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An Algorithm for Nonlinear Problems Based on Fixed Point Methodologies With Applications

Zhenhua Ma[1], Hamza Bashir[2], A. A. Alshejari[3,*], Junaid Ahmad[2,*], Muhammad Arshad[2]

[1]*Department of Mathematics and Physics, Hebei University of Architecture, Zhangjiakou 075024, China*

[2]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad - 44000, Pakistan*

[3]*Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P. O. Box 84428, Riyadh 11671, Saudi Arabia*

[*]*Corresponding author:* aaalshejari@pnu.edu.sa, ahmadjunaid436@gmail.com

Abstract. This research presents a highly efficient fixed point algorithm for the computation of fixed points for a very general class of nonexpansive mappings called generalized (α, β)-nonexpansive mappings within the context of uniformly convex Banach space. Our research establishes both weak and strong convergence theorems of the scheme. Furthermore, we demonstrate that the class of generalized (α, β)-nonexpansive mappings contain many classes of nonlinear mappings of the classical literature. Then, we perform various numerical computations to prove the efficiency of the proposed approach. We also study the convergence analysis of the scheme for two dimensional space with taxicab norm. Moreover, we show that our new result gives an alternative approach for solving Caputo fractional differential equation in a novel mappings setting.

1. Introduction

Throughtout the paper Ω denotes Banach space and Ψ is its nonempty closed convex subset. An element $\wp$ of Ψ is called fixed point of the mapping $M : \Psi \to \Psi$ if $M\wp = \wp$. If M posses a fixed point then $F_{ix}(M) = \{\wp \in \Psi : M\wp = \wp\}$ is the set of all fixed point of M. A mapping M is said to be contraction mapping if there exist some $\gamma \in [0, 1)$ such that $\|M\eta - M\mu\| \leq \gamma\|\eta - \mu\|$. The mapping M is called nonexpansive if $\|M\eta - M\mu\| \leq \|\eta - \mu\|$.

Fixed point theory plays an important role in the field of mathematical analysis, providing essential tools for finding the solution of those problems of mathematical sciences for which either

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analytical methods are time consumer or they failed to provide the solution. The search for finding these fixed points encouraged in the development of many mathematical techniques, and among them, iterative algorithms stand out as a versatile and powerful tool. The study of iterative algorithms for finding the fixed points gained fame in the mid$^{20\text{th}}$ century, driven by the need to solve complex mathematical problems in computational and systematic way. Many results in analysis like the Banach contraction mapping theorem and Picard-Lindelof iteration played pivot roles in the establishing the theoretical framework for iterative fixed point algorithms, [1,2]. Though, Picard iterative method was easy to but Krasnoselskii [3] noticed that it may diverged for nonexpansive mappings. The nonexpansive mappings are generalization of contractive mappings, they play a pivotal role in ensuring the existence and convergence of fixed points, making them indispensable in areas like functional analysis, convex optimization and signal processing.

The Banach contraction principle uses the Picard iterative method which is defined as follows:

$$\eta_{s+1} = M\eta_s \text{ for } s \in \mathbb{N}, \tag{1.1}$$

for contraction mappings but in case of nonexpansive mappings, this methods does not coverge to any fixed point in general. In 1953, Mann [4] proposed an iterative method which converges for the class of nonexpansive mappings but it may fails when mappings are pseudo-contractive. In 1974, Ishikawa [5] resolved that problem and proposed a two steps iterative method. Some other examples of commonly used iterative methods, to approximate the fixed points of nonexpansive mappings are by Noor [6], Agarwal [7], Abbas and Nazir [8], Thukar et al. [9], Ullah and Arshad [10], Ullah et al. [11], Saleem et al. [12], Abbas et al. [13], Ahamd et al. [14], JK iteration (see, also [15] and many others) proved the convergence results for Suzuki-type generalized nonexpansive mappings.

Let $\{a_s\}, \{b_s\}$ and $\{c_s\}$ are three sequences of real numbers in $(0,1)$ then Mann [4], Ishikawa [5], Noor [6], Agarwal [7], Abbas and Nazir [8], Thukar [9], Ullah and Arshad [10], Ullah et al. [11] and Abbas et al. [13] iterative methods are respectivley given below:

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)\eta_s + a_s M(\eta_s), \text{ for } s \in \mathbb{N}. \end{cases} \tag{1.2}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)\eta_s + a_s M(\mu_s), \\ \mu_s = (1-b_s)\eta_s + b_s M(\eta_s), \text{ for } s \in \mathbb{N}. \end{cases} \tag{1.3}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)\eta_s + a_s M(\mu_s), \\ \mu_s = (1-b_s)\eta_s + b_s M(\xi_s), \\ \xi_s = (1-c_s)\eta_s + c_s M(\eta_s), \ s \in \mathbb{N}. \end{cases} \tag{1.4}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)M(\eta_s) + a_s M(\mu_s), \\ \mu_s = (1-b_s)\eta_s + b_s M(\eta_s), \ s \in \mathbb{N}. \end{cases} \tag{1.5}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)M(\mu_s) + a_s M(\xi_s), \\ \mu_s = (1-b_s)M(\eta_s) + b_s M(\xi_s), \\ \xi_s = (1-c_s)\eta_s + c_s M(\eta_s), \ s \in \mathbb{N}. \end{cases} \tag{1.6}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)M(\xi_s) + a_s M(\mu_s), \\ \mu_s = (1-b_s)\xi_s + b_s M(\xi_s), \\ \xi_s = (1-c_s)\eta_s + c_s \eta_s, \ s \in \mathbb{N}. \end{cases} \tag{1.7}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = M(\mu_s), \\ \mu_s = M(\xi_s), \\ \xi_s = (1-a_c)\eta_s + a_n M(\eta_s), \ s \in \mathbb{N}. \end{cases} \tag{1.8}$$

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = M((1-a_s)M(\eta_s) + a_s M(\mu_s)), \\ \mu_s = M(\xi_s), \\ \xi_s = M((1-c_s)\eta_s + c_s M(\eta_s)), \ s \in \mathbb{N}. \end{cases} \tag{1.9}$$

Piri et. al. [16] introduced a new faster three-steps iterative process which converges faster than above mentioned. For two sequences of real numbers in $\{a_s\}$ and $\{b_s\}$ in $(0,1)$ then the sequence $\{\eta_s\}$ obtained by Piri et al. [16] is given as:

$$\begin{cases} \eta_1 \in \Psi, \\ \eta_{s+1} = (1-a_s)M(\xi_s) + a_s M(\mu_s)), \\ \mu_s = M(\xi_s), \\ \xi_s = M((1-b_s)\eta_s + b_s M(\eta_s)), \ s \in \mathbb{N}. \end{cases} \tag{1.10}$$

In this research article, we are focusing on the extension of the iterative process (1.10) from the case of generalized α-nonexpansive mappings to generalized (α, β)-nonexpansive mappings. We aim to provide a comprehensive understating of the theoretical foundation of this extension and its practical implications, with a specific focus on its application in solving Delay Caputo Fractional Differential Equation. We will also present the weak and strong convergence results and numerical example to showcase the effectiveness and potential of our proposed approach. By doing so, we aspire to contribute to the dynamic landscape of fixed point theory and provide valuable tools for solving complex mathematical problems in practical setting.

2. Preliminaries

The following definitions, theorems, propositions and lemmas help to prove our main results.

Definition 2.1. *Let* $M : \Psi \to \Psi$ *is a nonexapnsive mapping such that* $F_{ix}(M) \neq \emptyset$ *and* $\|M(\eta) - \wp\| \leq \|\eta - \wp\|, \forall \wp \in F_{ix}(M)$ *is true then* M *is called quasi -nonexpansive mapping.*

In 1965, Krik [17] showed that for a nonempty, bounded, closed and convex subset of a reflexive Banach space the nonexpansive mappings possess a fixed point. In 1965 Dietrich Göhde [18] and Felix E. Browder [19] separately proved the similar result for Uniformly Convex Banach space.

