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Multiple and Singular Soliton Solutions for Space–Time Fractional Coupled Modified Korteweg–De Vries Equations

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Abstract: The focus of this paper is on the nonlinear coupled evolution equations, specifically within the context of the fractional coupled modified Korteweg–de Vries (mKdV) equation, employing the conformable fractional derivative (CFD) approach. The primary objective of this paper is to thoroughly investigate the applicability of the Hirota bilinear method for deriving analytical solutions to the fractional mKdV equations. A range of exact analytical solutions for the fractional coupled mKdV equations is obtained. The findings in general indicate that the Hirota bilinear method is an effective approach for resolving the complexities associated with the fractional coupled mKdV equations.

1. Introduction

A plethora of nonlinear coupled evolution equations have emerged as pivotal tools in numerous disciplines, garnering considerable citations within the corpus of scientific research [1–9]. The coupled Korteweg–de Vries (KdV) equation is at the heart of this extensive research. The research regarding coupled KdV equation has been guided by two principal aims: one is to determine soliton solutions that typify the solitary wave characteristics inherent in these

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equations; the other is to verify the complete integrability of the coupled systems, an aspect that is essential [10, 11]. The widely recognized KdV equation, as presented in [2, 3], takes the form of

$$u_t + a\,uu_x + u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ symbolizes the wave's elongation at position x and time t, while a represents non-zero real constant. Various numerical and analytical methods have been utilized to investigate the solitary wave solutions that emerge from this equation [3, 5, 6, 7, 11]. According to [7, 12], the modified form of the KdV equation (mKdV) takes the form of

$$u_t + a\,u^2u_x + u_{xxx} = 0. \tag{2}$$

In this paper, the Hirota bilinear transformation method [12, 13] will be employed to investigate a variety of fractional coupled mKdV equations. The fractional coupled mKdV equations we target are represented by the following three systems

$$\begin{cases} D_t^\alpha u + 6auvD_x^\alpha u + D_x^{\alpha\alpha\alpha}u = 0, \\ D_t^\alpha v + 6auvD_x^\alpha v + D_x^{\alpha\alpha\alpha}v = 0; \end{cases} \tag{3}$$

$$\begin{cases} D_t^\alpha u + 6auvD_x^\alpha v + 6(u^2 - v^2)D_x^\alpha u + D_x^{\alpha\alpha\alpha}u = 0, \\ D_t^\alpha v + 24auvD_x^\alpha u + 6(u^2 - v^2)D_x^\alpha v + D_x^{\alpha\alpha\alpha}v = 0; \end{cases} \tag{4}$$

$$\begin{cases} D_t^\alpha u + a(v^2 - u^2)D_x^\alpha u + \frac{a}{4}D_x^{\alpha\alpha\alpha}u = 0, \\ D_t^\alpha v + a(v^2 - u^2)D_x^\alpha v + \frac{a}{4}D_x^{\alpha\alpha\alpha}v = 0. \end{cases} \tag{5}$$

This paper is dedicated to the derivation of precise wave solutions for systems (3) - (5) through the application of the conformable fractional derivative (CFD). Introduced by Khalil et al. [14], CFD represents a significant advancement in fractional calculus, exhibiting fundamental characteristics that are instrumental across various fields, including mathematics, engineering, and physics [14-23]. Within the above systems, the CFD related to time (t) and space (x) is denoted as D_t^α and D_x^α, respectively. We further elaborate on higher-order operations, for instance, $D_x^{\alpha\alpha}u = D_x^\alpha(D_x^\alpha u)$, to describe second-order CFDs. The use of CFD in soliton theory offers many advantages, particularly its effectiveness in characterizing soliton wave behaviors and providing profound physical insights. Given these advantages, this paper employs the Hirota bilinear method to obtain traveling wave solutions for the systems of fractional coupled mKdV equations (3) – (5).

