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APPLICATION OF E^p-STABILITY TO IMPULSIVE FINANCIAL MODEL

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Abstract. In this paper, we consider an impulsive stochastic model for an investment with production and saving profiles. The conditions for financial growth for the investment are investigated under impulsive action and results are obtained using the quantitative and Ep stability methods. The impulsive stochastic differential equation considered is assumed to be driven by a process with jump and non-linear gestation properties. One of the results established shows that, in the long run, it is impossible for a financial investment to grow or dominates the prescribed average financial investment but has a threshold value for which the investment cannot grow beyond. It is also established that an E^p- stable investment vector can be found which allows financial growth but this vector must be constrained to be in a given invariant set:It is advisable for the saving and depreciation to satisfy certain growth rates for proper income and investment growths.

1. Introduction

Impulsive differential equations (IDEs) are systems that are subject to rapid changes in the variables describing them. Impulses are noted to take place in different ways e.g. in the form of "shocks", "jumps", "mechanical impacts" etc. ([1]) and they take place for short moments during process of evolution ([1], [10], [14-17]). Many real life processes are impulsive in nature, examples are the biological bustling rhythms, the change in the states of the economy of some countries, the population under rapid changes, the outbreaks of earthquakes, eruption of epidemic in some ecological set-ups and so on ([1] & [16-17]).

In the recent times, financial markets are places where funds are sourced for investment. Many financial derivatives are traded under organized market system and trade over the counter. In the stock exchange market financial derivatives like plain vanillas, bonds and exotics options are traded. The volume of trade increases everyday and there is the need to analyze the performance of the market using some models. We need to determine the fair price of an option and payoff for the buyer. We need to understand the complex cash flow structures in the financial markets and the risks involve in managing the financial portfolios etc ([4], [6], [9],[13], [18] and [19]).

Most business organizations and many countries of the world usually set aside some substantial amount for investment or put in place some machineries to generate funds for investment. The instrument for raising money for investment often

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take the form of sinking funds, treasury bills, capital stocks, national reserves etc. The goal of entrepreneurs are to make the investment have appreciable financial growths.

It must be noted that sometimes, funds so invested experience favourable financial growth, if the investment atmosphere allows such growth. Under certain situations which are often not easily predicted, the business many suffer unexpected economic recession, in which case, profit is lot in the investment period. Some countries rely on natural resources such as fossil oil as the major export earner, for which the prices of crude oil tends to fluctuate every year, hence, the state of the economy of those countries tend to show impulsive behaviour ([14], [16]).

Furthermore, in the international market today, the price of fossil oil is impulsive because of rapid rise and falling of price of the oil within short period of times. Moreover, profits made on investment on crude oil goes up and down in yo-yo way. Therefore, the use of impulsive models would be useful for modelling prices of energy derivatives using different exotic options. The model also have potential applications for studying several real life problems in several fields of human endeavour that can be modeled using ISDEs with gestation function being taken to be zero.

In view of the above, an investment may also be affected by impulsive phenomenon and how can it experience growth under impulsive effect? the search for the answer to this question is the motivation for the study in this paper. We will consider an impulsive stochastic model in studying the financial investment portfolio with production and saving profiles. We will use the E^p-stability method, that is, stability relative to p-moment to the study model. Application of p-moment to SDEs are found in the literature but for the ISDEs we can say it is relatively new ([7]) if we consider the volume of publications made on SDEs.

It is worthy of note to say, the structure of the solution of ISDEs changes with equilibrium points of the model, hence the investment and national income equilibria depend on the process driving forces and the volatilities for the model. Therefore, the stability of the financial model cannot be deduce in the ordinary stochastic sense. Hence, we will exploit a broader stability concept, that is, stability relative to invariant sets to study the model. The importance of this approach were emphasized in many publications (see for examples [11-12] and [14], and references therein), however, we will unify this approach with E^p-stability, such kind of approach had appeared for stochastic processes [17]) but as for impulsive systems the unified approach seems to be relatively new.

2. Usefulness of Impulsive Phenomenon in the Financial Modeling

In mathematics a variety of models exist for financial investment using applications of some theories in chaos, control, neural network, ecophysics and so on. Most models from Economics, the central problem is the concern about the interactions of many complex variables for which fundamental "scarce" variable is the capita ([2],[4],[6],[9],[13] and [16]).

The model we will consider is one of the simplest impulsive mathematical model in economics, a "macro-model" with gestation lag and depreciation, it can be used as a simple model of company (or country) growth process and to demonstrate some fundamental relationships that exist among variables that quantify a financial investment using impulsive variables.

We note that the impulsive system theory offers viable techniques to handle dwindling effect of the investment, for example, through impulsive system theory, we can identify factors responsible for rapid and irregular growth of the investment and also factors responsible for sudden drop in the investment at fixed and non-fixed investment periods ([10],[14],[16] and [20]). We will consider an impulsive stochastic model of a financial investment with income, capita stock and depreciation vector containing gestation lag.

3. Preliminary Treatment

We shall denote by (Ω, ξ, p) a probability space Ω, being the set of points with events, which is a δ-algebra of subsets of Ω such that $\Omega \in \xi$ and p denotes the probability measure.

Let $x(t)$ be a random process at time t with expectation (mean) $Ex(t) = \bar{x}(t)$ and the variance is $\delta^2 x(t) = (x(t) - \bar{x}(t))^2$. The autocovariant vector $R(t, \tau) = E(x(t)x(\tau))$. We shall say ([5]) the sequence of random process $\{x_n(t)\}$ converges to $x(t)$ in probability almost surely as $n \to \infty$, if for any $\varepsilon > 0$, $\delta > 0$ there exists a number N such that

$$p(|x_n(t) - x(t)| > \delta) < \varepsilon, n > N;$$

and $\{x_n(t)\}$ is said to be convergent to $x(t)$ with probability 1 as $n \to \infty$ if

$$\lim_{n\to\infty} p(|x_n(t) - x(t)| = 0) = 1.$$

Crucial to our investigation is the following set $C([-h, 0], \Re)$ which is the space of continuous random processes on $[-h, 0]$ and taking values on $\Re = (-\infty, +\infty)$ and let $\Re^+ = [0, +\infty)$.

