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SOME RESULTS ON THE DRAZIN INVERSE OF A MODIFIED MATRIX WITH NEW CONDITIONS

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Abstract. In this article, we consider representations of the Drazin inverse of a modified matrix $M = A - CD^dB$ with the generalized Schur complement $Z = D - BA^dC$ under different conditions given in recent articles on the subject. Numerical example is given to illustrate our result.

1. Introduction

The importance of the Drazin inverse and its applications to singular differential equations and difference equations, to Morkov chains and iterative methods, to cryptography, to numerical analysis, to structured matrices and to perturbation bounds for the relative eigenvalue problems can be found in [1-3].

Let $\mathbb{C}^{m\times n}$ represent the set of $m \times n$ complex matrices. Let $A \in \mathbb{C}^{n\times n}$, then there exist a unique matrix $A^d \in \mathbb{C}^{n\times n}$ satisfying the following equations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA, \tag{1.1}$$

A^d is called the Drazin inverse of A, where $k = ind(A)$ is the index of A, the smallest nonnegative integer for which $rank(A^{k+1}) = rank(A^k)$ (see[1-3]). In particular, when $ind(A) = 1$, the Drazin inverse of A is called the group inverse of A. If A is nonsingular, it is clearly $ind(A) = 0$ and $A^d = A^{-1}$. Throughout this article, we denote by $A^\pi = I - AA^d$ and define $A^0 = I$, where I is the identity matrix with proper sizes.

In 1975, Shoaf [4] derived the result of the Drazin inverse of a modified square matrix, in 1994, Kala et al. [5] gave an explicit representation for the generalized inverse of a modified matrix, and in 2002, Wei [6] have discussed the expression of the Drazin inverse of a modified square matrix $A - CB$. Recently, in 2013, Dopazo et al. [7], Mosić [8] and Shakoor et al. [9] presented some new results for the Drazin inverse of a modified matrix $M = A - CD^dB$ in terms of the Drazin inverse of the matrix A and the generalized Schur complement $Z = D - BA^dC$ under the following conditions:

(1) $A^\pi C = 0,\ CD^\pi = 0,\ D^\pi B = 0,\ Z^\pi B = 0,\ CZ^\pi = 0$ (see [7]);
(2) $BA^\pi = 0,\ CD^\pi = 0,\ D^\pi B = 0,\ Z^\pi B = 0,\ CZ^\pi = 0$ (see [7]);
(3) $A^\pi C = CD^\pi,\ D^\pi B = 0,\ DZ^\pi = 0$ (see [8]);
(4) $BA^\pi = D^\pi B,\ CD^\pi = 0,\ Z^\pi D = 0$ (see [8]);
(5) $A^\pi C = 0,\ CD^\pi Z^dB = 0,\ CD^dZ^\pi B = 0,\ CZ^dD^\pi B = 0,\ CZ^\pi D^dB = 0$ (see [9]);

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(6) $BA^{\pi}=0,\ CD^{\pi}Z^{d}B=0,\ CD^{d}Z^{\pi}B=0,\ CZ^{d}D^{\pi}B=0\ \ CZ^{\pi}D^{d}B=0$ (see [9]).
Moreover, Shakoor et al. [9] gave some new results for the Drazin inverse of the modified matrix $M=A-CD^{d}B$, when the generalized Schur complement $Z=0$ under the following conditions:
(7) $A^{\pi}C=0,\ CD^{\pi}\Gamma^{d}B=0,\ CD^{d}\Gamma^{\pi}B=0,\ C\Gamma^{d}D^{\pi}B=0\ \ C\Gamma^{\pi}D^{d}B=0$ (see [9]);
(8) $BA^{\pi}=0,\ CD^{\pi}\Gamma^{d}B=0,\ CD^{d}\Gamma^{\pi}B=0,\ C\Gamma^{d}D^{\pi}B=0\ C\Gamma^{\pi}D^{d}B=0$ (see [9]).

In this article, we consider the Drazin inverse of a modified matrix $M=A-CD^{d}B$ in terms of the Drazin inverse of the matrix A and the generalized Schur complement $Z=D-BA^{d}C$ under conditions weaker than conditions (5) and (6) in [9], which extends some results in [7,8]. Furthermore, we consider some results for the Drazin inverse of the modified matrix $M=A-CD^{d}B$, when the generalized Schur complement $Z=0$ under different conditions in [9]. Finally, we give a numerical example to illustrate our result.

2. The Drazin inverse of a modified matrix

In this section, we consider the Drazin inverse of a modified matrix $M=A-CD^{d}B$ in terms of the Drazin inverse of the matrix A and the generalized Schur complement $Z=D-BA^{d}C$ is not necessarily invertible under different conditions presented in [7,8,9].