Definition 2.2. *[20, 21] A Banach space* Ω *is said to be uniformly convex Banach space if for every* $\epsilon \in (0, 2]$, *there exist a* $\delta \geq 0$, *such that for any two* $\eta, \mu \in \Omega$ *with* $\|\eta\| \leq 1, \|\mu\| \leq 1$ *and* $\|\eta - \mu\| \geq \epsilon \implies \|\frac{\eta + \mu}{2}\| \leq 1 - \delta.$

Definition 2.3. *[22] A Banach space* Ω *is said to satisfy the Opial's property if every weakly convergent sequence* $\{\eta_s\}$ *of* Ω *with the weak limit* η *and* $\forall \mu \in \Omega - \{\eta\}$ *satisfies the inequality;*

$$\limsup_{s\to\infty} \|\eta_s - \eta\| < \limsup_{s\to\infty} \|\eta_s - \mu\|.$$

The following lemma is famous as the *Characterization of unifrom convexity*

Lemma 2.1. *[23] Assume* Ω *is a uniformly convex Banach space and* $0 < t_s < 1, \forall s \in \mathbb{N}$. *For two sequences* $\{\eta_s\}$ *and* $\{\mu_s\}$ *in* Ω *such that* $\limsup_{s\to\infty} \|\eta_s\| \leq \vartheta$, $\limsup_{s\to\infty} \|\mu_s\| \leq \vartheta$ *and* $\limsup_{s\to\infty} \|t_s\eta_s + (1 - t_s)\mu_s\| = \vartheta$ *for some* $\vartheta \geq 0$ *then* $\lim_{s\to\infty} \|\eta_s - \mu_s\| = 0.$

Definition 2.4. *[24, 25] Let* Ψ *be a nonempty closed convex subset of a Banach space* Ω *and let* $\{\eta_s\}$ *be a bounded sequence in* Ω, *we set* $\gamma(\eta, \{\eta_s\}) = \limsup_{s\to\infty} \|\eta - \eta_s\|.$
The asymptotic radius of $\{\eta_s\}$ *relative to* Ψ *is given as:*
$\gamma(\Psi, \{\eta_s\}) = \inf\{\gamma(\eta, \{\eta_s\}) : \eta \in \Psi\}.$
The asymptotic center of $\{\eta_s\}$ *relative to* Ψ *is defined as:*
$\Lambda(\Psi, \{\eta_s\}) = \{\eta \in \Psi : \gamma(\eta, \{\eta_s\}) = \gamma(\Psi, \{\eta_s\})\}.$
In uniformly convex Banach spaces $\Lambda(\Psi, \{\eta_s\})$ *is nonempty and consist of only one point, when* Ψ *is weekly compact and convex then* $\Lambda(\Psi, \{\eta_s\})$ *is nonempty.*

Definition 2.5. *Let* $\emptyset \neq \Psi \subset \Omega$, *a selfmapping* $M : \Psi \to \Psi$ *is said to be Suzuki generalized nonexpansive mapping if for all* $\eta, \mu \in \Psi$ *such that*
whenever $\frac{1}{2}\|\eta - M(\eta)\| \leq \|\eta - \mu\| \implies \|M(\eta) - M(\mu)\| \leq \|\eta - \mu\|.$

Suzuki generlaized nonexpansive mappings are known as mapping satisfying Condition C. It is obvious that every Suzuki generalized nonexpansive mapping is also a nonexpansive, but Suzuki [26] established an example to show that the class of Suzuki generalized nonexpansive mappings are wider than nonexpansive mappings. He also proved that every Suzuki generalized nonexpansive mapping that possesses a fixed point is quasi-nonexpanive mapping.

Definition 2.6. *Let $\emptyset \neq \Psi$ is closed convex subtset of Banach space Ω. A mapping $M : \Psi \to \Psi$ is said to satisfy Condition I, if for an increasing function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ and $\gamma(\ell) > 0, \forall \ell > 0$, such that*

$$d(\eta, M\eta) \geq \gamma(d(\eta, M\eta)), \forall \eta \in \Psi,$$

where, $d(\eta, M\eta) = \inf_{\wp \in F_{ix}(M)}\{d(\eta, \wp)\}$.

In 2011, Koji Aoyama and Fumiaki Kohsaka [27] opened the new door for researchers by introducing with a new class of mappings known as α-nonexpansive mappings.

Definition 2.7. *Let $\emptyset \neq \Psi \subset \Omega$, a selfmapping $M : \Psi \to \Psi$ is said to be α-nonexpansive mapping if for all $\eta, \mu \in \Psi$ there is some $\alpha \in [0, 1)$, such that*

$$||M\eta - M\mu||^2 \leq \alpha||\eta - M\mu||^2 + \alpha||\mu - M\eta||^2 + (1 - 2\alpha)||\eta - \mu||^2.$$

In 2016, Ariza-Ruiz et al. [28] revealed the facts that for $\alpha < 0$ the concept of α-nonexansive mappings is trivial. It is straight forward that every nonexpansive mapping is 0-nonexpansive mapping and every α-nonexpansive mapping with fixed point is Quasi-nonexpansive. Suzuki generalized nonexpansive and α-nonexpansive mappings are not continuous mappings in general cite [26, 29].

In 2017, Pant and Shukla [29] defined a new class of mappings which contains the mapping satisfying Condition C which is called generalized α-nonexpansive mappings.

Definition 2.8. *Let $\emptyset \neq \Psi \subset \Omega$, a selfmapping $M : \Psi \to \Psi$ is said to be generalized α-nonexpansive mapping if for all $\eta, \mu \in \Psi$ there is some $\alpha \in [0, 1)$, such that whenever;*

$$\frac{1}{2}||\eta - M\eta|| \leq ||\eta - \mu||$$

implies,

$$||M\eta - M\mu|| \leq \alpha||\eta - M\mu|| + \alpha||\mu - M\eta|| + (1 - 2\alpha)||\eta - \mu||.$$

Every Suzuki's generalized nonexpansive mapping is generalized 0-nonexpansive mapping. In [29] they showed with an example that class of generalized α-nonexpansive mappings is bigger than Suzuki's mappings.

In 2019, Pandey et al. [30] proposed a wider class of mappings that properly coantins Suzuki's generalized nonexpansive mappings known as Reich-Suzuki-type-nonexpansive mappings.

Definition 2.9. *Let $\emptyset \neq \Psi \subset \Omega$, a selfmapping $M : \Psi \to \Psi$ is said to be β-Reich-Suzki-Type-nonexpansive mapping if for all $\eta, \mu \in \Psi$ there is some $\beta \in [0, 1)$, such that whenever;*

$$\frac{1}{2}\|\eta - M\eta\| \leq \|\eta - \mu\|$$

implies

$$\|M\eta - M\mu\| \leq \beta\|\eta - M\eta\| + \beta\|\mu - M\mu\| + (1 - 2\beta)\|\eta - \mu\|.$$

It is trivial to show that every Suzuki's nonexpansive mapping is 0-Reich-Suzuki-type-nonexpansive mapping. To show that β-Reich-type-nonexpansive mapping are wider than Suzuki's nonexpansive mapping, one can see [30].

In 2020, Ullah et al. [31] defined a wider class of mappings that properly contains Suzuki's generalized nonexpansive, generalized α-nonexpansive and β-Reich-Suzuki-type-nonexpansive mappings known as generalized (α, β)-nonexpansive mappings.

Definition 2.10. *Let $\emptyset \neq \Psi \subset \Omega$, a selfmapping $M : \Psi \to \Psi$ is said to be generalized (α, β)-nonexpansive mapping if for all $\eta, \mu \in \Psi$ there is some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that whenever*

$$\frac{1}{2}\|\eta - M\eta\| \leq \|\eta - \mu\|$$

implies

$$\|M\eta - M\mu\| \leq \alpha\|\eta - M\mu\| + \alpha\|\mu - M\eta\| + \beta\|\eta - M\eta\| + \beta\|\mu - M\mu\| + (1 - 2\alpha - 2\beta)\|\eta - \beta\|.$$

Proposition 2.1 provide many examples of generalized (α, β)-nonexpanisve mappings.

Proposition 2.1. *[31] Let $\emptyset \neq \Psi \subset \Omega$ then for a selfmapping $M : \Psi \to \Psi$, we have*

- *Every mapping with Condition C is generalized $(0, 0)$-nonexpansive mapping.*
- *Every generalized α-nonexpansive mapping is generalized $(\alpha,0)$-nonexpansive mapping.*
- *Every β-Reich-Suzuki-nonexpansive mapping is generalized $(0,\beta)$-nonexpansive mapping.*

Ullah et al. [31] and Ahmad et al. [32] provided some examples that the converse of Proposition 2.1 is not true.

Lemma 2.2. *[31] Let $\emptyset \neq \Psi \subset \Omega$ and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping with a fixed point $\wp$. Then, M is quasi-nonexpansive mapping.*

Lemma 2.3. *[31] Let $\emptyset \neq \Psi \subset \Omega$ and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping then $F_{ix}(M)$ is closed. Moreover, $F_{ix}(M)$ is convex if Ψ is strictly convex and Ω is convex.*

Lemma 2.4. *Let $\emptyset \neq \Psi \subset \Omega$ and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping then for all $\eta, \mu \in \Psi$ the following inequality holds,*

$$\|\eta - M\mu\| \leq \left(\frac{3 + \alpha + \beta}{1 - \alpha - \beta}\right)\|\eta - M\eta\| + \|\eta - \mu\|.$$

Theorem 2.1. *Let Ψ be a weakly compact convex subset of a uniformly convex Banach space and $M : \Psi \to \Psi$ be a mapping with Condition C then M has a fixed point.*

Theorem 2.2. *Let $\emptyset \neq \Psi$ is closed subset of Ω with Opial's property and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping. If $\{\eta_s\}$ converges weakly to a point τ and $\lim_{s\to\infty} \|M\eta_s - \eta_s\| = 0$ then, $M\tau = \tau$, that is, $(I - M)$ is demiclosed at zero, where I is the identity mapping on Ψ.*

3. Convergence Results

In this section, we prove the week and strong convergence theorems for the class of generalized (α, β)-nonexpansive mappings under our iteration process (1.10).