$PC([-h, 0], \Re) = \{x : [-h, 0] \to \Re$ is a piecewise continuous random process for $t \in [-h, 0]$ such that it is left continuous at t_k, $k = 1, 2, \dots\}$.
$K = \{a(r) : a \in C([-h, 0], \Re)$ are monotonically increasing in r and $\lim_{r\to+\infty} a(r) = \infty\}$.

Let $V(t, x(t))$ be a piecewise continuous random process on $\Re^+ \times C[-h, 0]$ and there exist $w_i \in K, i = 1, 2$ such that the following conditions are satisfied:

$$\bar{\omega}_1(|\varphi(0)|) \le V(t, \varphi) \le \bar{\omega}_2(|\varphi(0)|), \bar{\omega}_i \in K,\ i = 1, 2$$

$|V(t, \varphi_1) - V(t, \varphi_2)| \le C_1(|\varphi_1 - \varphi_2|)$, $C_1 =$ constant, $\varphi_i \in C([-h, 0], \Re^+)$, $i = 1, 2$.

Then the Dini derivative $D^+_{(\cdot)} V(t, x(t))$ of the function $V(t, \varphi)$ along the solution path $(\cdot)$ for the ISDE in equation $(\cdot)$ is defined as

$$D^+_{(\cdot)} V(t, x(t)) = \lim_{\delta\to 0} \sup \frac{1}{\delta}[V(t + \delta, x(t) + \delta f) - V(t, x(t))].$$

Consider the following ISDE

$$\begin{aligned} &dx(t) = f(t, x(t))dt + g(x(t-h))dw(t),\ t \ne t_k,\ k = 0, 1, 2, \dots \qquad (1)\\ &\triangle x(t_k) = I(x(t_k)) \end{aligned}$$

$0 < t_0 < t_1 < t_2 < \dots < t_k;\ \lim_{k\to\infty} t_k = +\infty$

where

$f : \Re^+ \times \Omega \to \Omega;\ g : \Omega \to \Omega,\ I : \Omega \to \infty$. The expectation of $V(t, x(t))$ is $EV(t, x(t))$ and the variance $\delta^2 V(t, x(t))$ and the p-moment $E^p V(t, x)$ about the origin.

3.1. **Comparison system.** Consider the following comparison impulsive stochastic differential equations (CISDE) corresponding to the eqn (1)

$$
\begin{aligned}
dx(t) &= e(t, u(t))dt + h(u(t))dw(t),\ t \neq \gamma_k,\ k = 0, 1, 2, \dots \qquad (2)\\
\triangle u(\gamma_k) &= I(u(\gamma_k))
\end{aligned}
$$

$0 < \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_k;\ \lim_{k\to\infty} \gamma_k = +\infty$

where $e : \Re^+ \times \Omega \to \Omega;\ h, I : \Omega \to \Omega$.

Assume that f, e, h and I are smooth enough as to guarantee the existence and uniqueness of solutions of eqn (1) and eqn (2) (see [10] and [19]).

We will make use of the following definitions:

Definition 1 ([11],[12] & [14])

Let $x(t)$ be the solution of eqn (1) passing through $(t_0, x(t_0 + 0) = x_0)$ then we say that the solution $x(t) = 0$ of the eqn (1).

1. E^p-uniformly stable (u.s.) with respect to the invariant $A = \{x \in \Omega : |x| \leq r\}$ if
   (a) $|x_0| = r$ implies $|Ex^p(t)| = r, t \geq t_0$;
   (b) $\forall\ \varepsilon > 0$ and $t_0 \in \Re^+$ there exists a real number $\delta = \delta(\varepsilon) > 0$ such that $r - \delta < |x_0| < \delta + r$ implies $r - \varepsilon |Ex^p(t)| < r + \varepsilon,\ t \geq t_0$;
2. E^p-uniformly asymptotically stable with respect to the invariant A if there exist real numbers $\delta_0 > 0$, and $T = T(\varepsilon) > 0$ such that $r - \delta_0 < |x_0| < r + \delta + 0$ implies $r - \varepsilon < |Ex^p(t)| < r + \varepsilon, t \geq t_0 + T$

Remark 1

E^p-stability is the unification of invariant stability and stability with respect to p-moment ([10-11] & [14]).

If $Ex(t) = \bar{x}(t)$ for $p = 1$ and $r = 0$, E^p-stability reduces to the usual stability in the Lyapunov's sense for the impulsive stochastic equations. If the underlying variable is deterministic then the system is simply the impulsive ordinary differential equations.

We will make use of the following auxiliary results.

Let x be a random variable such that

$$E(x) = \mu, E(x - \mu)^2 = \sigma^2, E|x - \mu|^r = \beta_r, E(|x - \mu|) = 0 \text{ and } \tilde{\delta} = \mu/\sigma.$$

4. Statement of the Problem

Consider a simple impulsive stochastic differential model (isdm)

$$dx(t) = \delta^{-1} y(t)dt - b + \alpha_1(t) g(x(t - h)) + \sigma_1 dw_1(t),\ t \neq t_k,\ k = 0, 1, 2, \dots \qquad (3)$$

$$dy(t) = \delta^{-1} y(t)dt + \alpha_2(t)V(t) + \sigma_2 dw_2(t), t \neq t_k,\ k = 01, 2, \dots \qquad (4)$$

$$dz(t) = \beta dx(t),\ t \neq t_k,\ k = 0, 1, 2, \dots \qquad (5)$$

$$\triangle x(t_k) = \beta_k x(t_k) \qquad (6)$$

Satisfying the initial conditions

$$x(t_0 + 0) = x_0, y(t_0 + 0) = y_0 \text{ and } z(t_0 + 0) = z_0 \qquad (7)$$

$$0 < t_1 < t_2 < t_3 < \cdots < t_k,\ t_k = +\infty,\ \text{as}\ \ k \to \infty$$

where

(1) $x(t)$ is the investment variable
(2) $y(t)$ is the national (company's) income
(3) $z(t)$ is the capital stock
(4) $v(t)$ is the fluctuation variable
(5) $w_i(t)$ are assume to be Brownian processes.

