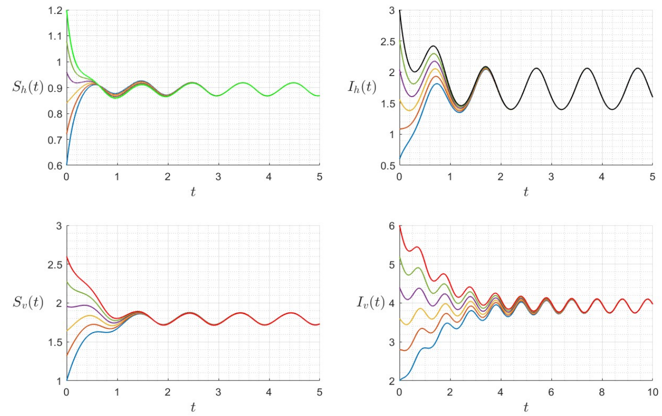

FIGURE 9. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 5$ and $S_{in}^{v0} = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

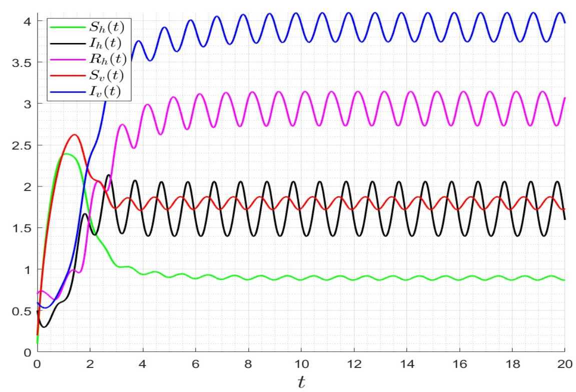

FIGURE 10. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 5$ and $S_{in}^{v0} = 4$ then $\mathcal{R}_0 \approx 4.4263 > 1$.

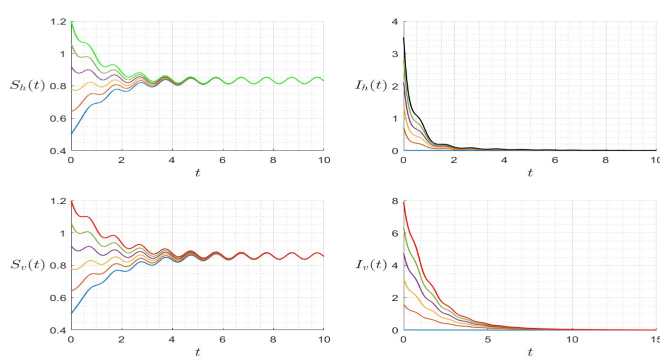

FIGURE 11. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 0.75$ and $S_{in}^{v0} = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

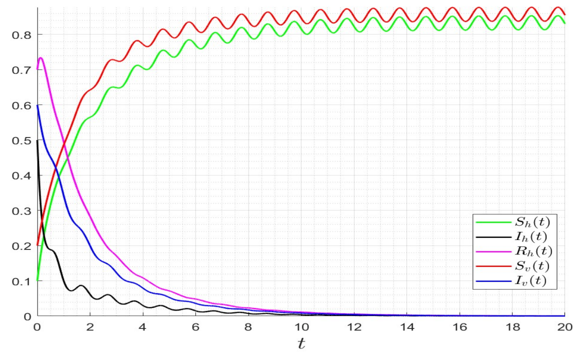

Figure 12. Behavior of the dynamics (6.1) for $S_{in}^{h0} = 0.75$ and $S_{in}^{v0} = 0.6$ then $\mathcal{R}_0 \approx 0.6639 < 1$.

7. Conclusions

In this paper, we studied a mathematical model for *Zika virus* dynamics when influenced by the seasonality. In the first step, we studied the autonomous system. The basic reproduction number was calculated and the equilibria of the dynamics were identified and characterised. In the second step, we studied on the non-autonomous dynamics and we defined the basic reproduction number, $\mathcal{R}_0$ by using a linear integral operator. It is deduced that if $\mathcal{R}_0 \leq 1$, all solution of the dynamics converge to the disease-free periodic trajectory and that the disease persists if $\mathcal{R}_0 > 1$. Both cases were performed by some numerical tests confirming the theoretical results highlighting the seasonality influence on the dynamics.

Acknowledgements: The author is grateful to the anonymous referees for their comments that helped to improve this article.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Magalhaes, B.D. Foy, E.T.A. Marques, G.D. Ebel, J. Weger-Lucarelli, Mosquito-Borne and Sexual Transmission of Zika Virus: Recent Developments and Future Directions, Virus Res. 254 (2018), 1–9. https://doi.org/10.1016/j.virusres.2017.07.011.