Let $A,B,C,D\in\mathbb{C}^{n\times n}$. Throughout this section, we use the following notations:

$$M=A-CD^{d}B,\ Z=D-BA^{d}C \tag{2.1}$$

and

$$K=A^{d}C,\ H=BA^{d},\ \Gamma=HK. \tag{2.2}$$

First, we present the following theorem.

Theorem 2.1. Let A, B, C, and D be complex matrices, where $ind(A)=k$. If $A^{\pi}C=CD^{\pi}$, $CD^{d}Z^{\pi}B=0$, $CZ^{d}D^{\pi}B=0$ and $CZ^{\pi}D^{d}B=0$, then

$$M^{d}=A^{d}+KZ^{d}H-\sum_{i=0}^{k-1}(A^{d}+KZ^{d}H)^{i+1}KZ^{d}BA^{i}A^{\pi} \tag{2.3}$$

and $ind(M)\leq ind(A)$.

Proof. Let $X=A^{d}+KZ^{d}H$. The assumption $A^{\pi}C=CD^{\pi}$ implies that $AA^{d}C=CDD^{d}$. Firstly, we note the facts:

$$\begin{aligned} MX &= AA^{d}+AA^{d}CZ^{d}BA^{d}-CD^{d}BA^{d}-CD^{d}(D-Z)Z^{d}BA^{d}\\ &= AA^{d}+CDD^{d}Z^{d}BA^{d}-CD^{d}DZ^{d}BA^{d}-CD^{d}Z^{\pi}BA^{d}\\ &= AA^{d} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} XM &= A^{d}A+A^{d}CZ^{d}BA^{d}A-A^{d}CD^{d}B-A^{d}CZ^{d}(D-Z)D^{d}B\\ &= A^{d}A-A^{d}CZ^{d}BA^{\pi}-A^{d}CZ^{\pi}D^{d}B\\ &= A^{d}A-KZ^{d}BA^{\pi}. \end{aligned} \tag{2.5}$$

From (2.4), we have

$$\begin{aligned} MM^d &= MX - MX\sum_{i=0}^{k-1} X^i KZ^d BA^i A^\pi \\ &= AA^d - \sum_{i=0}^{k-1} X^i KZ^d BA^i A^\pi \qquad (2.6) \end{aligned}$$

and using (2.5), we get

$$\begin{aligned} M^d M &= XM - \sum_{i=0}^{k-1} X^{i+1} KZ^d BA^i A^\pi M \\ &= A^d A - KZ^d BA^\pi - \sum_{i=0}^{k-1} X^{i+1} KZ^d BA^{i+1} A^\pi \\ &= A^d A - \sum_{i=0}^{k-1} X^i KZ^d BA^i A^\pi. \qquad (2.7) \end{aligned}$$

Thus

$$MM^d = M^d M.$$

Secondly, from (2.7) and $A^\pi M^d = 0$, we obtain

$$\begin{aligned} M^d MM^d &= (A^d A - \sum_{i=0}^{k-1} X^i KZ^d BA^i A^\pi) M^d \\ &= A^d AM^d \\ &= M^d. \end{aligned}$$

Finally, we shall prove that $M - M^2 M^d$ is a nilpotent matrix. From (2.6), we get

$$\begin{aligned} M - M^2 M^d &= [I - MM^d]M \\ &= (I - AA^d + \sum_{i=0}^{k-1} X^i KZ^d BA^i A^\pi) M \\ &= AA^\pi + \sum_{i=0}^{k-1} X^i KZ^d BA^{i+1} A^\pi. \end{aligned}$$

By induction on integer $n \geq 1$, we have

$$(M - M^2 M^d)^n = A^n A^\pi + \sum_{i=0}^{k-1} X^i KZ^d BA^{i+n} A^\pi. \qquad (2.8)$$

From (2.8), it gives that $(M - M^2 M^d)^k = 0$, where $k = ind(A)$. Therefore, we conclude that $M^{k+1} M^d = M^k$ and $ind(M) \leq ind(A)$, that completes the proof. $\square$

From Theorem 2.1, we obtain the following corollaries.

Corollary 2.2 ([8]). Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $A^{\pi}C = CD^{\pi}$, $Z^{\pi}B = 0$, $D^{\pi}B = 0$ and $CZ^{\pi} = 0$, then

$$M^d = A^d + KZ^dH - \sum_{i=0}^{k-1}(A^d + KZ^dH)^{i+1}KZ^dBA^iA^{\pi}$$

and $ind(M) \leq ind(A)$.

Corollary 2.3 ([8]). Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $A^{\pi}C = CD^{\pi}$, $DZ^{\pi} = 0$ and $D^{\pi}B = 0$, then

$$M^d = A^d + KZ^dH - \sum_{i=0}^{k-1}(A^d + KZ^dH)^{i+1}KZ^dBA^iA^{\pi}$$

and $ind(M) \leq ind(A)$.