It is assumed that $x(t)$ is the random variable representing in totality the amount of the investment (both liquid and solid asserts) which experience a growth rate of $\delta\beta^{-1} = \dfrac{\text{saving}}{\text{Capita output}}$

$\delta =$ saving ratio, δ^{-1} is the drift, b is given depreciation rate and $g(x(t-h))$ is the depreciation function with gestation lag h with the expectation Eg $(x(t-h))$. $g(x(t-h))$ is generally assume to be nonlinear continuous random process. The expectation of $g(x(t-h))$ denote by $g(x(t-h))$ is assumed to exists. $\alpha_i(t)$ are some jump parameters for $i = 1, 2$. The parameters $\beta_k = 1, 2, 3, \ldots,$ account for the impulses that happen during the investment period. These parameters can be investment for some period of times.

$V(t)$ is assume to be statistically independent with respect to the investment variable, hence $E(y(t), V(t)) = 0$ and $\delta^2(V(t), V(t)) = 1$.

We define

$$\begin{aligned} f_1(t, x(t)) &= \delta^{-1}x(t) - b\alpha_1(t)g(x(t-h)) + \sigma_1 dw_1(t) \text{ and} \\ f_2(t, y(t)) &= \delta^{-1}y(t) - b\alpha_2(t)v(t) + \sigma_2 dw_2(t) \end{aligned}$$

$dx(t) = a_1(t)dt + b_1(t)dw(t)$ and $dy(t) = a_2(t)dt + b_2(t)dw_2(t)$.

Then we define the stochastic differential equation corresponding to (isde) as

$$\begin{aligned} df_1 &= (f_{1t} + a_1(t)f_{1x} + \frac{1}{2}b_1^2(t)f_{1xx}dt + b_1(t)f_{1x}dw_1(t) \\ df_2 &= (f_{2t} + a_2(t)f_{2x} + \frac{1}{2}b_2^2(t)f_{2xx}dt + b_2(t)f_{2x}dw_2(t) \\ \triangle x(t_k) &= \beta_k x(t_k) \\ \triangle y(t_k) &= \beta_k y(t_k) \end{aligned}$$

and integration give

$$\begin{aligned} f_1(t, x(t) = f_1(0, x_0) &+ \int_0^t (f_{1s} + a_1(s)f_{1x} + \frac{1}{2}b_1^2(s)f_{1xx})ds \\ &+ \int_0^t b_1(s)f_{1x}dw_1(s) + \sum_{t_0 < t_k < t} \beta_k x(t_k) \end{aligned}$$

and

$$\begin{aligned} f_2(t, y(t) = f_2(0, y_0) &+ \int_0^t (f_{2s} + a_2(s)f_{2y} + \frac{1}{2}b_2^2(s)f_{2yy})ds \\ &+ \int_0^t b_2(s)f_{2y}dw_2(s) + \sum_{t_0 < t_k < t} \beta_k y(t_k) \end{aligned}$$

where $f_{1t} = \dfrac{\partial f_i}{\partial t}$, $f_{ix} = \dfrac{\partial f_i}{\partial x}$, $f_{itt} = \dfrac{\partial^2 f_i}{\partial t^2}$ and $f_{ixx} = \dfrac{\partial^2 f_i}{\partial x^2}$ for $i = 1, 2$.

If $\alpha_i(t) = 0$, $\triangle x(t_k) = 0$, the model is the Black Schole's model for pricing options where α^{-1} and σ_i are the drift and the volatility respectively ([2], [4] & [9]).

It is worthy of note to say that, it is not easy to obtain the equilibrium points for the problem. The solution structure of the problem changes with the equilibrium points. In general, but if $g(x(t-h)) = x(t-h)$ and $\triangle x(t_k) = C =$ constant, this give rise to the following nonlinear equations

$$\delta^{-1}x(t) - b\alpha_1(t)g(x(t-h)) + \sigma_1 dw_1(t) = 0$$
$$\delta^{-1}y(t) - b\alpha_2(t)v(t) + \sigma_2 dw_2(t) = 0$$

We solve for $x(t)$ and $y(t)$ to determine the equilibrium investment and the e-quilibrium national income.

Remark 2

The investment and national income equilibria change with time and both depend on the saving ratio. The two of them also depend on the jump and volatility parameters and the stochastic process driven forces as the above non-linear equations show. The above nonlinear equations can be solved using fixed point iterative process for stochastic processes. The major problem is the nonlinearity of the gestation radon variable $g(x(t-h))$.

Now let us consider the situation where $g(x(t-h)) = x(t-h)$ that is g has a fixed point. If the volatilities of the investment and national income are zeros, i.e. $\sigma_i = 0$ for $i = 1, 2$ then an equilibrium for the system is the point t^* such that $x(t^*) = b\alpha_1(t^*)\delta x(t^* - h)$ and $y(t^*) = -\alpha_2(t^*)\delta V(t^*)$ In order to determine the values of $x(t^*)$ and $y(t^*)$ set $s(t) = \lambda^t$ where $\lambda =$ constant and $t = 0, 1, 2, \ldots$ then it is easy to show that $\lambda = (b\alpha_1(t^*)\delta)^{1/h}$, thus $x(t^*)$ varies with t^*. Then the set of the equilibrium points for the model with zero volatilities ($\sigma_i = 0$) or simply the set E_1 is $E_1 = \{t^* \in I\!R^+ : x(t^*) = (b\alpha_1(t^*))^{t^*/h}$ and $y(t^*) = -\alpha_2(t^*)\delta V(t^*)$, $t^* \in (t_k, t_{k+1})$; $x(t_k) = x_0 + NC$, $\triangle x(t_k) = C$ and N is number of impulse points present in $(t_k, t_{k+1})\}$ where $x(t)$ is the solution of eqn (1) satisfying the initial condition $x(0) = x_0$ for finite N and C.