We now establish our key lemma as follows.

Lemma 3.1. *Let $\emptyset \neq \Psi$ is closed convex subset of Ω and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping with $F_{ix}(M) \neq \emptyset$. Let $\{\eta_s\}$ is a sequence generated by algorithm (1.10), then $\lim_{a\to\infty} \|\eta_s - \wp\|$ exits for all $\wp \in F_{ix}(M)$.*

Proof. Let $\eta \in \Psi$ and $\wp \in F_{ix}(M)$. By Lemma 2.2, M is Qusai-nonexpansive mapping,

$$\|M\eta - \wp\| \leq \|\eta - \wp\|$$

By using (1.10), we have

$$\begin{aligned} \|\xi_s - \wp\| &\leq \|M((1 - b_s)\eta_s + b_s\eta_s) - \wp\| \\ &\leq \|(1 - b_s)\eta_s + b_s\eta_s - \wp\| \\ &\leq (1 - b_s)\|\eta_s - \wp\| + b_s\|M\eta_s - \wp\|. \end{aligned} \tag{3.1}$$

As $\wp \in F_{ix}(M) \implies M\wp = \wp$ and M is generalized (α, β)-nonexpansive mapping, we have

$$\begin{aligned} \|M\eta_s - \wp\| &\leq \|M\eta_s - M\wp\| \\ &\leq \alpha\|\eta_s - M\wp\| + \alpha\|\wp - M\eta_s\| + \beta\|\eta_s - M\eta_s\| + \beta\|\wp - M\wp\| \\ &\quad +(1 - 2\alpha - 2\beta)\|\eta_s - \wp\| \\ &\leq \alpha\|\eta_s - \wp\| + \alpha\|M\eta_s - \wp\| + \beta\|M\eta_s - \wp\| + \beta\|\eta_s - \wp\| \\ &\quad +\beta\|\wp - \wp\| + (1 - 2\alpha - 2\beta)\|\eta_s - \wp\| \\ &\leq \alpha\|M\eta_s - \wp\| + \beta\|M\eta_s - \wp\| + (1 - \alpha - \beta)\|\eta_s - \wp\| \\ &\leq \|\eta_s - \wp\|. \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1), we have

$$\begin{aligned} \|\xi_s - \wp\| &\leq (1 - b_s)\|\eta_s - \wp\| + b_s\|\eta_s - \wp\| \\ &\leq \|\eta_s - \wp\|. \end{aligned} \tag{3.3}$$

Now

$$\|\mu_s - \wp\| \leq \|M\xi_s - \wp\| \leq \|\xi_s - \wp\|$$

by (3.3)

$$\|\mu_s - \wp\| \leq \|\eta_s - \wp\|. \tag{3.4}$$

It follows from (1.10), (3.3) and (3.4)

$$\begin{aligned} \|\eta_{s+1}-\wp\| &\leq \|(1-a_s)M\xi_s + a_sM\mu_s - \wp\| \\ &\leq (1-a_s)\|\xi_s-\wp\| + a_s\|\mu-\wp\| \\ &\leq \|\eta_s-\wp\|. \end{aligned} \tag{3.5}$$

Consequently, for each $\wp \in F_{ix}(M)$ the sequence $\{\|\eta_{s+1}-\wp\|\}$ is bounded and decreasing. It follows that $\lim_{s\to\infty}\|\eta_{s+1}-\wp\|$ exists for each $\wp \in F_{ix}(M)$. □

For generalized (α,β)-nonexpansive mapping on closed convex subset of a Banach space, we will prove the necessary and sufficient condition for the existence of fixed point in next theorem.

Theorem 3.1. *Let $\emptyset \neq \Psi$ is closed convex subset of Banach space Ω and $M : \Psi \to \Psi$ is generalized (α,β)-nonexpansive mapping. Let $\{\eta_s\}$ is a sequence generated by algorithm (1.10) then $F_{ix}(M) \neq \emptyset$ if and only if $\{\eta_s\}$ is bounded and $\lim_{s\to\infty}\|M\eta_s-\eta_s\|$.*

Proof. Let $F_{ix}(M) \neq \emptyset$ and $\wp \in F_{ix}(M)$ then, by Lemma 3.1 $\lim_{s\to\infty}\|\eta_s-\wp\|$ exists for each $\wp \in F_{ix}(M)$ and $\{\eta_s\}$ is bounded. Put

$$\lim_{s\to\infty}\|\eta_s-\wp\| = \varkappa. \tag{3.6}$$

By using (3.4), we have

$$\limsup_{s\to\infty}\|\mu_s-\wp\| \leq \limsup_{s\to\infty}\|\eta_s-\wp\| \leq \varkappa. \tag{3.7}$$

Using Lemma 2.2, we obtained

$$\limsup_{s\to\infty}\|M\eta_s-\wp\| \leq \limsup_{s\to\infty}\|\eta_s-\wp\| \leq \varkappa. \tag{3.8}$$

By using (3.3), we have $\|\xi_s-\wp\| \leq \|\eta_s-\wp\|$. Therefore,

$$\begin{aligned} \|\eta_{s+1}-\wp\| &= \|(1-a_s)M\xi_s + a_sM\mu_s - \wp\| \\ &\leq (1-a_s)\|\xi_s-\wp\| + a_s\|\mu_s-\wp\| \\ &\leq (1-a_s)\|\eta_s-\wp\| + a_s\|\mu_s-\wp\|. \end{aligned} \tag{3.9}$$

It follows that

$$\|\eta_{s+1}-\wp\| - \|\eta_s-\wp\| \leq \frac{\|\eta_{s+1}-\wp\| - \|\eta_s-\wp\|}{a_s} \leq \|\mu_s-\wp\| - \|\eta_s-\wp\|.$$

So, we have

$$\|\eta_{s+1}-\wp\| \leq \|\mu_s-\wp\|.$$

Now, from (3.6), we got

$$\varkappa \leq \liminf_{s\to\infty}\|\mu_s-\wp\|. \tag{3.10}$$

Thus, we obtained by (3.7) and (3.10)

$$\varkappa = \lim_{s\to\infty}\|\mu_s-\wp\|.$$

Therefore, from (3.6), we have

$$\begin{aligned}
\varkappa &= \lim_{s\to\infty}\|\mu_s-\wp\| = \lim_{s\to\infty}\|M\xi_s-\wp\| \\
&= \lim_{s\to\infty}\|M(M((1-b_s)\eta_s+b_sM\eta_s))-\wp\| \\
&\leq \lim_{s\to\infty}\|M((1-b_s)\eta_s+b_sM\eta_s)-\wp\| \\
&\leq \lim_{s\to\infty}\|(1-b_s)\eta_s+b_sM\eta_s-\wp\| \\
&= \lim_{s\to\infty}\|(1-b_s)(\eta_s-\wp)+b_s(M\eta_s-\wp)\| \\
&\leq \lim_{s\to\infty}(1-b_s)\|\eta_s-\wp\|+\lim_{s\to\infty}b_s\|M\eta_s-\wp\| \\
&\leq \varkappa. \qquad (3.11)
\end{aligned}$$

Hence,

$$\lim_{s\to\infty}\|(1-b_s)(\eta_s-\wp)+b_s(M\eta_s-\wp)\| = \varkappa. \qquad (3.12)$$

Using (3.7), (3.8), (3.12), and Lemma 2.1, we concluded that $\lim_{s\to\infty}\|M\eta_s-\eta_s\| = 0$. Now suppose conversely that $\{\eta_s\}$ is bounded and $\|M\eta_s-\eta_s\| = 0$.