In the same way, we give a new theorem.

Theorem 2.4. Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $BA^{\pi} = D^{\pi}B$, $CD^{\pi}Z^dB = 0$, $CD^dZ^{\pi}B = 0$ and $CZ^{\pi}D^dB = 0$, then

$$M^d = A^d + KZ^dH - \sum_{i=0}^{k-1}A^{\pi}A^iCZ^dH(A^d + KZ^dH)^{i+1}$$

and $ind(M) \leq ind(A)$.

Similarly, from Theorem 2.2. we have the following corollaries.

Corollary 2.5 ([8]). Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $BA^{\pi} = D^{\pi}B$, $CD^{\pi} = 0$, $Z^{\pi}B = 0$ and $CZ^{\pi} = 0$, then

$$M^d = A^d + KZ^dH - \sum_{i=0}^{k-1}A^{\pi}A^iCZ^dH(A^d + KZ^dH)^{i+1}$$

and $ind(M) \leq ind(A)$.

Corollary 2.6 ([8]). Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $BA^{\pi} = D^{\pi}B$, $CD^{\pi} = 0$ and $Z^{\pi}D = 0$, then

$$M^d = A^d + KZ^dH - \sum_{i=0}^{k-1}A^{\pi}A^iCZ^dH(A^d + KZ^dH)^{i+1}$$

and $ind(M) \leq ind(A)$.

Now, we consider some results for the Drazin inverse of the modified matrix $M = A - CD^dB$, when the generalized Schur complement $Z = 0$ under different conditions in [9].

Theorem 2.7. Let $A,\ B,\ C$, and D be complex matrices, where $ind(A) = k$. If $Z = 0,\ A^{\pi}C = CD^{\pi},\ D^{\pi}B = 0$ and $D\Gamma^{\pi} = 0$, then

$$M^d = (I - K\Gamma^d H)A^d(I - K\Gamma^d H) + \sum_{i=0}^{k-1}[(I - K\Gamma^d H)A^d]^{i+2}K\Gamma^d BA^i A^{\pi} \tag{2.9}$$

and $ind(M) \leq ind(A)$.

Proof. Let $X = (I - K\Gamma^d H)A^d(I - K\Gamma^d H)$. The assumptions $A^{\pi}C = CD^{\pi}$ and $D\Gamma^{\pi} = 0$ imply that $AA^dC = CDD^d$ and $D^d\Gamma^{\pi} = 0$. Firstly, we note the facts:

$$\begin{aligned} MX &= (A - AA^dC\Gamma^dBA^d - CD^dB + CD^dD\Gamma^dBA^d)A^d(I - K\Gamma^dH) \\ &= (A - CD^dB)(A^d - (A^d)^2C\Gamma^dBA^d) \\ &= AA^d - A^dC\Gamma^dBA^d - CD^dBA^d + CD^d\Gamma\Gamma^dBA^d \\ &= AA^d - K\Gamma^dH - CD^d\Gamma^{\pi}BA^d \\ &= AA^d - K\Gamma^dH. \end{aligned} \tag{2.10}$$

Since $DD^dB = B,\ A^dC = A^dCDD^d$ and $D\Gamma^{\pi} = 0$, then

$$\begin{aligned} XM &= (I - K\Gamma^dH)A^d(A - CD^dB - A^dC\Gamma^dBA^dA + A^dC\Gamma^dDD^dB) \\ &= (I - K\Gamma^dH)A^d(A - CD^dB - A^dC\Gamma^dBA^dA + A^dC\Gamma^dB) \\ &= (A^d - A^dC\Gamma^dB(A^d)^2)(A - CD^dB + A^dC\Gamma^dBA^{\pi}) \\ &= A^dA - A^dCD^dB + (A^d)^2C\Gamma^dBA^{\pi} - A^dC\Gamma^dBA^d + A^dC\Gamma^d\Gamma D^dB \\ &\quad - A^dC\Gamma^dB(A^d)^3C\Gamma^dBA^{\pi} \\ &= A^dA - K\Gamma^dH + (I - A^dC\Gamma^dBA^d)(A^d)^2C\Gamma^dBA^{\pi} - A^dCD^dD\Gamma^{\pi}D^dB \\ &= A^dA - K\Gamma^dH + (I - K\Gamma^dH)A^dK\Gamma^dBA^{\pi}. \end{aligned} \tag{2.11}$$