If the volatility is not equal to zero, that is, $\sigma_i \neq 0$, then we can obtain the set of the equilibrium points for the model for non-zero volatilities ($\sigma_i \neq 0$) or simply the set E_2 as

$$E_2 = \left\{ t^* \in J : \begin{pmatrix} x(t) = (b\alpha_1(t^*)\delta)^{t^*/h + \sigma_1 dw_1(t^*)} \\ y(t) = b\alpha_1(t^*)V(t^*)\delta - \sigma_2 dw_2(t^*) \end{pmatrix}, t \in (t_k, t_{k+1}); \right.$$
$$x(t_k) = x_0 + NC, \triangle x(t_k) = C, y(t_k) = y_0 + NC,\ \triangle y(t_k) = C$$

and N is the number of impulse points present in $(t_k, t_{k+1})\}$.

Thus, the equilibria also depend on the process driving forces $w_i(t)$ and the volatilities σ_i for the model for $i = 1, 2$. Since the equilibria for the (isdm) is generally changing from point to point, the stability of the financial model cannot be deduce in the ordinary sense. Hence, later on, we will exploit a broader stability concept, that is, stability relative to invariant sets to study the model. The importance of this approach were emphasized in many publications (see for examples [11-12] & [14], and reference therein), however, we will unify this approach with E^p stability, that is, stability relative to $p-$moment of vectors, such kind of approach had appeared for stochastic processes ([17]) but as for impulsive systems the unified approach seems to be relatively new.

4.1. **Discussion of the Variables Involved in the Model.** Assume that no gestation lag or depreciation, i.e., $b = 0$ or $g(x(t-h)) = 0$ and the rate of increase of the capita stock equals investment. Then the introduction of capita output ratio allows simple estimation of the production function of the business as express as ratio of the investment to the income. This simply mean that the investment (assumed equal to saving) shows a fixed ratio, δ to the income.

By Lord Keynes 'famous work of the general theory of employment interest and money; saving and investment are always equal'. Therefore, in term of consumption, the income can be expressed as

(1) income = consumption + saving
(2) income = consumption + investment

Hence, from the above, it is easy to show the ratio of the consumption to the investment is equal to $(\alpha^{-1} - 1)$.

In this model we shall assume that the depreciation function $g(x(t-h))$ is a nonlinear continuous random process.

4.2. **Nature of the Solution to the Investment Model.** The solution of equations (3 and 7) is a stochastic variable $q(t)$ such that

$$q(t) = q_0 + \int_{t_0}^{t} f(s, q(s))ds \ \text{ for } \ t \neq t_k, k = 0, 1, 2, \ldots \tag{9}$$

where

$$f(t, q(t)) = \begin{pmatrix} \delta^{-1} & 0 & 0 \\ \delta^{-1} & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + h(t, q(t))$$

$$h(t, q(t)) := \begin{pmatrix} -b\alpha_1(t)g(x(t-h)) + \sigma_1(t)w_1(t) \\ \\ \alpha_2(t)v(t) + \sigma_2(t)w_2(t) \\ \\ 0 \end{pmatrix},$$

$$q(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

In eqn (9) the integration is made in the Ito sense. The solution $x(t)$ of eqn (1) is a continuous random process that satisfies the equation for $t \neq t_k$, but for $t = t_k$ impulses take place and the solution starts to exhibit some kind of impulsive behaviour whose solution can also be obtained by slight modification of the results in ([10] & [17]) as follows:

It is a random process whose solution satisfies the equation

$$dq(t) = f(t, q(t))dt, \ t \neq t_k, \ k = 0, 1, 2, \ldots$$

such that

$$q(t_k^+) = x(t_k) + \triangle x(t_k)$$

where $\triangle x(t_k) = x(t_k+0) - x(t_k-0)$, $x(t_k+0)$ and $x(t_k-0)$ are the right and the left limits of $x(t)$ at $t = t_k$, $k = 1, 2, 3, \ldots$ respectively. Assume that $x(t_k) = x(t_k - 0)$.

For characterization of the solution of the eqn (1) in terms of non-fixed moments of some kind of impulsive operator (see [1] & [10]).

The appearance of impulses in this model, give rise to phenomenon such as “dying effect”, i.e., the solution cannot be continued across some barrier, “beating effect” i.e., pulses take place wherein the solution hits given hyper-surface several times and “lost of autonomy”, etc. (see [10] & [17] for detail) which financial models may likely prone to, such phenomena cannot be handled from the ordinary stochastic differential equations point of view (see [17]). The complexities in the behaviour of the solutions of most problems of impulsive family make it an interesting area of research focus of late. Some of the behaviour were communicated in many publications (see [1] & [8]) and even our recent papers ([14 - 15]).

The equation (3) and (7) can be used to find the solution to the model, since, the values for $y(t)$ and $z(t)$ can be obtained once $x(t)$ is known. In this direction, the solution can be found by slight modification of the result in (see [3], [10] & [17])

$$\begin{aligned} x(t) = &\prod_{t_0<t_k<t}(1+\beta_k)e^{\delta^{-1}(t+h)}e^{\delta^{-1}h}x(-h) \\ &+ be^{-\delta^{-1}h}\int_0^{t-h}\prod_{t_0<t_k<t}(1+\beta_k)\alpha_1(\tau)e^{\delta^{-1}\tau}g(x(\tau))d\tau \\ &+ b\int_0^{t}\prod_{t_0<t_k<t}(1+\beta_k)\alpha_1(\tau)e^{\delta^{-1}(\tau+h)}g(x(\tau))d\tau \\ &+ \int_0^{t}\sigma_i\prod_{t_0<t_k<t}(1+\beta_k)e^{\delta(\tau+h)}dw_1(\tau), t\geq 0,\ \tau:=t+s \end{aligned} \tag{10}$$

One crucial question one may ask at this point is: is it possible for the business set-up to have perpetual financial growth if we assume that the business environment is favourable? The next proposition affirms that it is impossible for the business to have indefinite financial growth.