Let $\wp \in \Lambda(\Psi,\{\eta_s\})$. By Lemma 2.4, we have

$$\begin{aligned}
\gamma(M\wp,\{\eta_s\}) &= \limsup_{s\to\infty}\|\eta_s-M\wp\| \\
&\leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\limsup_{s\to\infty}\|M\eta_s-\eta_s\|+\limsup_{s\to\infty}\|\eta_s-\wp\| \\
&= \limsup_{s\to\infty}\|\eta_s-\wp\| \\
&= \gamma(\wp,\{\eta_s\}).
\end{aligned}$$

Hence, we have $M\wp \in \Lambda(\Psi,\{\eta_s\})$. As Ω is uniformly convex, $\Lambda(\Psi,\{\eta_s\})$ is singleton set. It follows that $M\wp = \wp$. □

Theorem 3.2. *Let $\emptyset \neq \Psi$ is a closed convex subset of a uniformly convex Banach space Ω, Ω is uniformly convex and $M : \Psi \to \Psi$ with Opial's property is generalized (α,β)-nonexpansive mapping. Let $\{\eta_s\}$ is generated by algorithm (1.10) and $F_{ix} \neq \emptyset$. Then, $\{\eta_s\}$ converges weakly to the fixed point of M.*

Proof. Suppose $\wp \in F_{ix}(M)$. Then, by Theorem 3.1 the sequence $\{\eta_s\}$ is bounded and $\lim_{s\to\infty}\|M\eta_s-\eta_s\| = 0$. Since Ω is uniformly convex, Ω is reflexive. So, there exists a subsequence $\{\eta_{s_i}\}$ of $\{\eta_s\}$ such that $\{\eta_{s_i}\}$ converges weakly to some $\tau_1 \in \Psi$. By Lemma 2.2 $(I-M)\tau_1 = 0 \implies M\tau_1 = \tau_1$. Now it is sufficient to show that the sequence $\{\eta_s\}$ converges weakly to τ_1. Suppose on contrary, the sequence $\{\eta_s\}$ does not converges weakly to τ_1. Then, there exists a subsequence $\{\eta_{s_k}\}$ of $\{\eta_s\}$ and $\tau_2 \in \Psi$, such that $\{\eta_{s_k}\}$ converges weakly to τ_2 and $\tau_1 \neq \tau_2$. Again, by Lemma 2.2, $(I-M)\tau_2 = 0 \implies M\tau_2 = \tau_2$. By Lemma 3.1, $\lim_{s\to\infty}\|\eta_s\|$ exists for all $\wp \in F_{ix}(M)$. Now to prove $\tau_1 = \tau_2$, by Opial's property, we have

$$\begin{aligned}\lim_{s\to\infty}\|\eta_s-\tau_1\| &= \lim_{s_i\to\infty}\|\eta_{s_i}-\tau_1\|\\ &< \lim_{s_i\to\infty}\|\eta_{s_i}-\tau_2\|\\ &= \lim_{s\to\infty}\|\eta_s-\tau_2\|\\ &= \lim_{s_k\to\infty}\|\eta_{s_k}-\tau_2\|\\ &< \lim_{s_k\to\infty}\|\eta_{s_k}-\tau_1\|\\ &= \lim_{s\to\infty}\|\eta_s-\tau_1\|. \qquad (3.13)\end{aligned}$$

This is a contradiction. So, we have $\tau_1=\tau_2$. Thus, η_s converges weakly to $\tau_1\in F_{ix}(M)$. □

In the next theorem, we will prove necessary and sufficient condition for the convergence to fixed point.

Theorem 3.3. *Let $\emptyset\neq\Psi$ is a subset of uniformly convex convex Banach space Ω and $M:\Psi\to\Psi$ is generalized (α,β) nonexpansive mapping. Let $\{\eta_s\}$ is generated by algorithm (1.10) and $F_{ix}(M)\neq\emptyset$ then $\{\eta_s\}$ converges to fixed point of M if and only if $\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=0$. Where, $\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=\inf_{\wp\in F_{ix}(M)}\{\|\eta_s-\wp\|\}$.*

Proof. Suppose that $\{\eta_s\}$ converges to the fixed point of M that is for $\wp\in F_{ix}(M)$, $\{\eta_s\}\to\wp$ as $s\to\infty$. Then,

$$\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=0.$$

Suppose conversely, $\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=0$. By Lemma 3.1, $\lim_{s\to\infty}\|\eta_s-\wp\|$ exists for all $\wp\in F_{ix}(M)$. Therefore, $\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=0$. So, for given $\epsilon>0$ there exists $s_0\in\mathbb{N}$ such that for all $s\geq s_0$, $d(\eta_s,F_{ix}(M))<\frac{\epsilon}{2} \implies \inf_{\wp\in F_{ix}(M)}\{\|\eta_s-\wp\|\}<\frac{\epsilon}{2}$.

Now for $s,t\geq 0$, we have

$$\begin{aligned}\|\eta_{s+t}-\eta_s\| &\leq \|\eta_{s+t}-\wp\|+\|\eta_s-\wp\|\\ &\leq \|\eta_{s_0}-\wp\|+\|\eta_{s_0}-\wp\|\\ &= 2\|\eta_{s_0}-\wp\|\\ &< \epsilon.\end{aligned}$$

Hence, we concluded that the sequence $\{x_s\}$ is a Cauchy sequence in Ψ. As Ψ is closed subset of a Banach space Ω, there is a point $\tau\in\Psi$ such that $\lim_{s\to\infty}=\tau$. Now $\liminf_{s\to\infty} d(\eta_s,F_{ix}(M))=0$ gives that $d(\eta_s,F_{ix}(M))=0$. Hence, $\tau\in F_{ix}(M)$. □

In the next theorem, we will prove strong convergence to fixed point.

Theorem 3.4. *Let $\emptyset\neq\Psi$ be a compact convex subset of uniformly convex Banch space Ω and $M:\Psi\to\Psi$ is generalized (α,β)-nonexpansive mapping. Let $\{\eta_s\}$ is generated by algorithm (1.10). Then, the sequence $\{\eta_s\}$ converges stronlgy to a fixed point of M.*

Proof. From Theorem 2.1, we have $F_{ix}(M) \neq \emptyset$. Then, by Theorem 3.1, we have

$$\lim_{s\to\infty} \|M\eta_s - \eta_s\| = 0.$$

As Ψ is compact. So, there is a subsequence $\{\eta_{s_i}\}$ of $\{\eta_s\}$ that converges to some $\tau \in \Psi$. Then, by Lemma 2.4, we have

$$\|\eta_{s_i} - M\tau\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\|\eta_{s_i} - M\eta_{s_i}\| + \|\eta_{s_i} - \tau\| \quad \forall \geq 1.$$

By applying limit, we obtained $\eta_{s_i} \to M\tau$ as $i \to \infty$. This shows that $\tau \in F_{ix}(M)$. In addition, by Lemma 3.1 $\lim_{s\to\infty} \|\eta_s - \tau\|$ exists. So the sequence $\{\eta_s\}$ converges strongly to τ. □

Now, by using Condition I we shall prove the strong convergence theorem.

Theorem 3.5. *Let $\emptyset \neq \Psi$ be a closed convex subset of uniformly convex Banch space Ω and $M : \Psi \to \Psi$ is generalized (α, β)-nonexpansive mapping satisfying Condition I. Let $\{\eta_s\}$ is generated by algorithm (1.10) and $F_{ix}(M) \neq \emptyset$. Then, the sequence $\{\eta_s\}$ converges stronlgy to a fixed point of M.*

Proof. As proven in Theorem 3.1,

$$\lim_{s\to\infty} \|M\eta_s - \eta_s\| = 0.$$

using Condition I and (3.10), we have

$$0 \leq \lim_{s\to\infty} \gamma(d(\eta_s, F_{ix}(M))) \leq \lim_{s\to\infty} \|M\eta_s - \eta_s\| = 0.$$

which implies

$$\lim_{s\to\infty} \gamma(d(\eta_s, F_{ix}(M))) = 0.$$

Since, $\gamma : [0,\infty) \to [0,\infty)$ is an increaing function with $\gamma(0) = 0, \gamma(\ell) > 0, \forall \ell > 0$. From this, we have

$$\lim_{s\to\infty} (d(\eta_s, F_{ix}(M))) = 0.$$

Now, all conditions of Theorem 3.3 are satisfied. Consequently, the sequence $\{\eta_s\}$ converges strongly to the fixed point of M. □

4. Examples and Comperative Analysis

To comprehensively assess the significance of the iterative scheme (1.10) in approximating fixed points of generalized (α, β)-nonexpansive mappings against alternative iterative methods. We have chosen the following examples example to illuminate its efficacy.