The method of bilinearization introduced by Hirota [12], is a prominent technique for deriving traveling wave solutions for a variety of Nonlinear Partial Differential Equations (NPDEs). However, its application to the fractional coupled mKdV equation remains relatively unexplored research. Therefore, what is new in this research lies in the application of the Hirota method to the fractional coupled mKdV equation, as this method has not been widely applied to this particular equation. This not only promises to expand the repository of exact solutions available for the fractional coupled mKdV equation but also stands to offer deeper insights into the complex behaviors of fractional-order nonlinear systems. Previous scholarly endeavors have applied the Hirota method extensively to analyze an array of NPDEs, facilitating the derivation of traveling wave solutions; see for example [8, 10]. In undertaking this method, this study aims both to augment the compendium of exact solutions for mKdV equations and provide enhanced understanding of the intricate dynamics characteristic of fractional-order nonlinear phenomena.

The primary objective of this paper is to investigate the application of the Hirota bilinear method for deriving analytical solutions to the fractional coupled mKdV equation, as well as to examine the ramifications of these solutions for elucidating the characteristics of nonlinear dynamical systems. This study successfully yielded a broader spectrum of exact analytical solutions for the fractional coupled mKdV equation. The organization of this manuscript is delineated as follows: Section 2 discusses the conformable fractional derivative. The multiple and singular soliton solutions for the systems of fractional coupled mKdV equations (3) – (5) are articulated in Sections 3, 4, and 5 respectively. The manuscript culminates with a conclusion in Section 6.

2. Conformable fractional derivative (CFD)

We engage in an analytical exploration of the core principles of CFD, as discussed in [15-23]. We articulate the definition of the conformable derivative of order α, wherein the range for α is confined to $\alpha \in (0,1]$, with respect to an independent variable denoted as t. This is expressed mathematically as follows

$$D^{\alpha} f(t) = \lim_{\tau \to 0} \frac{f(t+\tau t^{1-\alpha}) - f(t)}{\tau}, \forall\, t > 0, \alpha \in (0,1], \;\; f^{\alpha}(0) = \lim_{t \to 0^{+}} D^{\alpha} f(t). \tag{6}$$

This fractional derivative upholds specific inherent characteristics that are vital. We are to examine a derivative with order α within the interval (0,1]. Assume that f and g signify functions that are α-differentiable for any positive value of t, with m and n being constant factors. Then:

- $D^{\alpha}t^{q} = qt^{q-\alpha}, q \in \mathbb{R}$.
- $D^{\alpha}m = 0$.
- $D^{\alpha}f(t) = t^{1-\alpha}\ \frac{df}{dt}$.
- $D^{\alpha}(mf + ng) = mD^{\alpha}f + nD^{\alpha}g$.
- $D^{\alpha}(fg) = fD^{\alpha}g + gD^{\alpha}f$.
- $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}f - fD^{\alpha}g}{g^2}$.
- $D^{\alpha}\Big(f\big(g(t)\big)\Big) = \frac{df}{dg}D^{\alpha}g(t) = t^{1-\alpha}\ \frac{df}{dg}\frac{dg}{dt}$.

These properties have been established and exhibit extensive congruities with integral derivatives as illustrated in [14]. It is observed that the conformable differential operator adheres to a multitude of essential axioms analogous to those governing the chain rule, Taylor series expansion, and Laplace transformation [20].

3. Multiple and singular soliton solutions of system (3)

We begin by examining the fractional coupled mKdV equations in system (3) given by

$$D_t^{\alpha}u + 6auvD_x^{\alpha}u + D_x^{\alpha\alpha\alpha}u = 0,$$
$$D_t^{\alpha}v + 6auvD_x^{\alpha}v + D_x^{\alpha\alpha\alpha}v = 0,$$

where a represents an arbitrary constant. It should be noted that when the condition $u = v$ is applied to this system, it simplifies to what is recognized as the first fractional coupled mKdV equation.