Proposition 1
Let the following conditions be satisfied:
There exist non-negative constants C_1, C_2, C_3 and C_4 such that:
$H_1:$ $x(t)$ has a distribution such that $Ex(t) = 0$ and $E|x^2(t)| < C_4$.
$H_2:$ $A^2E^2x(-h)\exp B^2\exp(e^{-a(t-h)}T) \leq C_1$
where

$$A = \prod_{t_0<t_k<t}(1+\beta_k)e^{\alpha^{-1}(t+h)}e^{\alpha^{-1}h}, B = \prod_{t_0<t_k<t}(1+\beta_k)e^{\alpha^{-1}h}b$$

$H_3:$ $A^4e^{-4\alpha^{-1}h}e^{-2\alpha^{-1}}Ex^2(-h) + 2AB^2e^{-2\alpha^{-1}h}|x(-h)|\exp\left(\frac{1-e^{-\alpha^{-1}t}}{\alpha^{-1}}\right) \leq C_2$

$H_4:$ $\sup E|x^2(\tau)| \leq \max[C_1, C_2] < C_4$.
Then

$$\lim_{t\to\infty}\frac{1}{t^2}\int_0^t P(|x(s)| \geq kE|x^2(s)|)ds = 0$$

Proof

Let $-h \le t \le 0$ and $\alpha_{10} = \max_{t\in(-h,0]} \alpha(t)$, replace $x(t)$ by $Ex^2(t)$ in eqn (10) and majorise the equation, then

$$Ex^2(t) \le A^2 Ex^2(-h) + 2ABx(-h)\int_0^{t-h} \alpha_1(\tau)e^{-\alpha^{-1}\tau}Ex(\tau)d\tau$$
$$+ b^2\int_0^{t-h} e^{2\alpha^{-1}\tau}Ex^2(\tau)d\tau + \int_0^{\infty} e^{2\alpha^{-1}\tau}Ex^2(\tau)dw_1(\tau).$$

By the application of the Gronwall-Bellman's equality, we have

$$Ex^2(t) \le A^2Ex^2(-h)\exp B^2\exp(\alpha_{10}e^{-\alpha^{-1}(t-h)}T)$$
$$\times\,\sigma_1\int_0^t e^{-2\alpha^{-1}\tau}Ex^2(\tau)dw_1(\tau) \le C_1$$

If $t > 0$, $\tau := t + s$.
Then

$$E|x^2(t)| \le A^4e^{4\alpha^{-1}h}e^{-2\alpha^{-1}t}|Ex^2(-h)|$$
$$+ 2\alpha_{10}A^2B^2AB^2e^{-2\alpha^{-1}h}|Ex(-h)|\sigma_1\int_0^t e^{-\alpha^{-1}\tau}Ex^2(\tau)d\tau$$
$$\le A^4e^{-4\alpha^{-1}h}e^{-2\alpha^{-1}\tau}C_2 + 2AB^2e^{-2\alpha^{-1}h}|Ex(-h)|\exp\left(\sigma_1\frac{1-e^{-\alpha^{-1}\tau}}{\alpha^{-1}}\right)$$
$$\le C_3.$$

It follows that

$$E|x^2(t)| \le \sup_{\tau\in(-h,0]} E|x^2(\tau)| \le \max(C_1, C_2) < C_4$$

Application of Chebyshev's inequality yields,

$$P(|x(t)| \ge kE|x^2(t)|)\frac{1}{k^2},\quad k > 0$$

Then,

$$\lim_{t\to\infty}\frac{1}{t^2}\int_0^t P(|x(s)| \ge kE|x^2(s)|) \le \lim_{t\to\infty}\frac{1}{t^2}\int_0^t \frac{1}{k^2}ds$$
$$= \lim_{t\to\infty}\frac{1}{t^2}\frac{1}{k^2}t = 0.$$

The result established above shows that, in the long run, "it is impossible for the investment to grow or dominates the prescribed average financial investment".

The behaviour of the financial investment under impulsive action is not necessarily be the same as the ordinary stochastic equations with prescribed probability distribution function, for example the Markov, Wienie, and Martingale processes etc. Let us assume that the impulsive processes have gamma distribution because of the relatively newness of the impulsive stochastic differential process and that their solutions behave as stated in the section 4.2. Although processes that allow jump behaviours as Poisson, Levy and Martingale can also be used to analyse the model.

Theorem 1

Suppose the random variable $x(t)$ in eqn (1) is a Stochastic process with gamma distribution with parameter $(n+1,\mu)$ and define the following constants:

$$H_1:\ A=\left[\prod_{t_0<t_k<t}(1+\beta_k)\right]^2 e^{2\alpha^{-1}h}x(-h)$$

$$H_2:\ E_1x^2(t)=\frac{\mu^{n+1}}{n(\mu-\alpha^{-1})^{n+1}}\left[\prod_{t_0<t_k<t}(1+\beta_k)\right]^2 e^{2\alpha^{-1}h}x(-h)$$

$$H_3:\ E_2x(t)=be^{-\alpha^{-1}h}\int_0^{\infty}\int_{-h}^{0}\prod_{t_0<t_k<t}(1+\beta_k)e^{-\alpha\tau}g(x(\tau))\frac{\mu(\mu t)^n}{n!}e^{-\mu t}dt$$

$$H_4:\ E_3x(t)=\int_0^{\infty}\int_{-h}^{0}\prod_{t_0<t_k<t}(1+\beta_k)e^{-\alpha^{-1}t+h}g(x(\tau))\frac{\mu(\mu t)^n}{n!}t^n e^{-\mu t}dt$$

