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An Augmented Mixed DG Scheme for the Electric Field

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Abstract. In this paper, a new augmented mixed DG formulation for the numerical approximation of the electrostatic field was introduced and studied. Its error analysis was carried out and an optimal error estimates as a function of the mesh size was obtained. Some numerical tests confirming the theoretical convergence were given.

1. Introduction

Along this paper, we consider the study and analysis of the mixed discontinuous Galerkin method for the three dimensional Maxwell's equations: Find u, p such that

$$\begin{aligned} \nabla \wedge (\mu^{-1} \nabla \wedge u) - \varepsilon \nabla p &= J \text{ in } \Omega, \\ \nabla \cdot (\varepsilon u) - \alpha p &= 0 \text{ in } \Omega, \\ n \wedge u &= 0 \text{ on } \partial\Omega, \\ p &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Assuming $\alpha = 0$, then u is linked to the electric field E by $E(t,x) = u(x)\cos(\omega t)$, with ω is non zero frequency and p is the Lagrange multiplier used for controlling the divergence of the electric field. The piecewise coefficients μ and ε are the magnetic permeability and electric permittivity of the media, we assume that these coefficients are sufficiently regular and there exists two lower bounds μ_ℓ, ε_ℓ and two upper bounds μ_u, ε_u such that $0 < \mu_\ell \le \mu(x) \le \mu_u$ and $0 < \varepsilon_\ell \le \varepsilon(x) \le \varepsilon_u$ for all $x \in \Omega$. α is any non negative function in $L^\infty(\Omega)$. We assume that Ω is a smooth subdomain of $\mathbb{R}^3$.

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In the last years, the Maxwell equations have been studied and analysed by using several numerical methods such as discontinuous Galerkin methods [5,6,9,10,13–16] and by weak Galerkin formulations [18]. Thanks to works of Cockburn et al. [1–3], DG methods are developed very well and it was applied to solve numerically many problems of partial differential equations like Poisson's equation [2], Stokes equations [7] , Maxwell's equations [5,6,8,10,13,17]. The researchers C. Daveau and A. Zaghdani study Maxwell equations and wave equation in [4–6] by using some new schemes of DG methods. In [16,17], A. Zaghdani et al. presented a mixed DG scheme for the numerical resolution of the electrostatic field. This work is an expansion of [5] where the problem (1.1) was considered with constant permeability and permittivity coefficients and when $\alpha = 0$, the problem (1.1) was also analysed in [9], however the formulation of equations exploited to find the error estimates is not consistent, this is due to the choice of the lifting operators. In our study, we present a new DG formulation using a symmetric principal bilinear form.
The outline of this paper is presented as follows. We start by giving some notations and some preliminaries results that are essential for our study. Next, we derive the DG formulation, we show that it is well posed and consistent. Then, we establish a priori error estimations and finally we present some numerical tests which confirm the theoretical study.

2. Mixed Formulation

2.1. **Functional spaces.** In the outline of this paper, Ω is an open bounded subset of $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$ and $\mathcal{T}_h$ be a quasi-uniform partition of Ω, more precisely

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T} \quad \text{and} \quad T_k \cap T_\ell = \emptyset \quad \text{for} \quad k \neq \ell.$$

Let $\mathcal{E}_h^I$ the set of each interior faces of the subdivision of Ω, $\mathcal{E}_h^D$ the set of all boundary faces and $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D$ the union of interior and boundary faces. If $0 < s$, we define the broken spaces

$$\mathbf{H}^s(\mathcal{T}_h) = \{u \in L^2(\Omega)^3 \text{ such that } \ u_{|T} \in H^s(T) \quad \forall\, T \in \mathcal{T}_h\}$$

and

$$\mathbf{H}^s(\nabla\wedge, \mathcal{T}_h) = \{u \ : \ u_{|T} \in H^s(T)^3 \text{ and } \nabla \wedge (u_{|T}) \in H^s(T)^3 \quad \forall T \in \mathcal{T}_h\}.$$

To establish a weak formulation of (1.1), we also define

$$V(h) = \mathbf{H}^1(\nabla\wedge, \mathcal{T}_h) \quad \text{and} \quad Q(h) = \mathbf{H}^1(\mathcal{T}_h).$$