3.1. Multiple-soliton solutions

We introduce the multiple-soliton solutions pertaining to the fractional coupled mKdV equation by assuming that the solutions of system (3) is given by

$$u(x,t) = e^{\theta_i}, v(x,t) = Ae^{\theta_i},\ \theta_i = (k_i x^{\alpha} - c_i t^{\alpha})/\alpha,\ \ i = 1, 2, 3\ , \cdots, N, \tag{7}$$

where A signifies a constant parameter. Substituting Equation (7) into the linear term of system (3), presents the dispersion relation

$$c_i = k_i^3. \tag{8}$$

As a result of this equation, we acquire

$$\theta_i = \left(k_i x^\alpha - \mathrm{k}_i^3 t^\alpha\right)/\alpha. \tag{9}$$

Following [11, 24], the multiple-soliton solutions are characterized by the auxiliary functions $u(x,t)$ and $v(x,t)$ as follows

$$\begin{cases} u(x,t) = \beta D_x^\alpha \left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \beta\, \frac{g(x,t)D_x^\alpha f(x,t) - fD_x^\alpha g(x,t)}{f(x,t)^2+g(x,t)^2}, \\ \qquad v(x,t) = \beta_1 D_x^\alpha \left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \beta_1 \frac{g(x,t)D_x^\alpha f(x,t) - f(x,t)D_x^\alpha g(x,t)}{f(x,t)^2+g(x,t)^2}, \end{cases} \tag{10}$$

where the constants $\beta$and β_1 require determination. For the single-soliton solution, the auxiliary functions $f(x,t)$ and $g(x,t)$ are defined thusly:

$$f(x,t) = e^{\theta_1} = e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} \text{ and } g(x,t) = 1. \tag{11}$$

By substituting the relations from Equation (11) into Equation (10) and subsequently substituting the result into system (3), one can infer values for β and β_1 as $\beta = c$ and $\beta_1 = \frac{4}{ac}$, with c representing a constant. Therefore, combining Equations (8) – (11), the single-soliton solution of system (3) is explicitly given by

$$u_1(x,t) = \frac{ck_1 e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}{1 + e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}},\ v_1(x,t) = \frac{4k_1 e^{e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}}{ac\left(1 + e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}\right)}.$$

By considering the quotient of u_1 and v_1, the following relationship is deduced

$$\frac{u_1(x,t)}{v_1(x,t)} = \frac{a\,c^2}{4}. \tag{12}$$

In the determination of the two-soliton solutions, the auxiliary functions, $f(x,t)$ and $g(x,t)$ are given by

$$\begin{cases} f(x,t) = e^{\theta_1} + e^{\theta_2} = e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} + e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha}, \\ g(x,t) = 1 - a_{12} e^{\theta_1 + \theta_2} = 1 - a_{12} e^{\left((k_1+k_2)x^\alpha - (K_1^3 + K_2^3)t^\alpha\right)/\alpha}, \end{cases} \tag{13}$$

where the term a_{12} symbolizes the temporal displacement inherent to a wave formation and is referred to as the phase shift. Deduction of the phase shift can be achieved through the substitution of Equation (13) into Equation (10), followed by substituting the result into system (3). This procedure gives

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{14}$$

This relationship can further be extended to encompass a generic case as depicted below

$$a_{ij} = \frac{\left(k_i-k_j\right)^2}{\left(k_i+k_j\right)^2}, 1 \le i < j \le 3. \tag{15}$$

Through incorporating equations (14) and (13) into equation (10), one arrives at the two-soliton solutions characterized as

$$u_2(x,t) = \beta D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = cD_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}+e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha})/\alpha}}\right)\right],$$

$$v_2(x,t) = \beta_1 D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \frac{4}{\mathrm{a}\,c}D_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}+e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha})/\alpha}}\right)\right].$$

It is imperative to note, following the demonstration in [24], that the value of a_{12} outlined in Equation (14) is neither zero nor infinity provided that $|k_1| \neq |k_2|$. Under such conditions, the fractional coupled mKdV system presented in system (3) is exempt from generating resonant phenomena [25].