Then,

$$Ex^2(t)=E_1x^2(t)+E_2x^2(t)+E_3x^2(t)$$

Proof

From the eqn (10) and definitions of E_1 and E_2 we have

$$\begin{aligned}
E_1x^2(t)&=\frac{\mu^{n+1}}{n(\mu-\alpha^{1})^{n+1}}\left[\prod_{t_0<t_k<t}(1+\beta_k)\right]^2 e^{2\alpha^{-1}h}x(-h)\\
E_2x^2(t)&=\frac{-b}{n!}\frac{e^{-\alpha^{-1}h}}{\mu^{n+1}}\int_{-h}^{0}\prod_{t_0<t_k<t}(1+\beta_k)e^{-\alpha^{-1}\tau}g(x(\tau))dt\int_0^{\infty}\beta^n e^{-\beta}d\beta\\
&=\frac{-b}{n!}\frac{e^{-\alpha^{-1}h}}{\mu^{n+1}}\Gamma(n-1)\int_{-h}^{0}\prod_{t_0<t_k<t}(1+\beta_k)e^{-\alpha^{-1}\tau}g(x(\tau))d\tau
\end{aligned}$$

Furthermore, we have the estimation

$$\begin{aligned}
Ex^2(t)&=b\int_0^{\infty}\int_0^{t}\prod(1+\beta_k)e^{-\alpha^{-1}(\tau+h)}g(x(\tau)d\tau\frac{\mu^{n+1}}{n!}t^n e^{\mu t}dt\\
&=\frac{b}{n!}\Gamma(n)\int_0^{t}\prod(1+\beta_k)e^{-\alpha^{-1}(\tau+h)}g(x(\tau))d\tau\\
&=\frac{b}{n!}\Gamma(n)\prod(1+\beta_k)\int_0^{t}e^{-\alpha^{-1}\tau}g(x(\tau))d\tau
\end{aligned}$$

But

$$Ex^2(t) = A^4e^{2\alpha t}Ex^2(-h) - 2A^2e^{\alpha t}\int_{-h}^{0} Be^{-\alpha\tau}E(x(h))g(x(\tau))d\tau$$
$$\int_{-h}^{0}\int_{-h}^{0} Be^{-\alpha(\tau_1+\tau)}E(g(x(\tau_1)))g(x(\tau))d\tau_1 d\tau$$
$$+\int_{0}^{t}\int_{0}^{t} B^2e^{-\alpha^{-1}(\tau_1+\tau)}E(g(x(\tau_1)))g(x(\tau))d\tau_1 d\tau$$
$$+\int_{-h}^{0}\int_{0}^{t} B^2e^{-\alpha^{-1}(\tau_1+\tau)}E(g(x(\tau_1)))g(x(\tau))d\tau d\tau_1$$
$$+\sigma_1^2\int_{0}^{t}\int_{0}^{t} e^{-\alpha(\tau+\tau_1)}R(\tau,\tau_1)d\tau d\tau_1$$

where $R(\tau,\tau_1)$ is the autocovariant function.
Therefore, $Ex^2(t) = E_1x^2(t) + E_2x^2(t) + E_3x^2(t)$.

An analogous result can be established for the model with gamma distribution with $(n+1,\mu)$ parameter using the Peek's inequality as follows:

If $Ex^2(t)$ is bounded by a constant C_5 such that $\zeta(t)$ has asymptotic expansion and define $\dfrac{Ex^2(0)}{Ex^2(T)} = A+\zeta(T), \zeta(T) := a_0 + fraca_1T + \dfrac{a_1}{T^2} + \cdots,\ a_i,\ i = 0,1,2,\dots$ are some constants and $A = \exp(1+a_0) - a_0$ then by Peek's inequality it follows that

$$P(x(t) \geq \lambda Ex^2(t)) < \frac{1-Ex^2(t)}{\lambda^2 - 2\lambda Ex^2(t) + 1}$$

Therefore,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T P(x(s) = \lambda Ex^2(s))ds &< \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\frac{1-Ex^2(s)}{\lambda^2-2\lambda Ex^2(s)+1}\right]ds\\ &\leq \lim_{T\to\infty}\frac{1}{T}\left(\int_0^T\frac{1-Ex^2(s)ds}{-2\lambda Ex^2(s)}\right)\\ &= \frac{1}{\lambda}\left[\lim_{T\to\infty}\log e\left(\frac{Ex^2(0)}{Ex^2(T)}\right) - 1 + \zeta(T)\right]\\ &= \frac{1}{\lambda}\lim_{T\to\infty}[\ln(A+\zeta(T)) - 1 + \zeta(T)]\\ &= \frac{1}{\lambda}[\ln(A+a_0) - 1 + a_0] = 0.\end{aligned}$$

Therefore,

$$Ex^2(T) = Ex^2(0)\exp -\lambda(A+1)\exp(-\zeta(T))$$

Hence, $Ex^2(T) \to 0$ as $T \to \infty$, a contradiction of $0 < |Ex^2(t)| < C_5$ for $t \in [-h,T]$ for all T, which also contradicts the fact that $|Ex(t)| < C_1$, therefore $\lim_{T\to\infty} P(x(t) \leq \lambda Ex(t)) = 1$. If the process has a gamma distribution as against the arbitrary situation proved in proposition 1, then the financial variable grows and tends to be bounded by λC_5 in the long run. This simply affirms that the investment has a threshold value scaling factor of $\lambda C_5'$ for which the investment cannot grow beyond.

The question of boundedness of the growth of the investment posses the question of investigating the problem from qualitative point of view, to be precise the stability of solution of the model will be investigated in relation to a given invariant set. In this direction, we will discuss the stability of the problem in relation to $p-$moment.