Multiplying the first and the second equations in (1.1) by v and ψ and integrating over T, we get

$$\int_T \mu^{-1}(\nabla \wedge u)\cdot(\nabla \wedge v)\,dx - \int_{\partial T} v\cdot((\mu^{-1}\nabla \wedge u)\wedge n_T)ds + \int_T p\nabla\cdot \varepsilon v\,dx - \int_{\partial T}(\varepsilon v\cdot n_T)p\,ds = \int_T J\cdot v\,dx \quad \text{for all} \quad v \in V(h) \tag{2.1}$$

and

$$-\int_T \varepsilon u \nabla\psi\,dx - \int_T \alpha p\psi\,dx + \int_{\partial T}(\varepsilon u\cdot n_T)\psi\,ds = 0 \quad \text{for all} \quad \psi \in Q(h). \tag{2.2}$$

The primal DG method consists to replace the traces of functions used in (2.1)-(2.2) by numerical fluxes, this is due to the discontinuity of solution (u,p) on interfaces of the triangulation. For the definitions of averages and jumps of a discontinuous function on the interfaces we refer to [16]. As in [1], the numerical fluxes are chosen to provide a numerical scheme consistent and conservative. In this sense, denote by η_a and η_c the stabilization parameters introduced in [16] and we adapt the numerical fluxes given in [3] for the laplacien and for the curl-curl operators in [2, 10] for defining the fluxes as

$$\mu^{-1}\widehat{\nabla \wedge u} = \mu^{-1}\{\nabla \wedge u\} - \eta_a[u]_T, \tag{2.3}$$

$$\widehat{\varepsilon u} = \{\varepsilon u\} - \eta_c[p]_N \quad \text{in} \quad \mathcal{E}_h^I \quad \text{and} \quad \varepsilon u - \eta_c p n \qquad \text{in} \quad \mathcal{E}_h^D.$$

and

$$\widehat{p} = \{p\} - \eta_a[\varepsilon u]_N \quad \text{in} \quad \mathcal{E}_h^I \quad \text{and} \quad 0 \qquad \text{in} \quad \mathcal{E}_h^D.$$

Now, equations (2.1)-(2.2) can be replaced by

$$\int_T (\mu^{-1}\nabla \wedge u)\cdot(\nabla \wedge v)\,dx - \int_{\partial T} v\cdot(\mu^{-1}(\widehat{\nabla \wedge u})\wedge n_T)\,ds + \int_T p\nabla\cdot\varepsilon v\,dx$$
$$- \int_{\partial T}(\varepsilon v\cdot n_T)\widehat{p}ds = \int_T J\cdot v\,dx \tag{2.4}$$

and

$$-\int_T \varepsilon u\cdot\nabla v\,dx - \int_T \alpha p\psi\,dx + \int_{\partial T}(\widehat{\varepsilon u}\cdot n_T)v\,ds = 0. \tag{2.5}$$

we integrate back by parts the equation (2.5) and arrive at

$$\int_T v\nabla\cdot\varepsilon u\,dx - \int_T \alpha p\psi\,dx + \int_{\partial T}((\widehat{\varepsilon u} - \varepsilon u)\cdot n_T)v\,ds = 0. \tag{2.6}$$

2.2. **Discontinuous Galerkin scheme.**

Let us first remark that the equations (see [13]),

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi(t\wedge n_T)ds = \int_{\mathcal{E}_h}[\phi]_T\{t\} - \int_{\mathcal{E}_h^I}[t]_T\{\phi\}\,ds$$

and

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} v(\phi\cdot n_T)ds = \int_{\mathcal{E}_h^I}(\{v\}[\phi]_N + [v]_N\{\phi\})ds + \int_{\mathcal{E}_h^D} v(\phi\cdot n)\,ds$$

are valid for any t and ϕ in $\Pi_{T\in\mathcal{T}_h}L^2(\partial T)^3$ and for any $v\in\Pi_{T\in\mathcal{T}_h}L^2(\partial T)$. In order to simplify notations, we define the following three bilinear forms

$$s_T(u,v) := \int_{\mathcal{E}_h}\eta_a[u]_T[v]_T\,ds,$$

$$s_N(u,v) := \int_{\mathcal{E}_h^I}\eta_a[\varepsilon u]_N[\varepsilon v]_N\,ds,$$

$$\mathcal{S}(u,v) := \int_{\mathcal{E}_h}\mu^{-1}[u]_T\nabla\wedge v\,ds.$$

By summation of the equations (2.4), (2.6) over all elements of $\mathcal{T}_h$, using the introduced numerical fluxes and the last formulas, we obtain