From (2.10), we have

$$\begin{aligned} MM^d &= MX + M(I - K\Gamma^dH)A^d\sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+1}K\Gamma^dBA^iA^{\pi} \\ &= AA^d - K\Gamma^dH + (AA^d - CD^dBA^d)\sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+1}K\Gamma^dBA^iA^{\pi} \\ &= AA^d - K\Gamma^dH + \sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+1}K\Gamma^dBA^iA^{\pi} \end{aligned} \tag{2.12}$$

and using (2.11), we get

$$\begin{aligned} M^dM &= XM + \sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+2}K\Gamma^dBA^iA^{\pi}M \\ &= A^dA - K\Gamma^dH + (I - K\Gamma^dH)A^dK\Gamma^dBA^{\pi} + \sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+2} \\ &\quad \times K\Gamma^dBA^{i+1}A^{\pi} \\ &= A^dA - K\Gamma^dH + \sum_{i=0}^{k-1}[(I - K\Gamma^dH)A^d]^{i+1}K\Gamma^dBA^iA^{\pi}. \end{aligned} \tag{2.13}$$

Thus

$$MM^d = M^dM.$$

From (2.13) and $A^\pi M^d = 0$, we obtain

$$\begin{aligned} M^dMM^d &= (A^dA - K\Gamma^dH + \sum_{i=0}^{k-1}[(I-K\Gamma^dH)A^d]^{i+1}K\Gamma^dBA^iA^\pi)M^d \\ &= (A^dA - K\Gamma^dH)M^d \\ &= M^d. \end{aligned}$$

By using (2.12) and $D^\pi B = 0$, we get

$$M - M^2M^d = AA^\pi - \sum_{i=0}^{k-1}[(I-K\Gamma^dH)A^d]^iK\Gamma^dBA^{i+1}A^\pi.$$

By induction on integer $n \geq 1$, we have

$$(M - M^2M^d)^n = A^nA^\pi - \sum_{i=0}^{k-1}[(I-K\Gamma^dH)A^d]^iK\Gamma^dBA^{i+n}A^\pi.$$

From above expression, it follows that $(M - M^2M^d)^k = 0$, where $k = ind(A)$. Therefore, we obtain that $M^{k+1}M^d = M^k$ and $ind(M) \leq ind(A)$, which completes the proof. $\square$

In similar way, we present another result of this paper.

Theorem 2.8. Let A, B, C, and D be complex matrices, where $ind(A) = k$. If $Z = 0$, $BA^\pi = D^\pi B$, $CD^\pi = 0$ and $\Gamma^\pi D = 0$, then

$$M^d = (I - K\Gamma^dH)A^d(I - K\Gamma^dH) + \sum_{i=0}^{k-1}A^\pi A^iC\Gamma^dH[A^d(I-K\Gamma^dH)]^{i+2}$$

and $ind(M) \leq ind(A)$.

In the end of this section, we give a numerical example to demonstrate our result of a modified matrix. This numerical example describes matrices A, B, C and D which do not satisfy the conditions of [7, Theorem 2.1] nor the conditions of [8, Theorem 1] but they satisfy the conditions of Theorem 2.1. Therefore, we can apply the formula given in Theorem 2.1 to obtain the Drazin inverse of a modified matrix M.

Numerical example 2.9. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $ind(A) = 1$ and $ind(D) = 1$, then we obtain

$$A^d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A^\pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; D^d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; D^\pi = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now we have

$$M = A - CD^dB = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z = D - BA^dC = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}, \; Z^d = \frac{1}{9}\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}, \; Z^\pi = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix}.$$

It can be calculated that:
(i) $CZ^\pi \neq 0$, so the conditions given in [7, Theorem 2.1] are not satisfied.
(ii) $DZ^\pi \neq 0$, thus the conditions given in [8, Theorem 1] are not satisfied.

On the other hand, we can observe that $A^\pi C = CD^\pi$, $CD^dZ^\pi B = 0$, $CZ^dD^\pi B = 0$ and $CZ^\pi D^dB = 0$. Then applying Theorem 2.1, we obtain

$$M^d = \frac{1}{9}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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References

[1] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer, New York, 2003.
[2] S.L. Campbell, C.D. Meyer, Generalized Inverse of Linear Transformations, Dover, New York, 1991.
[3] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Science Press, Beijing/New York, 2004.
[4] J.M. Shoaf, The Drazin inverse of a rank-one modification of a square matrix, Ph.D. Dissertation, North Carolina State University, 1975.
[5] R. Kala, K. Klaczyński, Generalized inverses of a sum of matrices, Sankhya Ser. A 56 (1994) 458-464.
[6] Y. Wei, The Drazin inverse of a modified matrix, Appl. Math. Comput. 125 (2002) 295-301.
[7] E. Dopazo, M.F. Martínez-Serrano, On deriving the Drazin inverse of a modified matrix, Linear Algebra Appl. 438 (2013) 1678-1687.
[8] D. Mosić, Some results on the Drazin inverse of a modified matrix, Calcolo. 50 (2013) 305?11.
[9] A. Shakoor, H. Yang, I. Ali, Some representations for the Drazin inverse of a modified matrix, Calcolo. DOI 10.1007/s10092-013-0098-0 (2013).

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