Example 4.1. *Consider the mapping $M : [0,\infty) \to [0,\infty)$ defined by*

$$M\eta = \begin{cases} 7, & \eta \in [0,2), \\ \dfrac{10\eta + 11}{11}, & \eta \in [2,\infty). \end{cases}$$

Here, M does not satisfy Condition C. However, M is generalized (α,β)-nonexpansive mapping. Let $\eta = \frac{7}{3}$ and $\mu = \frac{17}{6}$ then $M\eta = \frac{103}{33}$. So,

$$\frac{1}{2}\left|\eta - M\eta\right| = \frac{1}{2}\left|\frac{7}{3} - \frac{103}{33}\right| = \frac{1}{2}\left|\frac{26}{33}\right| = \frac{13}{33}.$$

And, $$|\eta-\mu| = \left|\frac{7}{3} - \frac{17}{6}\right| = \frac{1}{2} \implies \frac{1}{2}\left|\eta - M\eta\right| \leq |\eta-\mu|.$$

However, $$|M\eta - M\mu| = \left|\frac{103}{33} - 11\right| = \frac{260}{33},$$

$\implies$ $$|M\eta - M\mu| \geq |\eta-\mu|.$$

Hence M does not satisfy Condition C.

Now take $\alpha = \frac{10}{21}$ and $\beta = \frac{1}{42}$. Clearly $\alpha+\beta = \frac{1}{2} < 1$, the the following cases arise.

Case 1: If $\eta,\mu \in [0,2)$, then we have

$$\frac{10}{21}|\eta - M\mu| + \frac{10}{21}|\mu - M\eta| + \frac{1}{42}|\eta - M\eta| + \frac{1}{42}|\mu - M\mu)| \geq 0 \geq |M\eta - M\mu|.$$

Case 2: If $\mu \in [0,2)$, and $\eta \in [2,\infty)$, then we have

$$\begin{aligned}
&\frac{10}{21}|\eta - M\mu| + \frac{10}{21}|\mu - M\eta| + \frac{1}{42}|\eta - M\eta| + \frac{1}{42}|\mu - M\mu| \\
=\ & \frac{10}{21}\left|\eta - 11\right| + \frac{10}{21}\left|\mu - \frac{10\eta+11}{11}\right| + \frac{1}{42}\left|\eta - \frac{10\eta+11}{11}\right| + \frac{1}{42}|\mu + 11| \\
=\ & \frac{10}{21}\left|\eta - 11\right| + \frac{10}{21}\left|\mu - \frac{10\eta+11}{11}\right| + \frac{1}{42}\left|\frac{\eta-11}{11}\right| + \frac{1}{42}|\mu - 11| \\
\geq\ & \frac{10}{11}\left|\eta - 11\right| \\
=\ & |M\eta - M\mu|.
\end{aligned}$$

Case 3: If $\eta,\mu \in [0,2)$, then we have

$$\begin{aligned}
&\frac{10}{21}|\eta - M\mu| + \frac{10}{21}|\mu - M\eta| + \frac{1}{42}|\eta - M\eta| + \frac{1}{42}|\mu - M\mu| \\
=\ & \frac{10}{21}\left|\eta - \frac{10\mu+11}{11}\right| + \frac{10}{21}\left|\mu - \frac{10\eta+11}{11}\right| + \frac{1}{42}\left|\eta - \frac{10\eta+11}{11}\right| + \frac{1}{42}\left|\mu - \frac{10\mu+11}{11}\right| \\
=\ & \frac{10}{21}\left|\frac{11\eta - 10\mu - 11}{11}\right| + \frac{10}{21}\left|\frac{11\mu - 10\eta - 11}{11}\right| + \frac{1}{42}\left|\frac{\eta-11}{11}\right| + \frac{1}{42}\left|\frac{\mu-11}{11}\right| \\
\geq\ & \frac{20}{42}\left|\frac{21\eta - 21\mu}{11}\right| + \frac{1}{42}|\frac{\eta-\mu}{11}| \\
\geq\ & \frac{421}{42}\left|\frac{\eta-\mu}{11}\right| \\
\geq\ & \frac{10}{11}|\eta-\mu| = |M\eta - M\mu|.
\end{aligned}$$

Hence, M is generalized $\left(\frac{10}{21}, \frac{1}{42}\right)$-nonexpansive mapping.

Now to establish the fact that the iterative scheme (1.10) is faster than that of Mann iteration (1.2), S iteration (1.5), Noor iteration (1.4), Abbas and Nazir iteration (1.6), and Thakur iteration (1.7). Now for the initial guess $\{\eta_s\} = 16.35312$ by asuming $\{a_s\} = 0.56$, $\{b_s\} = 0.87$ and $\{c_s\} = 0.29$ the comparision are in the Table 1 and Figure 1.

Table 1. Convergence comparison of different schemes with Piri iterative scheme.

s	Piri	Thakur	Abbas	S	Noor	Mann
1	16.35312	16.35312	16.35312	16.35312	16.35312	16.35312
2	14.86675	15.22812	15.50545	15.65093	15.80823	16.08060
3	13.79309	14.33955	14.79201	15.04085	15.31881	15.82195
4	13.01755	13.63772	14.19154	14.51080	14.87920	15.57647
5	12.45735	13.08338	13.68616	14.05028	14.48434	15.34348
6	12.05270	12.64554	13.26080	13.65016	14.12968	15.12236
7	11.76040	12.29972	12.90280	13.30253	13.81111	14.91250
8	11.54926	12.02657	12.60149	13.0005	13.52497	14.71331
9	11.39675	11.81083	12.34790	12.73809	13.26796	14.52427
10	11.28659	11.64043	12.13446	12.51009	13.03711	14.34486
11	11.20701	11.50583	11.95481	12.31201	12.82975	14.17457
12	11.14953	11.39953	11.80362	12.13991	12.6435	14.01296
13	11.10801	11.31556	11.67636	11.99038	12.47621	13.85957
14	11.07802	11.24925	11.56926	11.86047	12.32595	13.71399
15	11.05636	11.19687	11.47912	11.74760	12.19098	13.57583
16	11.04071	11.15549	11.40325	11.64953	12.06976	13.44469
17	11.02941	11.12282	11.33940	11.56433	11.96087	13.32024
18	11.02124	11.09700	11.28565	11.49031	11.86306	13.20211
19	11.01534	11.07662	11.24042	11.42599	11.77521	13.09001
20	11.01108	11.06052	11.20235	11.37011	11.69630	12.98361
21	11.00801	11.04780	11.17031	11.32156	11.62543	12.88262
22	11.00578	11.03775	11.14334	11.27938	11.56177	12.78678
23	11.00418	11.02982	11.12064	11.24274	11.50458	12.69582
24	11.00302	11.02355	11.10154	11.21090	11.45322	12.60948
25	11.00218	11.01860	11.08546	11.18323	11.40709	12.44978
26	11.00157	11.01469	11.07193	11.15920	11.36565	12.37597
27	11.00114	11.01161	11.06054	11.13831	11.32843	12.30592
28	11.00082	11.00917	11.05095	11.12017	11.29500	12.23944
29	11.00059	11.00724	11.04288	11.10441	11.26498	12.17634
30	11.00042	11.00572	11.03609	11.09071	11.23800	12.11646

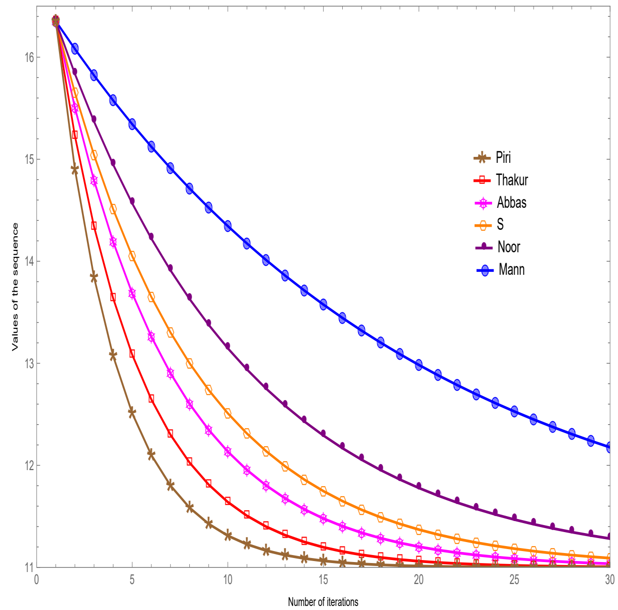

FIGURE 1. Behaviors of various iterative processes using Example 4.1.

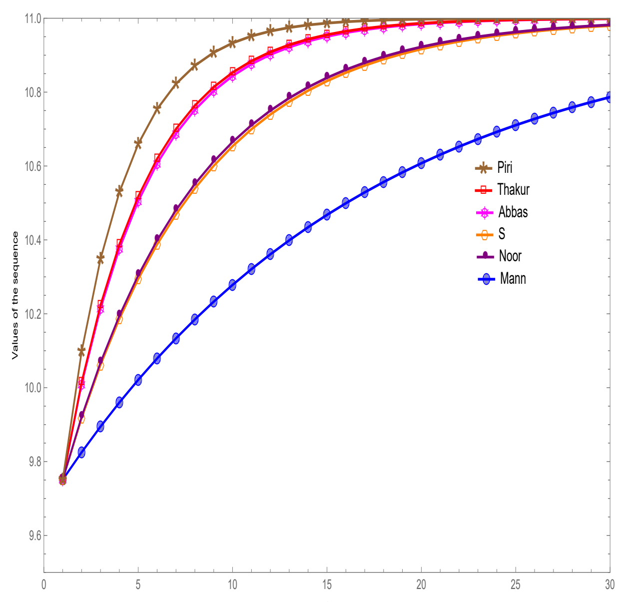

FIGURE 2. Behaviors of various iterative processes using Example 4.1.