$$\int_\Omega \mu^{-1}\nabla \wedge u \nabla \wedge v \, dx - \int_{\mathcal{E}_h} [v]_T \mu^{-1}\{\nabla \wedge u\} \, ds + s_T(u,v) + s_N(u,v) + \int_\Omega p\nabla \cdot \varepsilon v \, dx - \int_{\mathcal{E}_h^I} [\varepsilon v]_N \{p\} \, ds = \int_\Omega J \cdot \psi \, dx$$

and

$$\int_\Omega \varepsilon u \nabla \psi \, dx - \int_\Omega \alpha p \psi dx - \int_{\mathcal{E}_h^I} [\varepsilon u]_N \{\psi\} ds - \int_{\mathcal{E}_h^I} \eta_c [p]_N [\psi]_N \, ds - \int_{\mathcal{E}_h^D} \eta_c [\psi]_N [p]_N \, ds = 0.$$

Using the fact that for the true solution u we have $n \wedge u = 0$ in $\mathcal{E}_h^D$, $[u]_T = 0$ in $\mathcal{E}_h^I$ and $\nabla \cdot \varepsilon u = 0$ in Ω, we can add a term of penalization as

$$r \int_\Omega \nabla \cdot (\varepsilon u) \nabla \cdot (\varepsilon v) \, dx - \mathcal{S}(u,v).$$

We note that the first quantity $r \int_\Omega \nabla \cdot (\varepsilon u) \nabla \cdot (\varepsilon v) \, dx$ is added to maintain the coercivity of A_s on the whole discrete space which will be defined later, while $\mathcal{S}(u,v)$ is added for symmetrizing the principal bilinear form A_s that we are going to introduce. Now, one define the bilinear forms

$$A_s(u,v) =: s(u,v) - \mathcal{S}(v,u) - \mathcal{S}(u,v), \quad B(v,p) =: \int_\Omega p \nabla \cdot \varepsilon v \, dx - \int_{\mathcal{E}_h^I} [\varepsilon v]_N \{p\} \, ds$$

and

$$C(p,\psi) =: \int_\Omega \alpha p \psi dx + \int_{\mathcal{E}_h} \eta_c [\psi][p] \, ds.$$

where we have denoted by

$$s(u,v) := \int_\Omega \mu^{-1} (\nabla \wedge u) \cdot (\nabla \wedge v) \, dx + s_T(u,v) + s_N(u,v) + r \int_\Omega \nabla \cdot (\varepsilon u) \nabla \cdot (\varepsilon v) \, dx$$

The considered DG formulation associated to (1.1) is to find $u \in V(h)$ and $p \in Q(h)$ satisfying

$$A_s(u,v) + B(v,p) = L(v) \quad \text{for any} \quad v \in V(h), \tag{2.7}$$

$$B(u,\psi) - C(p,\psi) = 0 \quad \text{for any} \quad \psi \in Q(h). \tag{2.8}$$

2.3. **Discrete scheme.** Given $T \in \mathcal{T}_h$ and $\mathbb{P}_k(T)$ the set of polynomials of degree at most k on T. We define the finite space by

$$D_k = \{p : p_{|T} \in \mathbb{P}_k(T)^3, T \in \mathcal{T}_h\}.$$

The numerical discretization for the scheme (2.7)-(2.8) consist first to discretize the space $Q(h) \times V(h)$ by the finite dimensional space $\mathcal{Q}_h \times \mathcal{V}_h =: D_1 \times D_2$. The mixed discontinuous Galerkin method take the discrete form: find $u_h \in \mathcal{V}_h$ and $p_h \in \mathcal{Q}_h$ satisfying

$$A_s(u_h, v) + B(v, p_h) = L(v) \quad \text{for any} \quad v \in \mathcal{V}_h, \tag{2.9}$$

$$B(u_h, \psi) - C(p_h, \psi) = 0 \quad \text{for any} \quad \psi \in \mathcal{Q}_h. \tag{2.10}$$

In the next theorem, we prove that our mixed DG scheme is consistent and well posed.