[2] L.R. Petersen, D.J. Jamieson, A.M. Powers, M.A. Honein, Zika Virus, N. Engl. J. Med. 374 (2016), 1552–1563. https://doi.org/10.1056/nejmra1602113.

[3] G.W.A. Dick, S.F. Kitchen, A.J. Haddow, Zika Virus (I). Isolations and Serological Specificity, Trans. R. Soc. Trop. Med. Hyg. 46 (1952), 509–520. https://doi.org/10.1016/0035-9203(52)90042-4.

[4] P.S. Mead, N.K. Duggal, S.A. Hook, M. Delorey, M. Fischer, et al. Zika Virus Shedding in Semen of Symptomatic Infected Men, N. Engl. J. Med. 378 (2018), 1377–1385. https://doi.org/10.1056/nejmoa1711038.

[5] N.C. Grassly, C. Fraser, Mathematical Models of Infectious Disease Transmission, Nat. Rev. Microbiol. 6 (2008), 477–487. https://doi.org/10.1038/nrmicro1845.

[6] M.E. Hajji, A.H. Albargi, A Mathematical Investigation of an "SVEIR" Epidemic Model for the Measles Transmission, Math. Biosci. Eng. 19 (2022), 2853–2875. https://doi.org/10.3934/mbe.2022131.

[7] A.A. Alsolami, M. El Hajji, Mathematical Analysis of a Bacterial Competition in a Continuous Reactor in the Presence of a Virus, Mathematics. 11 (2023), 883. https://doi.org/10.3390/math11040883.

[8] A.H. Albargi, M. El Hajji, Bacterial Competition in the Presence of a Virus in a Chemostat, Mathematics. 11 (2023), 3530. https://doi.org/10.3390/math11163530.

[9] F. Brauer, C. Castillo-Chavez, A. Mubayi, S. Towers, Some models for epidemics of vector-transmitted diseases, Infect. Dis. Model. 1 (2016), 79–87. https://doi.org/10.1016/j.idm.2016.08.001.

[10] Y. Nakata, T. Kuniya, Global Dynamics of a Class of SEIRS Epidemic Models in a Periodic Environment, J. Math. Anal. Appl. 363 (2010), 230–237. https://doi.org/10.1016/j.jmaa.2009.08.027.

[11] S.K. Sasmal, I. Ghosh, A. Huppert, J. Chattopadhyay, Modeling the Spread of Zika Virus in a Stage-Structured Population: Effect of Sexual Transmission, Bull. Math. Biol. 80 (2018), 3038–3067. https://doi.org/10.1007/s11538-018-0510-7.

[12] A. Alshehri, M. El Hajji, Mathematical Study for Zika Virus Transmission With General Incidence Rate, AIMS Math. 7 (2022), 7117–7142. https://doi.org/10.3934/math.2022397.

[13] D. Xiao, Dynamics and Bifurcations on a Class of Population Model With Seasonal Constant-Yield Harvesting, Discr. Contin. Dyn. Syst. - Ser. B. 21 (2015), 699–719. https://doi.org/10.3934/dcdsb.2016.21.699.

[14] N. Bacaër, S. Guernaoui, The Epidemic Threshold of Vector-Borne Diseases With Seasonality, J. Math. Biol. 53 (2006), 421–436. https://doi.org/10.1007/s00285-006-0015-0.

[15] N. Bacaër, Approximation of the Basic Reproduction Number R_0 for Vector-Borne Diseases with a Periodic Vector Population, Bull. Math. Biol. 69 (2007), 1067–1091. https://doi.org/10.1007/s11538-006-9166-9.

[16] N. Bacaër, R. Ouifki, Growth Rate and Basic Reproduction Number for Population Models With a Simple Periodic Factor, Math. Biosci. 210 (2007), 647–658. https://doi.org/10.1016/j.mbs.2007.07.005.

[17] J. Ma, Z. Ma, Epidemic Threshold Conditions for Seasonally Forced SEIR Models, Math. Biosci. Eng. 3 (2006), 161–172. https://doi.org/10.3934/mbe.2006.3.161.

[18] T. Zhang, Z. Teng, On a Nonautonomous SEIRS Model in Epidemiology, Bull. Math. Biol. 69 (2007), 2537–2559. https://doi.org/10.1007/s11538-007-9231-z.

[19] S. Guerrero-Flores, O. Osuna, C. Vargas-de-Leon, Periodic Solutions for Seasonal Siqrs Models With Nonlinear Infection Terms, Electron. J. Diff. Equ. 2019 (2019), 92.