The auxiliary functions $f(x,t)$ and $g(x,t)$ can be used to construct the three-soliton solutions given by

$$\begin{cases} f(x,t) = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}, \\ g(x,t) = 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3}, \end{cases} \tag{16}$$

where $\theta_1 = (k_1x^{\alpha} - \mathrm{k}_1^3t^{\alpha})/\alpha, \theta_2 = (k_2x^{\alpha} - \mathrm{k}_2^3t^{\alpha})/\alpha$, and $\theta_3 = (k_3x^{\alpha} - \mathrm{k}_3^3t^{\alpha})/\alpha$. The phase shifts a_{ij}, where $1 \le i < j \le 3$, are explained previously in Equation (15).

The three-soliton solutions of system (3) are obtained by substituting Equation (16) and Equation (15) into Equation (10)

$$u_3(x,t) = \beta D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = cD_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\theta_1}+e^{\theta_2}+e^{\theta_3}-a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}}{1-a_{12}e^{\theta_1+\theta_2}-a_{13}e^{\theta_1+\theta_3}-a_{23}e^{\theta_2+\theta_3}}\right)\right],$$

$$v_3(x,t) = \beta_1 D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \frac{4}{\mathrm{a}\,c} D_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\theta_1}+e^{\theta_2}+e^{\theta_3}-a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}}{1-a_{12}e^{\theta_1+\theta_2}-a_{13}e^{\theta_1+\theta_3}-a_{23}e^{\theta_2+\theta_3}}\right)\right],$$

where $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$, $a_{13} = \frac{(k_1-k_3)^2}{(k_1+k_3)^2}$, and $a_{23} = \frac{(k_2-k_3)^2}{(k_2+k_3)^2}$.

The discovery of the N-soliton solution is evidently achievable for any distinct positive integer within a finite range.

3.2. Singular-soliton solutions

The singular-soliton solutions of system (3) are derived using the Hirota transformation method [12, 24], as presented below

$$\begin{cases} u(x,t) = \beta D_x^{\alpha}\left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \beta\,\frac{g(x,t)D_x^{\alpha}f(x,t)-f(x,t)D_x^{\alpha}g(x,t)}{f(x,t)g(x,t)}, \\ v(x,t) = \beta_1 D_x^{\alpha}\left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \beta_1\frac{g(x,t)D_x^{\alpha}f(x,t)-f(x,t)D_x^{\alpha}g(x,t)}{f(x,t)g(x,t)}. \end{cases} \tag{17}$$

As stated in [10], the auxiliary functions $f(x,t)$ and $g(x,t)$ take the forms of

$$\begin{cases} f(x,t) = 1 + \sum_{n=1}^{\mathrm{N}} f_n(x,t)\,, \\ g(x,t) = 1 - \sum_{n=1}^{\mathrm{N}} g_n(x,t). \end{cases} \tag{18}$$

As previously discussed, the concept of the dispersion relation is introduced by

$$c_i = k_i^3, i = 1,2, \ldots N, \tag{19}$$

and therefore, we have

$$\theta_i = \left(k_i x^{\alpha} - \mathrm{k}_i^3 t^{\alpha}\right)/\alpha. \tag{20}$$

If we set $N = 1$ in Equation (18), then

$$\begin{cases} f(x,t) = 1 + e^{\theta_1} = 1 + e^{(k_1 x^{\alpha} - K_1^3 t^{\alpha})/\alpha}, \\ g(x,t) = 1 - e^{\theta_1} = 1 - e^{(k_1 x^{\alpha} - K_1^3 t^{\alpha})/\alpha}. \end{cases} \tag{21}$$

In order to calculate β and $\beta_1$in this scenario, we first insert Equation (21) into Equation (17), and then substitute the resulting equation into system (3). This gives $\beta = c$ and $\beta_1 = -\frac{1}{\mathrm{ac}}$, where c is constant. Combining the above-mentioned results, the single-soliton solution of system (3) is given by

$$u_4(x,t) = \frac{2ck_1 e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}{1-e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}; \text{ and } \quad v_4(x,t) = -\frac{2k_1 e^{e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}}{ac\left(1-e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}\right)}.$$