Theorem 2.1. *One can find a positive constant κ_0 such that for all κ greater than κ_0, the problem (2.9)-(2.10) is consistent and well posed.*

Proof. First, we notice that the exact solution (u, p) of (1.1) is in the space $H_0(\nabla\wedge, \Omega) \cap H(\nabla_\varepsilon\cdot, \Omega) \times H_0^1(\Omega)$, we integrating by parts (2.1)-(2.2) to show easily that the true solution of (1.1) verify (2.9))-(2.10), which proves the consistency. Next, we demonstrate that (2.9)-(2.10) has a unique solution. To do this, we use the fact that (2.9)-(2.10) is a linear and finite dimensional problem. Assume that J is null in Ω and let $v = u$ and $\psi = p$ in (2.9)-(2.10), substracting the last equation of (2.9)-(2.10) from the previous, we obtain

$$C(p, p) + s(u, u) - 2\mathcal{S}(u, u) = 0.$$

That means

$$C(p,p) + \frac{1}{\mu}\int_\Omega (\nabla \wedge u)\cdot(\nabla \wedge u)\,dx + s_T(u,u) + s_N(u,v) + \int_\Omega r\nabla\cdot(\varepsilon u)\nabla\cdot(\varepsilon u)\,dx - 2\int_{\mathcal{E}_h} \mu^{-1}[u]_T \nabla \wedge u\,ds = 0.$$

Using the boundedness of μ and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} 2\mathcal{S}(u,u) &\le 2\delta \int_{\mathcal{E}_h} \eta_a [u]_T^2\,ds + \frac{2C}{\delta}\int_{\mathcal{E}_h} \frac{1}{\eta_a}|\{\nabla \wedge u^h\}|^2\,ds \quad \text{for any} \quad \delta > 0 \\ &\le 2\delta s_T(u,u) + \frac{2C}{\delta}\int_{\mathcal{E}_h} \frac{1}{\eta_a}|\{\nabla \wedge u^h\}|^2\,ds \quad \text{for any} \quad \delta > 0. \end{aligned}$$

Applying the following inverse inequality [8, 11, 12, 16]

$$\|q\|_{0,\partial T}^2 \le C\frac{1}{h_T}\|q\|_{0,T}^2, \text{ for every } q \in P_k(T) \tag{2.11}$$

and using the fact that $\nabla \wedge \mathcal{V}_h \subset \mathcal{V}_h$, we get

$$\int_{\mathcal{E}_h} |\frac{1}{\sqrt{\eta_a}}\{\nabla \wedge v\}|^2 ds \le \frac{C}{\kappa}\int_\Omega |\nabla \wedge v|^2\,dx \quad \text{for any} \quad v \in \mathcal{V}_h$$

which means

$$2\mathcal{S}(u,u) \le 2\delta s_T(u,u)\,ds + \frac{2C}{\delta\kappa}\int_\Omega |\nabla \wedge u|^2\,dx.$$

Therefore, we obtain

$$\begin{aligned} C(p,p) + s(u,u) - 2\mathcal{S}(u,u) \ge\, & C(p,p) + (1 - \frac{2C}{\delta\kappa})\int_\Omega (\nabla \wedge u^h)^2\,dx \\ &+ \int_\Omega r(\nabla\cdot\varepsilon u^h)^2\,dx + (1-2\delta)s_T(u,u) + s_N(u,u) \end{aligned}$$

If we consider κ and δ such that $\min(1-2\delta, 1-\frac{2C}{\delta\kappa}) > 0$, we get

$$\nabla \wedge u = 0, \quad \nabla \cdot (\varepsilon u) = 0 \quad \text{in} \quad \Omega$$

and

$$s_N(u,u) = 0 \quad \text{on} \quad \mathcal{E}_h^I$$
$$s_T(u,u) = 0 \quad \text{on} \quad \mathcal{E}_h \quad \text{and} \quad C(p,p) = 0.$$

This means that

$$[u]_T = 0 \quad \text{on} \quad \mathcal{E}_h \quad \text{and} \quad [\varepsilon u]_N = 0 \quad \text{on} \quad \mathcal{E}_h^I.$$

The equations $\nabla \wedge u = 0$ and $[u]_T = 0$ on $\mathcal{E}_h$ implies that u is in $\mathbf{H}_0(\nabla \wedge 0, \Omega)$. Similarly $\nabla \cdot (\varepsilon u) = 0$ in Ω and $[\varepsilon u]_N = 0$ on $\mathcal{E}_h^I$ means that u belongs to $\mathbf{H}(\nabla_\varepsilon \cdot 0, \Omega)$. We deduce that u is null in Ω. The equation $C(p,p) = 0$ implies that $[p]_N = 0$ on $\mathcal{E}_h$ and then p is in $H_0^1(\Omega)$ and it is clear that if $\alpha \neq 0$, then $p = 0$ in Ω. If $\alpha = 0$, we integrate by parts the equation (2.10) yields $-\int_\Omega v \nabla p \, dx = 0$ for all $v \in \mathcal{V}_h$ which gives that p is null in Ω. □