Now for the initial guess $\{\eta_s\} = 9.75123$ by asuming $\{a_s\} = 0.65$, $\{b_s\} = 0.78$ and $\{c_s\} = 0.92$ the comparision are in the table 2 and figure 2.

TABLE 2. Convergence comparison of different schemes with Piri iterative scheme.

s	Piri	Thakur	Abbas	S	Noor	Mann
1	9.751230	9.75123	9.751230	9.751230	9.751230	9.751230
2	10.09780	10.01553	10.00883	9.917079	9.921108	9.825021
3	10.34819	10.22389	10.21329	10.06090	10.06788	9.894452
4	10.52908	10.38815	10.37558	10.18562	10.19468	9.959779
5	10.65978	10.51764	10.50439	10.29378	10.30423	10.02125
6	10.75420	10.61973	10.60662	10.38757	10.39888	10.07908
7	10.82242	10.70021	10.68777	10.46891	10.48065	10.13350
8	10.87170	10.76366	10.75218	10.53944	10.55130	10.18470
9	10.90731	10.81368	10.8033	10.60061	10.61234	10.23288
10	10.93303	10.85312	10.84388	10.65365	10.66508	10.27821
11	10.95162	10.88420	10.87608	10.69965	10.71064	10.32086
12	10.96505	10.90871	10.90164	10.73954	10.75000	10.36099
13	10.97475	10.92803	10.92193	10.77413	10.78401	10.39875
14	10.98175	10.94326	10.93804	10.80413	10.81339	10.43428
15	10.98682	10.95527	10.95082	10.83014	10.83878	10.46771
16	10.99048	10.96474	10.96096	10.85270	10.86071	10.49916
17	10.99312	10.97220	10.96902	10.87226	10.87966	10.52876
18	10.99503	10.97808	10.97541	10.88923	10.89603	10.55660
19	10.99641	10.98272	10.98048	10.90394	10.91017	10.58280
20	10.99741	10.98638	10.98451	10.91670	10.92239	10.60746
21	10.99813	10.98926	10.98770	10.92776	10.93295	10.63065
22	10.99865	10.99153	10.99024	10.93736	10.94207	10.65248
23	10.99903	10.99333	10.99225	10.94568	10.94995	10.67301
24	10.99929	10.99474	10.99385	10.95289	10.95676	10.69233
25	10.99949	10.99585	10.99512	10.95915	10.96264	10.71051
26	10.99963	10.99673	10.99613	10.96457	10.96772	10.72762
27	10.99973	10.99742	10.99693	10.96928	10.97212	10.74372
28	10.99981	10.99797	10.99756	10.97336	10.97591	10.75886
29	10.99996	10.99840	10.99806	10.97690	10.97919	10.77311
30	11.00000	10.99874	10.99846	10.97996	10.98202	10.78652

Example 4.2. *Let $\Psi = [0,2]$ with taxicab norm. Consider a mapping $M : \Psi \times \Psi \to \Psi \times \Psi$ defined by $M(\eta,\mu) = \left(\frac{\eta}{2}, \frac{\mu+1}{2}\right)$, for any $(\eta,\mu) \in \Psi \times \Psi$. Here M is generalized (α,β)-nonexpansive mapping. For (η_1,μ_1) and (η_2,μ_2) in $\Psi \times \Psi$, whenever $\frac{1}{2}\|(\eta_1,\mu_1) - M(\eta_1,\mu_1)\| \leq \|(\eta_1,\mu_1) - (\eta_2,\mu_2)\|$. For $\alpha = \frac{1}{4}$*

and $\beta = \frac{1}{4}$, *we have*

$$\frac{1}{4}\|(\eta_1,\mu_1) - M((\eta_2,\mu_2))\| + \frac{1}{4}\|(\eta_2,\mu_2) - M((\eta_1,\mu_1))\| + \frac{1}{4}\|(\eta_1,\mu_1) - \eta((\eta_1,\mu_1))\| + \frac{1}{4}\|(\eta_2,\mu_2) - M((\eta_2,\mu_2))\|$$
$$= \frac{1}{4}\|(\eta_1,\mu_1) - (\frac{\eta_2}{2},\frac{\mu_2+1}{2})\| + \frac{1}{4}\|(\eta_2,\mu_2) - (\frac{\eta_1}{2},\frac{\mu_1+1}{2})\| + \frac{1}{4}\|(\eta_1,\mu_1) - (\frac{\eta_1}{2},\frac{\mu_1+1}{2})\| + \frac{1}{4}\|(\eta_2,\mu_2) - (\frac{\eta_2}{2},\frac{\mu_2+1}{2})\|$$
$$= \frac{1}{4}\|(\frac{2\eta_1-\eta_2}{2},\frac{2\mu_1-\mu_2-1}{2})\| + \frac{1}{4}\|(\frac{2\eta_2-\eta_1}{2},\frac{2\mu_2-\mu_1-1}{2})\| + \frac{1}{4}\|(\frac{\eta_1}{2},\frac{\mu_1-1}{2})\| + \frac{1}{4}\|(\frac{\eta_2}{2},\frac{\mu_2-1}{2})\|$$
$$\geq \frac{1}{4}\left\{\|(\frac{3\eta_1-3\eta_2}{2},\frac{3\mu_1-3\mu_2}{2})\| + \|(\frac{\eta_1-\eta_2}{2},\frac{\mu_1-\mu_2}{2})\|)\right\}$$
$$\geq \frac{1}{4}\left\{\|(\frac{4\eta_1-4\eta_2}{2},\frac{4\mu_1-4\mu_2}{2})\|\right\}$$
$$= \frac{1}{4}\left\{|\frac{4\eta_1-4\eta_2}{2}| + |\frac{4\mu_1-4\mu_2}{2})|\right\}$$
$$= |\frac{\eta_1-\eta_2}{2}| + |\frac{\mu_1-\mu_2}{2}|$$
$$= \|(\frac{\eta_1-\eta_2}{2},\frac{\mu_1-\mu_2}{2})\|$$
$$= \|M(\eta_1,\mu_1) - M(\eta_2,\mu_2)\|.$$

Now, we will draw graphs and tables to show that the sequence $\{\eta_s\}$ of the Piri iterative scheme (1.10) moves faster to the fixed point of example 4.2 as compared to Mann iteration (1.2), Ishikawa iteration (1.3), Noor (1.4) and M-iteration (1.8). By assuming $\{a_s\} = 0.59$, $\{b_s\} = 0.48$ and $\{c_s\} = 0.39$ and by taking the initial guess $(1.5234, 1.8987)$ the observations are provided in Table 3 and Figure 3, which show that Piri iterative scheme (1.10) is faster than above mentioned.

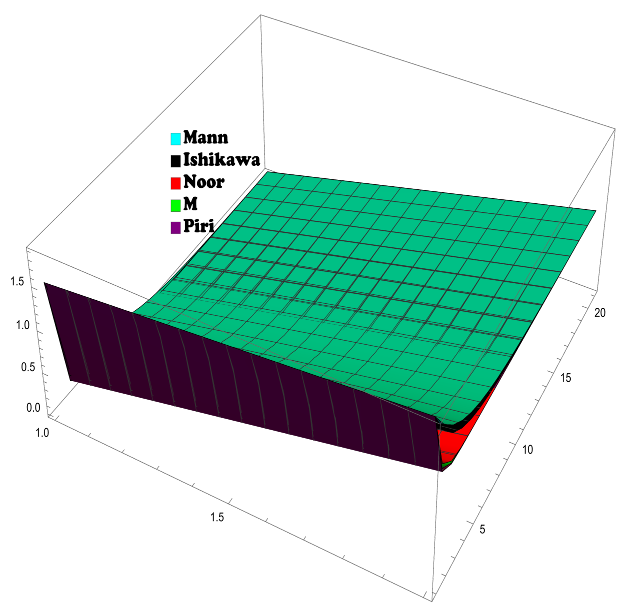

Figure 3. Behaviors of various iterative processes using Example 4.2.