Dividing u_4 by v_4 gives

$$\frac{u_4(x,t)}{v_4(x,t)} = -a\, c^2. \tag{22}$$

The following auxiliary functions $f(x,t)$ and $g(x,t)$ are used to determine the singular two-soliton solutions

$$\begin{cases} (x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}; \\ g(x,t) = 1 - e^{\theta_1} - e^{\theta_2} + b_{12} e^{\theta_1+\theta_2}. \end{cases} \tag{23}$$

By Substituting Equation (23) into Equation (17) and then substituting the result into system (3), we derive

$$a_{12} = b_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{24}$$

Next, substituting Equation (24) into Equation (23), we obtain the auxiliary functions $f(x,t)$ and $g(x,t)$ as

$$\begin{cases} f(x,t) = 1 + e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} + e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha}, \\ g(x,t) = 1 - e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} - e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha}. \end{cases} \tag{25}$$

The singular two-soliton solutions are obtained by substituting Equation (25) into Equation (17) to get

$$u_5(x,t) = \beta D_x^\alpha \left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right]$$

$$= cD_x^\alpha \left[ln\left(\frac{1+e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} + e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha}}{1-e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} - e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha.}} \right)\right],$$

$$v_5(x,t) = \beta_1 D_x^\alpha \left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right]$$

$$= -\frac{1}{a\,c} D_x^{\alpha}\left[ln\left(\frac{1+e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}+e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha}}{1-e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}-e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha.}}\right)\right].$$

To construct a singular three-soliton solution, the following auxiliary functions are used

$$\begin{cases} f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + f_3(x,t), \\ g(x,t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + g_3(x,t). \end{cases} \tag{26}$$

By substituting Equation (26) into Equation (17) then substituting the result in system (3), we obtain

$$\begin{cases} f_3(x,t) = b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ g_3(x,t) = -b_{123}e^{\theta_1+\theta_2+\theta_3},\ b_{123} = a_{12}a_{13}a_{23}, \end{cases} \tag{27}$$

where a_{ij}, $1 \le i < j \le 3$, are derived above in Equation (15). Then, the auxiliary functions $f(x,t)$ and $g(x,t)$ are given by

$$\begin{cases} f(x,t) = 1 + e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha} + e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha} + e^{\left(k_3x^{\alpha}-K_3^3t^{\alpha}\right)/\alpha} \\ \qquad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{\left((k_1+k_3)x^{\alpha}-(K_1^3+K_3^3)t^{\alpha}\right)/\alpha} \\ \qquad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_2+k_3)x^{\alpha}-(K_2^3+K_3^3)t^{\alpha}\right)/\alpha} \\ \qquad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_1+k_2+k_3)x^{\alpha}-(K_1^3+K_2^3+k_3^3)t^{\alpha}\right)/\alpha}, \\ g(x,t) = 1 - e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha} - e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha} - e^{\left(k_3x^{\alpha}-K_3^3t^{\alpha}\right)/\alpha} \\ \qquad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{\left((k_1+k_3)x^{\alpha}-(K_1^3+K_3^3)t^{\alpha}\right)/\alpha} \\ \qquad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_2+k_3)x^{\alpha}-(K_2^3+K_3^3)t^{\alpha}\right)/\alpha} \\ \qquad - \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_1+k_2+k_3)x^{\alpha}-(K_1^3+K_2^3+k_3^3)t^{\alpha}\right)/\alpha}. \end{cases} \tag{28}$$

The singular three-soliton solution is then obtained by substituting Equation (28) into Equation (17).

4. Multiple and singular soliton solutions of system (4)

We next consider the fractional coupled mKdV equation of system (4) given by

$$D_t^{\alpha}u + 6auvD_x^{\alpha}v + 6(u^2 - v^2)D_x^{\alpha}u + D_x^{\alpha\alpha\alpha}u = 0,$$
$$D_t^{\alpha}v + 24auvD_x^{\alpha}u + 6(u^2 - v^2)D_x^{\alpha}v + D_x^{\alpha\alpha\alpha}v = 0,$$

where a is constant. When $u = v$, the above system becomes the second fractional coupled mKdV equation.