The objective of the following section is to study the convergence of our numerical scheme. For this, let us introduce the mesh dependent norm on the discrete spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$. Given $v \in \mathcal{V}_h$ and $\phi \in \mathcal{Q}_h$, then we define

$$||v||_h^2 := s(v,v) + ||\frac{1}{\sqrt{\eta_a}}\{\mu^{-\frac{1}{2}}\nabla \wedge v\}||_{0,\mathcal{E}_h}^2 \tag{2.12}$$

and

$$||\phi||_h^2 := ||\alpha^{\frac{1}{2}}\phi||_{0,\Omega}^2 + ||\sqrt{\eta_c}[\phi]||_{0,\mathcal{E}_h}^2 + ||\sqrt{\eta_c}\{\phi\}||_{0,\mathcal{E}_h}^2. \tag{2.13}$$

It is straightforward to prove that (2.12) and (2.13) are two norms on the element spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$ respectively.

3. Error Estimates

Let us start by studying some properties of the three bilinear forms A, B and C. It is obvious that the continuity of these bilinear forms follow from the definitions of mesh-dependent norm and the Cauchy Shwarz inequality. Now, we demonstrate an inf-sup condition for B.

3.1. An inf-sup condition.

Let us start by introducing the following lemma which we need for proving the inf-sup condition for B.

Lemma 3.1. *Given u in $H^1(\mathcal{T}_h)^3$, we can find an interpolant $R^h \in D_2$ satisfying*

$$\int_T (\nabla \cdot \varepsilon u - R^h)\psi_h \, dx = 0 \quad \text{and} \quad \int_e (\varepsilon u - R^h) \, ds = 0$$

for any ψ_h in $P_1(T)$, for all T in $\mathcal{T}_h$ and for all e in $\mathcal{E}_h$.

Proof. The proof can be deduced from theorem 1 in [4] since εu is in $\mathbf{H}^1(\mathcal{T}_h)^3$ for all u in $\mathbf{H}^1(\mathcal{T}_h)^3$ if ε is sufficiently smooth. □

Now, An inf-sup condition for B is given in the following theorem.

Theorem 3.1. *For any p in Q_h, there exists $\beta > 0$ independent h and satisfying*

$$\sup_{u\in V_h\setminus\{0\}} \frac{B(u,p)}{||u||_h} \geq \beta ||p||_h. \tag{3.1}$$

Proof. Using the inverse inequality (2.11) and the fact that $\eta_c = \frac{1}{\eta_a}$, we obtain

$$\begin{aligned} ||p_h||^2_{0,\Omega} \geq & \frac{1}{3}||p_h||^2_{0,\Omega} + C||\frac{1}{\sqrt{\eta_a}}[p]||^2_{0,\mathcal{E}_h} + C||\frac{1}{\sqrt{\eta_a}}\{p\}||^2_{0,\mathcal{E}_h} \\ \geq & \frac{1}{3}||p_h||^2_{0,\Omega} + C||\sqrt{\eta_c}[p]||^2_{0,\mathcal{E}_h} + C||\sqrt{\eta_c}\{p\}||^2_{0,\mathcal{E}_h} \geq C||p||^2_h. \end{aligned}$$

Fix $p_h \in Q_h$, it is well known that there exists $\widetilde{u}_h \in H^1(\Omega)^3$ satisfying

$$\nabla \cdot \varepsilon \widetilde{u}_h = p_h \quad \text{and} \quad ||\widetilde{u}_h||_{1,\Omega} \leq C||p_h||_{0,\Omega}.$$

Using Lemma 3.1 and setting $u_h = R^h(\widetilde{u}_h)$, yields

$$B(u_h, p_h) = \sum_{T\in\mathcal{T}_h} \int_T p_h \nabla \cdot R^h(\widetilde{u}_h) = \sum_{T\in\mathcal{T}_h} \int_T p_h \nabla \cdot \varepsilon \widetilde{u}_h = ||p_h||^2_{0,\Omega} \geq C||p_h||^2_h.$$

Since R^h is a continuous operator, we obtain $||u_h||_h = ||R^h(\widetilde{u}_h)||_h \leq C||\widetilde{u}_h||_{1,\Omega} \leq C||p_h||_{0,\Omega} \leq C||p_h||_h$ and the proof of (3.1) can be easily deduced. □

For the coercivity of the bilinear form A_s, it is well known that it is sufficient to demonstrate the coercivity of A_s on $ker(B)$. However, we can demonstrate that A_s is coercive on the whole discrete space $\mathcal{V}_h \times \mathcal{V}_h$.