TABLE 3. Convergence comparison of different schemes with Piri iterative scheme.

s	Piri	M	Noor	Ishikawa	Mann
1	(1.5234,1.8987)	(1.5234,1.8987)	(1.5234,1.8987)	(1.5234,1.8987)	(1.5234,1.8987)
2	(0.2041,1.1204)	(0.2685,1.1584)	(0.9451,1.5580)	(0.9661,1.5700)	(1.0740,1.6335)
3	(0.0273,1.0161)	(0.0473,1.0279)	(0.5863,1.3459)	(0.6127,1.3614)	(0.7571,1.4467)
4	(0.0037,1.0022)	(0.0083,1.0049)	(0.3638,1.2146)	(0.3880,1.2292)	(0.5338,1.3149)
5	(0.0005,1.0002)	(0.0015,1.0009)	(0.2257,1.1331)	(0.2464,1.1454)	(0.3763,1.2220)
6	(0.0000,1.0000)	(0.0003,1.0002)	(0.1400,1.0826)	(0.1563,1.0922)	(0.2653,1.1565)
7	(0.0000,1.0000)	(0.0000,1.0000)	(0.0869,1.0512)	(0.0991,1.0585)	(0.1870,1.1103)
8	(0.0000,1.0000)	(0.0000,1.0000)	(0.0539,1.0318)	(0.0629,1.0371)	(0.1319,1.0778)
9	(0.0000,1.0000)	(0.0000,1.0000)	(0.0334,1.0197)	(0.0399,1.0235)	(0.0930,1.0548)
10	(0.0000,1.0000)	(0.0000,1.0000)	(0.0207,1.0122)	(0.0253,1.0149)	(0.0655,1.0387)
11	(0.0000,1.0000)	(0.0000,1.0000)	(0.0129,1.0076)	(0.0160,1.0095)	(0.0462,1.0273)
12	(0.0000,1.0000)	(0.0000,1.0000)	(0.0080,1.0047)	(0.0102,1.0060)	(0.0326,1.0192)
13	(0.0000,1.0000)	(0.0000,1.0000)	(0.0050,1.0030)	(0.0064,1.0040)	(0.0230,1.0135)
14	(0.0000,1.0000)	(0.0000,1.0000)	(0.0031,1.0020)	(0.0041,1.0024)	(0.0162,1.0096)
15	(0.0000,1.0000)	(0.0000,1.0000)	(0.0019,1.0011)	(0.0026,1.0015)	(0.0114,1.0067)
16	(0.0000,1.0000)	(0.0000,1.0000)	(0.0012,1.0007)	(0.0016,1.0010)	(0.0080,1.0047)
17	(0.0000,1.0000)	(0.0000,1.0000)	(0.0007,1.0004)	(0.0010,1.0006)	(0.0057,1.0033)
18	(0.0000,1.0000)	(0.0000,1.0000)	(0.0005,1.0003)	(0.0007,1.0004)	(0.0040,1.0024)
19	(0.0000,1.0000)	(0.0000,1.0000)	(0.0003,1.0002)	(0.0004,1.0002)	(0.0029,1.0017)
20	(0.0000,1.0000)	(0.0000,1.0000)	(0.0002,1.0001)	(0.0002,1.0001)	(0.0020,1.0012)

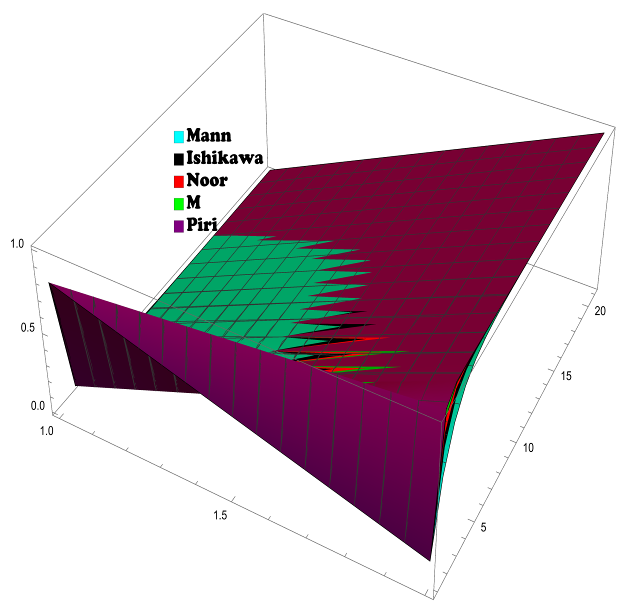

FIGURE 4. Behaviors of various iterative processes using Example 4.2.

By assuming $\{a_s\} = 0.95$, $\{b_s\} = 0.84$ and $\{c_s\} = 0.93$ and by taking the initial guess $(0.7921, 0.1472)$ the observations are provided in Table 4 and Figure 4, which show that Piri iterative scheme (1.10) is faster than above mentioned.

Table 4. Convergence comparison of different schemes with Piri iterative scheme.

s	Piri	M	Noor	Ishikawa	Mann
1	(0.7921,0.1472)	(0.7921,0.1472)	(0.7921,0.1472)	(0.7921,0.1472)	(0.7921,0.1472)
2	(0.0603,0.9351)	(0.1040,0.8881)	(0.1843,0.8015)	(0.2578,0.7224)	(0.4159,0.5528)
3	(0.0046,0.9951)	(0.01365,0.9853)	(0.0429,0.9538)	(0.0839,0.9096)	(0.2183,0.7650)
4	(0.0003,0.9996)	(0.0018,0.9981)	(0.0100,0.9892)	(0.0273,0.9706)	(0.1146,0.8766)
5	(0.0000,0.9999)	(0.0002,0.9993)	(0.0023,0.9975)	(0.0090,0.9904)	(0.0602,0.9352)
6	(0.0000,1.0000)	(0.0000,0.9999)	(0.0005,0.9994)	(0.0029,0.9969)	(0.0316,0.9660)
7	(0.0000,1.0000)	(0.0000,1.0000)	(0.0001,0.9997)	(0.0009,0.9990)	(0.0166,0.9821)
8	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,0.9999)	(0.0003,0.9997)	(0.0087,0.9906)
9	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0001,0.9999)	(0.0046,0.9951)
10	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0024,0.9974)
11	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0013,0.9986)
12	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0007,0.9993)
13	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0003,0.9996)
14	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0002,0.9998)
15	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0001,0.9999)
16	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)
17	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)
18	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)
19	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)
20	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,1.0000)

5. Application

The fundamental idea behind fractional calculus is to extend the notation of differentiation and integration by allowing the order of differentiation and integration to be real or complex numbers instead of positive integers. One of the most intriguing aspects of fractional calculus is its wide range of applications across various scientific and engineering disciplines. Fractional differential equations (FDEs), which involves fractional derivatives, are essential tools for modeling and solving real-world problems that exhibits complex behaviors, such as anomalous diffusion, viscoelasticity, and non-local phenomena. These equations have found applications in physics, biology, engineering, finance (see, for more details [33–35] and others).

Mandelbort [36] noted that there are numerous fractional dimension wonders existing in nature and technology. Various physical systems have fractional-order dynamical manners because of the natural properties and singular ingredients. In [37], Richard anticipated the presence of delay phenomenon in several physical systems. In this section, by using our proposed iterative scheme (1.10), we shall give the solution of Delay Caputo Fractional Differential Equation.

Consider the following Delay Caputo Fractional Differentional Equation;

$$^{c}\mathcal{D}h(u) = g(u, h(u), h(u-v)),\ u \in [u_0, G], \tag{5.1}$$

with initial conditions

$$h(u) = \varphi(u),\ u \in [u_0 - w, u_0], \tag{5.2}$$

where the constant v stands for time delay, $v > 0, K > 0, w > 0, \varphi \in C([u_0 - w, u_0] : \mathbb{R}^k, h \in \mathbb{R}^k$ and $g : [u_0, K] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ are continuous mappings. Consider the following assumptions are true: $(\mathcal{A}_1)$ There exists a Lipschitz constant $L_g > 0$ such that

$$\|g(u, m_1, n_1) - g(u, m_2, n_2)\| \leq L_g(\|m_1 - m_2\| + \|n_1 - n_2\|,\ \forall m_1, n_1, m_2, n_2 \in \mathbb{R}^k.$$

$(\mathcal{A}_2)$ There exists a constant $\delta_L > 0$ with $\dfrac{2L}{\delta_L} < 1$.