4.1. Multiple-soliton solutions

As given earlier, the dispersion relation $c_i = k_i^3$ and $\theta_i = \left(k_i x^\alpha - \mathrm{k}_i^3 t^\alpha\right)/\alpha$. The multiple-soliton solutions of system (4) are assumed earlier in Equation (10) where the auxiliary functions are as given above in Equation (11). Substituting Equation (11) into Equation (10), and then substituting the result into system (4) gives β and β_1 as

$$\beta = \pm \frac{2}{\sqrt{4a-3}}\ and\ \ \beta_1 = \pm \frac{4}{\sqrt{4a-3}}, \qquad a > \frac{3}{4}.$$

By combining these results, the single-soliton solution can be expressed as follows:

$$u_6(x,t) = \pm \frac{2k_1 e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}{\sqrt{4a-3}\left(1+e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}\right)},\ v_6(x,t) = \pm \frac{4k_1 e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}}{\sqrt{4a-3}\left(1+e^{2\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha}\right)};\ \ a > \frac{3}{4}.$$

These results provide the relation between $u_6(x,t)$ and $v_6(x,t)$ as follows

$$\frac{u_6(x,t)}{v_6(x,t)} = \pm \frac{1}{2}. \tag{29}$$

The two-soliton solutions are determined by substituting Equation (13) into Equation (10) and then substituting the result into system (4). This gives the following phase shift

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{30}$$

The generalization of Equation (30) provides

$$a_{ij} = \frac{\left(k_i-k_j\right)^2}{\left(k_i+k_j\right)^2}, 1 \le i < j \le 3. \tag{31}$$

The two-soliton solutions are obtained by substituting Equations (30) and (13) into Equation (10)

$$u_7(x,t) = \beta D_x^\alpha \left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \pm \frac{2}{\sqrt{4a-3}} D_x^\alpha \left[tan^{-1}\left(\frac{e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} + e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha}}\right)\right],$$

$$v_7(x,t) = \beta_1 D_x^\alpha \left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right] = \pm \frac{4}{\sqrt{4a-3}} D_x^\alpha \left[tan^{-1}\left(\frac{e^{\left(k_1 x^\alpha - K_1^3 t^\alpha\right)/\alpha} + e^{\left(k_2 x^\alpha - K_2^3 t^\alpha\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha}}\right)\right].$$

The three-soliton solutions of system (4) are then determined by substituting Equations (31) and Equation (16) into Equation (10). The N-soliton solution is evidently achievable for any distinct positive integer within a finite range.

4.2. Singular-soliton solutions

Proceeding as before, the dispersion relation $c_i = k_i^3$ and $\theta_i = \left(k_i x^\alpha - \mathrm{k}_i^3 t^\alpha\right)/\alpha$. Substituting Equation (21) into (17) and then substituting the result into system (4), β and $\beta_1$are given by

$$\beta = \pm\frac{1}{\sqrt{3-4a}} \text{ and } \beta_1 = \pm\frac{2}{\sqrt{3-4a}}; a < \frac{3}{4}.$$

By synthesizing this result with the results of the auxiliary functions presented in Equation (21), the singular-soliton solutions are given by

$$u_8 = \pm\frac{2k_1 e^{\left(k_1 x^\alpha - k_1^3 t^\alpha\right)/\alpha}}{\sqrt{3-4a}\left(1-e^{2\left(k_1 x^\alpha - k_1^3 t^\alpha\right)/\alpha}\right)}, \quad v_8 = \pm\frac{4\,k_1 e^{\left(k_1 x^\alpha - k_1^3 t^\alpha\right)/\alpha}}{\sqrt{3-4a}\left(1-e^{2\left(k_1 x^\alpha - k_1^3 t^\alpha\right)/\alpha}\right)}.$$