Proposition 3.1. *We have*

$$A_s(v,v) \geq C||v||^2_h \quad \textit{for any} \quad v \in \mathcal{V}_h \tag{3.2}$$

with a constant $C > 0$ independent of h.

Proof. With the same steps and techniques presented in [16] the proof follows without difficulty. □

Theorem 3.2. *Let (u,p) be the true solution of (1.1) and (u_h,p_h) be the discrete solution of (2.9)-(2.10), we assume that $(u,p) \in \mathbf{H}^{t+1}(\mathcal{T}_h)^3 \times \mathbf{H}^{s-1}(\mathcal{T}_h)$ with $2 \leq s$ and $1 \leq t$ then there exists a constant C independent of h such that*

$$||p-p_h||^2_h + ||u-u_h||^2_h \leq C\left(h^{2[\min(s,2)-1]}||p||^2_{s,\mathcal{T}_h} + h^{2\min(t,2)}||u||^2_{t+1,\mathcal{T}_h}\right)$$

Proof. The proof is an easy consequence of the known Babuska-Brezzi theory and is well detailed in [13]. □

4. Numerical Tests

The numerical tests are conducted for the Maxwell Eqs (1.1) on the unit cube $\Omega = [0,1]^3$. For the sake of simplicity, we assume that $\varepsilon = 1$, $\mu = 1$ and $\alpha(x)$ is null on Ω which means that the term αp vanishes from the equation (1.1). Note that, in the numerical scheme (2.9)-(2.10) there is two parameters κ and r. We can choose r as any non negative real number but not too large while for κ, it is not chosen too small to ensure the coercivity result of A_s and also not too large to maintain the good conditioning of the principal matrix of A_s. In our numerical code, we have taken $(r, \kappa) = (1, 100)$ for the following two examples.

Example 4.1. In this example, we assume that the true solution (u, p) is given by

$$u = \begin{pmatrix} yz(y-1)(z-1)\sin(zy) \\ xz(x-1)(z-1)\sin(zx) \\ xy(x-1)(y-1)\sin(yx) \end{pmatrix} \quad \text{and} \quad p = xyz(x-1)(y-1)(z-1)\sin(zyx).$$

For this example, the numerical results are presented in table 1, confirming the theoretical convergence estimates as proved in theorem 3.2.

Table 1. Numerical results for Example 4.1.

h	$\|\|u-u^h\|\|_h$	rate	$\|\|p-p^h\|\|_h$	rate
0.4367	0.1102	-	0.7005	-
0.2184	0.03162	1.80	0.2201	1.75
0.1733	0.001808	2.41	0.1504	1.64
0.09268	0.004451	2.23	0.06111	1.43
0.07703	0.003052	2.04	0.04803	1.30

Example 4.2. Here we suppose that the exact solution (u, p) is

$$u = \begin{pmatrix} yz(y-1)(z-1)\exp(zy) \\ xz(x-1)(z-1)\exp(zx) \\ xy(x-1)(y-1)\exp(yx) \end{pmatrix} \quad \text{and} \quad p = xyz(x-1)(y-1)(z-1)\exp(zyx).$$

The values of errors associated to this example are given in table 2. We remark that the numerical solution u^h converge to the true solution u with respect to the rate $O(h^2)$ and for p with rate $O(h)$ as derived in theoretical study.

5. Remarks and Conclusion.

From the two previous error tables, we confirm the convergence results developed in earlier sections, though the numerical results show an excellent approximation to the exact solution. As expected, the convergence rate for approximating u is in order $O(h^2)$ and $O(h)$ for p and then our theoretical results are numerically confirmed. We believe that our DG method is of superconvergent for Maxwell's equations.

Table 2. Numerical results for Example 4.2.

h	$\|u - u^h\|_h$	rate	$\|p - p^h\|_h$	rate
0.4367	0.1305	-	0.8891	-
0.2184	0.03875	1.7524	0.2229	1.9966
0.1733	0.02051	2.7506	0.1700	1.1713
0.09268	0.004521	2.4161	0.06283	1.5904
0.07703	0.003111	2.0209	0.04660	1.6157

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