[20] W. Wang, X.Q. Zhao, Threshold Dynamics for Compartmental Epidemic Models in Periodic Environments, J. Dyn. Diff. Equ. 20 (2008), 699–717. https://doi.org/10.1007/s10884-008-9111-8.

[21] M. El Hajji, D.M. Alshaikh, N.A. Almuallem, Periodic Behaviour of an Epidemic in a Seasonal Environment with Vaccination, Mathematics. 11 (2023), 2350. https://doi.org/10.3390/math11102350.

[22] M. El Hajji, Periodic Solutions for an "SVIQR" Epidemic Model in a Seasonal Environment with General Incidence Rate, Int. J. Biomath. In Press.

[23] B.S. Alshammari, D.S. Mashat, F.O. Mallawi, Mathematical and Numerical Investigations for a Cholera Dynamics With a Seasonal Environment, Int. J. Anal. Appl. 21 (2023), 127. https://doi.org/10.28924/2291-8639-21-2023-127.

[24] M. El Hajji, R.M. Alnjrani, Periodic Trajectories for HIV Dynamics in a Seasonal Environment With a General Incidence Rate, Int. J. Anal. Appl. 21 (2023), 96. https://doi.org/10.28924/2291-8639-21-2023-96.

[25] M. El Hajji, N.S. Alharbi, M.H. Alharbi, Mathematical Modeling for a CHIKV Transmission Under the Influence of Periodic Environment, Int. J. Anal. Appl. 22 (2024), 6. https://doi.org/10.28924/2291-8639-22-2024-6.

[26] M. El Hajji, Periodic Solutions for Chikungunya Virus Dynamics in a Seasonal Environment With a General Incidence Rate, AIMS Math. 8 (2023), 24888–24913. https://doi.org/10.3934/math.20231269.

[27] J. Murray, Mathematical Biology II: Spatial Models and Biomedical Applications, 2nd ed., Biomathematics Texts, Vol. 18, 1980.

[28] A.K. Chakraborty, M.A. Haque, M.A. Islam, Mathematical Modelling and Analysis of Dengue Transmission in Bangladesh With Saturated Incidence Rate and Constant Treatment, Commun. Biomath. Sci. 3 (2020), 101–113.

[29] N.K. Goswami, B. Shanmukha, Stability and Optimal Control Analysis of Zika Virus With Saturated Incidence Rate, Malaya J. Mat. 8 (2020), 331–342. https://doi.org/10.26637/mjm0802/0004.

[30] E. Bonyah, K.O. Okosun, Mathematical modeling of Zika virus, Asian Pac. J. Trop. Dis. 6 (2016), 673–679. https://doi.org/10.1016/s2222-1808(16)61108-8.

[31] O. Diekmann, J.A.P. Heesterbeek, J.A.J. Metz, On the Definition and the Computation of the Basic Reproduction Ratio R_0 in Models for Infectious Diseases in Heterogeneous Populations, J. Math. Biol. 28 (1990), 365–382. https://doi.org/10.1007/bf00178324.

[32] P. van den Driessche, J. Watmough, Reproduction Numbers and Sub-Threshold Endemic Equilibria for Compartmental Models of Disease Transmission, Math. Biosci. 180 (2002), 29–48. https://doi.org/10.1016/s0025-5564(02)00108-6.

[33] J. LaSalle, The Stability of Dynamical Systems, SIAM, Philadelphia, 1976.

[34] M. El Hajji, A.Y. Al-Subhi, M.H. Alharbi, Mathematical Investigation for Two-Bacteria Competition in Presence of a Pathogen With Leachate Recirculation, Int. J. Anal. Appl. 22 (2024), 45. https://doi.org/10.28924/2291-8639-22-2024-45.

[35] A.H. Albargi, M.E. Hajji, Mathematical Analysis of a Two-Tiered Microbial Food-Web Model for the Anaerobic Digestion Process, Math. Biosci. Eng. 20 (2023), 6591–6611. https://doi.org/10.3934/mbe.2023283.

[36] M. El Hajji, Mathematical Modeling for Anaerobic Digestion Under the Influence of Leachate Recirculation, AIMS Math. 8 (2023), 30287–30312. https://doi.org/10.3934/math.20231547.

[37] M. El Hajji, Influence of the Presence of a Pathogen and Leachate Recirculation on a Bacterial Competition, Int. J. Biomath. In Press.

[38] F. Zhang, X.Q. Zhao, A Periodic Epidemic Model in a Patchy Environment, J. Math. Anal. Appl. 325 (2007), 496–516. https://doi.org/10.1016/j.jmaa.2006.01.085.

[39] X.Q. Zhao, Dynamical Systems in Population Biology, Springer, 2003. https://doi.org/10.1007/978-3-319-56433-3.