Corollary 1
H_1 : Suppose all the conditions in Theorem 1 are satisfied
$H_2 : E^k := \sum\sum\sum\cdots\sum_{\theta} E_k E_\theta, \theta$ are permutation over $1, 2, \ldots, n$.
Then

$$E^2x(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i E_{i-j} x(t)$$

Proof
Straight forward like Theorem 1.
Suppose that the depreciation variable $g(x(t))$ is such that $g(x(t)) = rx(t)$; or

$$g(x(t)) = rx(t)x(t-h) \leq rx^2(t).$$

Then,

$$E(x(-h)g(x(t)) \leq rE(x(-h)x(t)) \leq rC_4$$

and

$$E(x(-h)h(x(t)) \leq rx^2(t).$$

Then

$$E(x(-h))g(x(t)) \leq rE(x(-h)x(t)) \leq rC_4.$$

We are now in the position to investigate E^p-stability of the financial model. In the practical term, this implies given a solution $x(t)$ to the financial investment in the invariant set, is it possible for the p-moment of the solution about the origin $E^p x(t)$ to be found in the invariant set which satisfies the E-stability properties in the definition 1. In real life situation, this simply means that if right economic strategies that needed for the growth of the investment are used, then the investment can be made to grow so long as it is confined to the invariant set. In practice, the invariant set is the set in which the investment is constrained to and outside of it the investment may not be favourable.

Theorem 2
Let $x(t) \in PC([-h, 0], \Re)$ such that $V(t, x(t))$ is a continuous random process in $[-h, 0]$ with expectation $EV(t, x(t))$ such that

(1) $H_1 : |EV(t, x_1(t)) - EV(t, x_2(t))| \leq K_1|x_1(t) - x_2(t)|, K_1 =$constant
(2) There exist $\beta_1, \beta_2 \in K$ such that
$\beta_2(E|x(t)|^p) \leq EV(t, x(t)) \leq \beta_1(E|x(t)|^p)$
(3) $dEV(t, x(t)) \leq g(EV(t, x(t)))dt$
(4) $EV(t_k + 0, x(t_k) + \triangle x(t_k)) \leq \varphi_k(x(t_k))$
(5) $\beta_k u = u + \varphi_1(u),\ \varphi_1 \in K$

The zero solution $x(t) = 0$ of eqn (1) is E-stable with respect to the invariant set A.

Proof.

Suppose that the zero solution $x(t) = 0$ of eqn (1) is E-stable with respect to the invariant set A, then $\forall\varepsilon > 0$, $\exists\delta = \delta(x_0, \varepsilon)$ such that $|Ex(t)| < \varepsilon$ implies that $|x_0| < \delta_0$, there is a finite number $p > 0$ such that $r - \varepsilon < |Ex^p(t)| < r + \varepsilon$ for $\delta_0 + r_0 < |x_0| < \delta_0 + r_0$.

Let

$$|x_0|^p < r - \delta_0, a_2(\varepsilon - r) > \beta_1(r - 2\delta_0), \text{ for } a_2 \text{ and } \beta_0 \in K$$

Let $m(t)$ be the solution of eqn (1) such that

$$\begin{aligned}\dot{m}(t) &\le g(t, m(t)),\ t \ne t_k,\ k = 0, 1, 2, \dots\\ m(t_k^+) &\le m(t) + \varphi_k(m(t)),\ t = t_k,\ k = 0, 1, 2, \dots\\ m(t_k^+) &\le u(t_0)\end{aligned}$$

If $r(t) = r(t, t_0 + 0, u_0)$ is a random process which is the maximal solution of the impulsive stochastic differential equation in eqn (1) then by standard results, $V(t, x(t)) \le r(t)$ we can show that $EV(t, x(t)) \le r(t)$.

From eqn (2)

$$\begin{aligned}EV(t, x(t)) &\le EV(t_0 + 0, x_0) \le r(t) \le \beta_1(E|x_0|^p) = \beta_1(|x_0|)^p)\\ &\le \beta_1(r - \delta_0) < \beta_2(r - \varepsilon)\end{aligned}$$

Therefore, $E|x(t)|^p < \varepsilon + r$ for $t \ge t_0$, by similar estimation we have

$$E|x(t)|^p > \varepsilon - r \text{ for } r - \delta < |x_0 < \delta + r.$$

Therefore the zero solution $x(t) = 0$ of the eqn (1) is E-stable with respect to the invariant set Ω.

The general form of distribution function governing an impulsive stochastic model is unknown, such distribution if exists may be continuous or discrete or even possess piecewise continuous property. Recently, the theory of Time Scale have been exploited to study systems which are either continuous or discrete or both simultaneously ([11-12]).

In the quest for an ideal distribution for impulsive stochastic system and the correspond p-moments are open problems. Meanwhile, we define the characteristic function for $x(t)$ taking into consideration impulsive tendency as

$$C_x(\varepsilon) = E[x(t)e^{-i\varepsilon t}]$$

From the characteristic function we can obtain the k-moment as

$$E\left\{x^k(t)\right\} = \frac{1}{(i\varepsilon)^k}\left[\frac{d^k C_x(\varepsilon)}{d\varepsilon^k}\right],\ \ k = 1, 2, \dots$$

The construction of the ideal distribution may be made by the formulation of impulsive analogue of the Chapman-Kolmogorov equation if the underlying stochastic process is a Markov process ([3]). The construction of an ideal distribution function for impulsive stochastic systems is one of the fundamental problems future research should focus on. Because of the peculiar nature of the problem of how to determine $C_x(\varepsilon)$ we resort to investigate the behaviour of the solution of the model using qualitative approach, hence the problem will be studied from stability point of view.

Remark 3

We propose the monkey function which somehow shows the behaviour of a monkey in the game and can be used to mimic the dynamics of the financial market.