It is now obvious that

$$\frac{u_8(x,t)}{v_8(x,t)} = \pm\frac{1}{2}. \tag{32}$$

The singular two-soliton solutions are constructed by substituting Equation (23) into Equation (17) and then substituting the result into system (4). Therefore, the phase shifts are

$$a_{12} = b_{12} = \frac{(k_1-k_2)^2}{(k_1-k_2)^2}. \tag{33}$$

In case of the two-soliton solutions, we use Equation (33) to obtain the auxiliary functions presented in Equation (25). As a result of this, the singular two-soliton solutions are obtained if we substitute Equation (25) into Equation (17)

$$u_9(x,t) = \beta D_x^\alpha\left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right]$$

$$= \pm\frac{1}{\sqrt{3-4\mathrm{a}}} D_x^\alpha\left[ln\left(\frac{1+e^{\left(k_1x^\alpha-K_1^3t^\alpha\right)/\alpha}+e^{\left(k_2x^\alpha-K_2^3t^\alpha\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha-(K_1^3+K_2^3)t^\alpha\right)/\alpha}}{1-e^{\left(k_1x^\alpha-K_1^3t^\alpha\right)/\alpha}-e^{\left(k_2x^\alpha-K_2^3t^\alpha\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha-(K_1^3+K_2^3)t^\alpha\right)/\alpha.}}\right)\right],$$

$$v_9(x,t) = \beta_1 D_x^\alpha\left[ln\left(\frac{f(x,t)}{g(x,t)}\right)\right]$$

$$= \pm\frac{2}{\sqrt{3-4\mathrm{a}}} D_x^\alpha\left[ln\left(\frac{1+e^{\left(k_1x^\alpha-K_1^3t^\alpha\right)/\alpha}+e^{\left(k_2x^\alpha-K_2^3t^\alpha\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha-(K_1^3+K_2^3)t^\alpha\right)/\alpha}}{1-e^{\left(k_1x^\alpha-K_1^3t^\alpha\right)/\alpha}-e^{\left(k_2x^\alpha-K_2^3t^\alpha\right)/\alpha}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha-(K_1^3+K_2^3)t^\alpha\right)/\alpha.}}\right)\right].$$

To specify the singular three-soliton solutions of system (4), we substitute Equation (26) into Equation (17) and then substitute the result into system (4) to obtain

$$\begin{cases} f_3(x,t) = b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ g_3(x,t) = -b_{123}e^{\theta_1+\theta_2+\theta_3}, \text{and } b_{123} = a_{12}a_{13}a_{23}. \end{cases} \tag{34}$$

For the singular three-soliton solutions, we use the result of Equation (34) to obtain

$$\begin{cases} f(x,t) = 1 + e^{\left(k_1x^\alpha - K_1^3t^\alpha\right)/\alpha} + e^{\left(k_2x^\alpha - K_2^3t^\alpha\right)/\alpha} + e^{\left(k_3x^\alpha - K_3^3t^\alpha\right)/\alpha} \\ \quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{\left((k_1+k_3)x^\alpha - (K_1^3+K_3^3)t^\alpha\right)/\alpha} \\ \quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_2+k_3)x^\alpha - (K_2^3+K_3^3)t^\alpha\right)/\alpha} \\ \quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_1+k_2+k_3)x^\alpha - (K_1^3+K_2^3+k_3^3)t^\alpha\right)/\alpha}, \\ g(x,t) = 1 - e^{\left(k_1x^\alpha - K_1^3t^\alpha\right)/\alpha} - e^{\left(k_2x^\alpha - K_2^3t^\alpha\right)/\alpha} - e^{\left(k_3x^\alpha - K_3^3t^\alpha\right)/\alpha} \\ \quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^\alpha - (K_1^3+K_2^3)t^\alpha\right)/\alpha} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{\left((k_1+k_3)x^\alpha - (K_1^3+K_3^3)t^\alpha\right)/\alpha} \\ \quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_2+k_3)x^\alpha - (K_2^3+K_3^3)t^\alpha\right)/\alpha} \\ \quad - \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{\left((k_1+k_2+k_3)x^\alpha - (K_1^3+K_2^3+k_3^3)t^\alpha)t^\alpha\right)/\alpha}. \end{cases} \tag{35}$$

Equation (35) in turn gives the three singular-soliton solutions of system (4) if we substitute it into Equation (17).