If $\wp \in (C([u_0 - w, K] : \mathbb{R}^k) \cap (C^1([u_0, K] : \mathbb{R}^k)$ is a function satisfying (5.1) and (5.2), then $\wp$ is called the solution of the problem (5.1) and (5.2). The solution of the following integral equation is equivalent to the solution of the problem (5.1) and (5.2).

$$h(u) = \varphi(u_0) + \frac{1}{\Gamma(r)} \int_{u_0}^{u} (u-w)^{(r-1)} g(w, h(w), h(w-t))\mathrm{d}w,\ u \in [u_0, K] \tag{5.3}$$

where $h(u) = \varphi(u),\ \forall u \in [u_0 - w, u_0]$. Let the norm $\|.\|_{\delta_L}$ on $C([u_0 - \wp, u_0]) : \mathbb{R}^k)$ be defined by,

$$\|\varphi\|_{\delta_L} = \frac{\sup \|\varphi(u)\|}{E_r(\delta_L u_r)}\ \forall \varphi \in C([u_0 - \wp, u_0] : \mathbb{R}^k). \tag{5.4}$$

where $E_r : \mathbb{R} \to \mathbb{R}$ is called the Mittag-Leffler function. For all $r \in \mathbb{R}$ the Mittag-Leffler function is defined by

$$E_r(u) = \sum_{i=0}^{\infty} \frac{u_i}{\Gamma(r^i + 1)}.$$

Obviously, $(C([u_0 - \wp, u_0])$ is Banach Space.

In the next theorem, we obtain an approximatate solution of Caputo Fractional Differential Equation using iterative scheme (1.10).

Theorem 5.1. *Let the function h and φ be the same as defined above. If the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ satisfied then the problem (5.1) and (5.2) has a unique solution $\wp \in (C([u_0 - w, K] : \mathbb{R}^k) \cap (C^1([u_0, K] : \mathbb{R}^k) = S$ and the sequence $\{\eta_n\}$ defined by (1.10) converges to $\wp$.*

Proof. Define an operator M on S as:

$$Mh(u) = \begin{cases} \varphi(u_0) + \dfrac{1}{\Gamma(r)} \int_{u_0}^{u} (u-w)^{(r-1)} g(w, h(w), h(w-t))\mathrm{d}w,\ u \in [u_0, K], \\ \varphi(u),\ u \in [u_0 - w, u_0]. \end{cases}$$

Now, we will show that $h_i \to \wp$ as $i \to \infty$. For $u \in [u_0 - w, u_0]$. It is easy to verify that $h_i \to \wp$ as $i \to \infty$. Now for $u \in [u_0, K]$ then by using (1.10), Lemma 3.1 and by assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we have

$$\begin{aligned} \|\xi_s - \wp\| &= \|M((1-b_s)\eta_s + b_s M(\eta_s)) - \wp\| \\ &\leq \|(1-b_s)\eta_s + b_s M(\eta_s) - \wp\| \\ &\leq (1-b_s)\|\eta_s - \wp\| + b_s\|M(\eta_s) - wp\|. \end{aligned} \tag{5.5}$$

Using supremum over $[u_0 - w, K]$ on both sides of (5.5), we got

$$\begin{aligned} &\sup_{u\in[u_0-w,K]} \|\xi_s - \wp\| \qquad (5.6)\\ \leq\ & \sup_{u\in[u_0-w,K]} \left((1-b_s)\|\eta_s - \wp\| + b_s\|M(\eta_s) - \wp\|\right) \\ =\ & (1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\| + b_s \sup_{u\in[u_0-w,K]}\|M(\eta_s) - M(\wp)\| \\ =\ & (1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\| + b_s \sup_{u\in[u_0-w,K]}\Big\|\varphi(u_0) + \frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)} \\ & g(w, \eta_s(w), \eta_s(w-t))\mathrm{d}w - \varphi(u_0) - \frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)} g(w, \wp(w), \wp(w-t))\mathrm{d}w\Big\| \\ =\ & (1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\| + b_s \sup_{u\in[u_0-w,K]}\frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)} \\ & (\|g(w, \eta_s(w), \eta_s(w-t))\mathrm{d}w - g(w, \wp(w), \wp(w-t))\mathrm{d}w\|)\mathrm{d}w \\ \leq\ & (1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\| + b_s \sup_{u\in[u_0-w,K]}\frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)} \\ & L_g(\|\eta_s(w) - \wp(w)\| + \|\eta_s(w-t) - \wp(w-t)\|)\mathrm{d}w \\ =\ & (1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\| + b_s\frac{L_g}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)}\mathrm{d}w \\ & \Big(\sup_{u\in[u_0-w,K]}\|\eta_s(w) - \wp(w)\| + \sup_{u\in[u_0-w,K]}\|\eta_s(w-t) - \wp(w-t)\|\Big). \qquad (5.7) \end{aligned}$$

Dividing both sides of (5.7) with $E_r(\delta_L u_r)$, we have

$$\begin{aligned} \frac{\sup_{u\in[u_0-w,K]}\|\xi_s - \wp\|}{E_r(\delta_L u_r)} &\leq \frac{(1-b_s)\sup_{u\in[u_0-w,K]}\|\eta_s - \wp\|}{E_r(\delta_L u_r)} + b_s\frac{L_g}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)}\mathrm{d}w \\ & \left(\frac{\sup_{u\in[u_0-w,K]}\|\eta_s(w) - \wp(w)\|)}{E_r(\delta_L u_r)} + \frac{\sup_{u\in[u_0-w,K]}\|\eta_s(w-t) - \wp(w-t)\|}{E_r(\delta_L u_r)}\right) \\ \|\xi_s - \wp\|_{\delta_L} &\leq (1-b_s)\|\eta_s - \wp\|_{\delta_L} + b_s\frac{L_g}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)}\mathrm{d}w \\ & (\|\eta_s(w) - \wp(w)\|_{\delta_L} - \|\eta_s(w-t) - \wp(w-t)\|_{\delta_L}) \\ &= (1-b_s)\|\eta_s - \wp\|_{\delta_L} + 2L_g b_s\|\eta_s - \wp\|_{\delta_L}\frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)}\mathrm{d}w \end{aligned}$$

$$\begin{aligned} &= (1-b_s)\|\eta_s-\wp\|_{\delta_L}+\frac{2L_g b_s}{E_r(\delta_L u_r)}\|\eta_s-\wp\|_{\delta_L} \\ &\quad \frac{1}{\Gamma(r)}\int_{u_0}^{u}(u-w)^{(r-1)}E_r(\delta_L u_r)\mathrm{d}w \\ &= (1-b_s)\|\eta_s-\wp\|_{\delta_L}+\frac{2L_g b_s}{E_r(\delta_L u_r)}\|\eta_s-\wp\|_{\delta_L}{}^{c}I^{0}\left({}^{c}\mathcal{D}\frac{E_r(\delta_L u_r)}{\delta_L}\right) \\ &= (1-b_s)\|\eta_s-\wp\|_{\delta_L}+\frac{2L_g b_s}{\delta_L}\|\eta_s-\wp\|_{\delta_L}. \end{aligned}$$

Since $\dfrac{2L_g}{\delta_L} < 1$, we obtained

$$\|\xi_s-\wp\|_{\delta_L} \le \|\eta_s-\wp\|_{\delta_L}. \tag{5.8}$$

Now

$$\|\mu_s-\wp\|_{\delta_L} \le \|M\xi_s-\wp\|_{\delta_L} \le \|\xi-\wp\|_{\delta_L} \le \|\eta_s-\wp\|_{\delta_L}. \tag{5.9}$$

Also,

$$\begin{aligned} \|\eta_{s+1}-\wp\|_{\delta_L} &= \|(1-a_s)M\xi_s+a_sM\mu-\wp\|_{\delta_L} \\ &\le (1-a_s)\|\xi-\wp\|_{\delta_L}+a_s\|\mu-\wp\|-\delta_L \end{aligned}$$

Using (5.8) and (5.9), we got

$$\|\eta_{s+1}-\wp\|_{\delta_L} \le \|\eta_s-\wp\|_{\delta_L}.$$

If we put $\|\eta_s-\wp\|_{\delta_L} = \nu_s$, then we get $\nu_{s+1} \le \nu_s, \forall s \in \mathbb{N}$. Thus, $\{\nu_s\}$ is monotonically dercreasing sequence. Additionally, it is bounded sequence. So, we can conclude that $\lim_{s\to\infty} \nu_s = \inf\{\nu_s\} = 0$. Hence, $\lim_{s\to\infty} \|\eta_s-\wp\|_{\delta_L} = 0$. □

6. Conclusion

In this research, we employed an iterative algorithm proposed by Piri et. al. [16] to approximate fixed points associated with generalized (α,β)-nonexpansive mappings. Our study establishes both weak and strong convergence results for mappings within uniformly convex Banach spaces that exhibit generalized (α,β)-nonexpansiveness. Notably, the Piri-iterative scheme for generalized (α,β)-nonexpansive mappings demonstrated superior convergence rates compared to certain existing algorithms, as evidenced by a numerical example. Through the utilization of an Piri-iterative scheme, we established convergence properties for generalized (α,β)-nonexpansive mappings towards the solution of Caputo fractional differential equation.

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