We define the monkey process as

$$M(t_1,t) = \begin{bmatrix} f_k(t) & t \in [t_1, t_k),\ k = 1, 2, \ldots \\ f_s(t) & t \in [t_k, t_N),\ f_s(t) \in C^\infty(\Omega) \\ \delta(t_N - t) & t = t_N \\ f_j(t) & t \in (t_{k+1}, t_s)\ 0 \le t_1 < t_k < t_N \le t_s \end{bmatrix}$$

$f_k(t)$ is uniformly and identically distributed in the given interval
$f_s(t)$ is a smooth random function in the given interval, and
$f_j(t)$ is a continuous random process in the given interval
$\delta(t)$ is the Dirac function of t
The monkey function should be constructed to have the following properties:
$H_1:\ M(t_1,t) \ge 0$ for $t_1 \ge t$
$H_2:\ \displaystyle\int_0^t M(t_1,s)ds = 1$
$H_3:\ M(t_1,0) = \delta(-t_1)$
$H_4:\ dM = M(t_1, t + \triangle t) - M(t_1,t)$

The construction of an ideal distribution for an impulsive stochastic system using the monkey function and the correspond p-moments are open problems.

5. Application of the E-stability

Without loss of generality if the depreciation variable is chosen in such a way that $g(x(t-h)) \ge x(t)$ and the fluctuation variable is selected to be bounded ($V(t) \le k =$ constant) then the comparison equations corresponding to the isdm is

$$\begin{aligned} dx(t) &\le (\delta^{-1} - b\alpha_1)x(t)dt + \sigma_1 dw_1(t),\ t \ne \gamma_k,\ k = 0,1,2,\ldots \\ dy(t) &\le \delta^{-1}y(t)dt + K + \sigma_2 dw_2(t),\ t \ne \gamma_k,\ k = 0,1,2,\ldots \\ \triangle x(\gamma_k) &\le \beta_k x(\gamma_k). \end{aligned}$$

If $m_1(t)$ and $m_2(t)$ are the maximal solution to the comparison's equation above respectively.
Then

$$\begin{aligned} dm_1(t) &\le (\delta^{-1} - b\alpha_1)m_1(t)dt + \sigma_1 dw_1(t),\ t \ne \gamma_k,\ k = 0,1,2,\ldots \\ dm_2(t) &\le \delta^{-1}m_2(t)dt + K + \sigma_1 dw_1(t),\ t \ne \gamma_k,\ k = 0,1,2,\ldots \\ \triangle x(t_k) &\le \beta_k^1 m_1(\gamma_k) \\ \triangle y(t_k) &\le \beta_k^2 m_2(\gamma_k) \\ m_1(\gamma_k + 0) = x_0 &\ \text{ and }\ m_2(\gamma_k + 0) = y_0 \end{aligned}$$

whose solutions are found to be

$$m_1(t) = r_{m_1}(t) = \prod_{t_0<t_k<t} (1+\beta_k^1) e^{-(\delta^{-1}-b\alpha_1)(t-t_0)} x_0 + \int_{t_0}^{t} \sigma_1 \prod_{t_0<t_k<t} (1+\beta_k^1) e^{-(\delta^{-1}-b\alpha_1)(s-t_0)} dw_1(s)$$

$$m_2(t) = r_{m_2}(t) = \prod_{t_0<t_k<t} (1+\beta_k^2) e^{-(\delta^{-1}-b\alpha_1)(t-t_0)} y_0 + K[\delta-(t-t_0)] \int_{t_0}^{t} \sigma_2 \prod_{t_0<t_k<t} (1+\beta_k^2) e^{-(\delta^{-1}-b\alpha_1)(s-t_0)} dw_2(s).$$

If $(\delta^{-1} - b\alpha_1) > 0$, as time $t \to \infty$, the investment is bounded above by

$$\lim_{t\to\infty} \int_{t_0}^{t} \sigma_1 \prod_{t_0<t_k<t} (1+\beta_k^1) e^{-(\delta^{-1}-b\alpha_1)(s-t_0)} dw_2(s)$$

while the saving is unbounded below because of the linear term $-Kt$. It is advisable to maintain the saving and the depreciation rates such that $(\delta^{-1} - b\alpha_1) < 0$ for proper income and investment growths. The question we need to ask is: what conditions would guarantee financial E-stability? We will use the proposition 1 to investigate the required conditions to answer the question:

Let

$$V(t,x(t)) = \frac{1}{2}\int_{t_0}^{t} (s^2 + x^2(s))ds + \sum_{t_0<t_k<t} p_k x_k^2(t_k)$$

Then,

$$EV(t,x(t)) = \int_0^\infty V(t,x(t))dF$$

where dF is the distribution of the process describing the random process. Therefore,

$$EV(t,x(t)) = \frac{1}{2}\int_0^\infty \int_0^t (s^2 + x^2(s))dsdF + \sum_{t_0<t_k<t} p_k \int_0^\infty x_k^2(s)ds$$

and

$$Ex^p = \int_0^\infty x^p(s)dF + \sum_{t_0<t_k<t} p_k x_k^{2p}(t_k)$$

It is easy to show that

$$|EV(t,x_1) - EV(t,x_2)| \le \frac{1}{2}L_1|x_1-x_2||dF| + \frac{1}{2}L_2|p_k||x_1-x_2| = \frac{1}{2}(L_1|F| + L_2|p_k|)|x_1-x_2|$$

Pick $\beta_1(r) = r^2$ and $\beta_2(r) = r^p$ for $p \ge 2$ and $-\triangle x(t_k) \le x(t_k+0) \le 1 - \triangle x(t_k)$ it is not difficult to show that

$$\beta_1(E|x(t)|^2) \le \beta_2(E|x(t)|^p) \le EV(t,x(t)) \le \beta_1(E|x(t)|^p).$$

By Theorem 2, this implies that the investment is E-stable with respect to the invariant set A. This result simply shows that we can find an E-stable investment vector which is constrained to a given invariant set in A which is E-stable.

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