5. Multiple and singular soliton solutions of system (5)

We study now the fractional coupled mKdV of system (5) given by

$$D_t^\alpha u + a(v^2 - u^2)D_x^\alpha u + \frac{a}{4}D_x^{\alpha\alpha\alpha}u = 0,$$

$$D_t^\alpha v + a(v^2 - u^2)D_x^\alpha v + \frac{a}{4}D_x^{\alpha\alpha\alpha}v = 0,$$

where a is an arbitrary constant.

5.1. Multiple-soliton solutions

Note that this section follows the same methodology as the previous sections. Some details are omitted to avoid redundancy. By substituting Equation (11) into Equation (10), and then substituting the result into system (5) we obtain β and β_1 as

$$\beta = \sqrt{\frac{6}{c^2-1}} \; and \; \beta_1 = \sqrt{\frac{6}{c^2-1}}c; \; c > 1.$$

Here, c is an arbitrary constant. Combining this result with the result of Equation (10), the single-soliton solution is

$$u_{10}(x,t)=\frac{\sqrt{\frac{6}{c^2-1}}k_1e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}}{1+e^{2\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}},\ v_{10}(x,t)=\frac{\sqrt{\frac{6}{c^2-1}}ck_1e^{e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}}}{1+e^{2\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}}.$$

The two-soliton solutions are constructed by substituting Equation (13) into Equation (10) and then substituting the result into system (5). Thus, the phase shifts are

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{36}$$

Equation (36) can be extended to other related scenarios, where

$$a_{ij}=\frac{\left(k_i-k_j\right)^2}{\left(k_i+k_j\right)^2},\ \text{where } 1\le i<j\le 3. \tag{37}$$

The two-soliton solutions are then obtained by substituting Equations (36) and (13) into Equation (10) as follows

$$u_{11}(x,t)=\beta D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right]=\sqrt{\frac{6}{c^2-1}}D_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}+e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha}}\right)\right],$$

$$v_{11}(x,t)=\beta_1 D_x^{\alpha}\left[tan^{-1}\left(\frac{f(x,t)}{g(x,t)}\right)\right]=\sqrt{\frac{6}{c^2-1}}c\,D_x^{\alpha}\left[tan^{-1}\left(\frac{e^{\left(k_1x^{\alpha}-K_1^3t^{\alpha}\right)/\alpha}+e^{\left(k_2x^{\alpha}-K_2^3t^{\alpha}\right)/\alpha}}{1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\left((k_1+k_2)x^{\alpha}-(K_1^3+K_2^3)t^{\alpha}\right)/\alpha}}\right)\right].$$

The three-soliton solutions of system (5) are then obtained by substituting Equations (37) and Equation (16) into Equation (10). The N-soliton solution is evidently achievable for any positive integer within a finite range.

5.2. Singular soliton solutions

The single, two and three-soliton solutions of system (5) can be obtained by following the same steps that were used in the previous sections.

6. Conclusion:

In this paper, the application of the Hirota bilinear method has been meticulously employed to generate analytical solutions for the space-time fractional coupled modified Korteweg–de Vries (mKdV) equations, whereby the conformable fractional derivative (CFD) has been incorporated. A spectrum of both multiple and singular soliton solutions for the fractional coupled mKdV equations have been acquired. The proficiency, simplicity, and practicality of the Hirota bilinear method are evident from the results presented within this study. Such a method offered promising avenues for addressing complex problems that span various scientific domains. The derived soliton solutions in this study were match the solutions provided [25] when the value of α equals 1. These results bear considerable implications for computational and empirical research in wave dynamics. All calculations within this study were conducted using MAPLE software. Future studies could explore higher-order fractional derivatives within the fractional coupled mKdV equation to broaden our comprehension of such phenomena.

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