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Uncertainty Principle for the Weinstein-Gabor Transforms

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Abstract. In this paper, we present the localization of the ν-entropy for the Weinstein Gabor transform. Through the utilization of the ν-entropy, we establish an alternative expression for the Heisenberg uncertainty principle for the Weinstein Gabor transform. In addition, we further extend our study by elaborating on an L^p version of the Heisenberg uncertainty principle for the Weinstein Gabor transform.

1. Introduction

The uncertainty principle is a fundamental concept in physics and signal processing that states that certain pairs of physical properties, such as position and momentum, cannot both be precisely determined simultaneously. In the context of signal processing, the uncertainty principle manifests as a trade-off between the precision with which a signal can be localized in time and frequency domains. This principle is closely related to the Heisenberg Uncertainty Principle in quantum mechanics.

The lastest years, the uncertainty principles was investigate in many setting such as free metaplectic transformation [18], quadratic-phase Fourier transforms [15], linear canonical Fourier-Bessel wavelet transform [2], linear canonical Dunkl setting [11], and in Weinstein setting [12,13].

In recent years, the behavior of Weinstein transform was investigated by many researchers, in relation to different problems already studied in classical Fourier transform. For instance, Wigner and Weyl transform [6,14], wavelet transform [9,10], reproducing kernels [8], pseudo differential operators [17], inequalities ans uncertainty principles [4,5,7].

In this paper, we explore the localization of ν-entropy within the Weinstein Gabor transform. By harnessing the ν-entropy, we introduce a fresh perspective on the Heisenberg uncertainty principle specifically tailored for the Weinstein Gabor transform. Additionally, we broaden our investigation

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by delving into an L^r variant of the Heisenberg uncertainty principle applicable to the Weinstein Gabor transform. we prove an L^r local uncertainty inequalities in the Weinstein setting.

The layout of this article is as follows. In section 2 we give a brief overview of the Weinstein Gabor transform and some of its fundamental properties. In section 3, our study focuses on determining the localization of the ν-entropy associated to the Weinstein Gabor transform. By employing the ν-entropy, we introduce a different formulation of the Heisenberg uncertainty principle for the Weinstein Gabor transform. In the last section 4, we extend our study by elaborating on an L^r version of the Heisenberg uncertainty principle for the Weinstein Gabor transform.

2. Preliminaires

2.1. **Weinstein transform.** The Weinstein operator Δ_ν^d defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\infty)$, by

$$\Delta_\nu^d = \Delta_d + L_\nu,\ \nu > -1/2,$$

where Δ_d denoted the Laplacian operator on $\mathbb{R}^d$ and L_ν represent the Bessel operator for the last variable given on $(0,\infty)$ by

$$L_\nu w = \frac{\partial^2 g}{\partial x_{d+1}^2} + \frac{2\nu+1}{x_{d+1}}\frac{\partial g}{\partial x_{d+1}}.$$

For all $\lambda = (\lambda_1, ..., \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the following system of equations

$$\frac{\partial^2 g}{\partial x_j^2}(x) = -\lambda_j^2 g(x), \quad \text{if } 1 \le j \le d$$

$$L_\nu g(x) = -\lambda_{d+1}^2 g(x),$$

$$g(0) = 1, \quad \frac{\partial g}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial g}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \le j \le d$$

has a unique solution indicated by $\mathscr{K}_\nu(\lambda, .)$, and denoted by

$$\mathscr{K}_\nu(\lambda, x) = e^{-i<x',\lambda'>} j_\nu(x_{d+1}\lambda_{d+1}) \tag{2.1}$$

where $\lambda = (\lambda', \lambda_{d+1}),\ x = (x', x_{d+1})$ and j_ν represent the normalized Bessel function defined by

$$j_\nu(x) = \nu(\nu+1)\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \nu(\nu+k+1)}.$$

$(\lambda, x) \mapsto \mathscr{K}_\nu(\lambda, x)$ is named the Weinstein kernel and satisfies for all $(\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$

$$\left|\mathscr{K}_\nu(\lambda, x)\right| \le 1. \tag{2.2}$$

Through this paper, we note by $L_\nu^r(\mathbb{R}_+^{d+1}),\ 1 \le r \le \infty$, the space of all measurable functions g on $\mathbb{R}_+^{d+1}$ satisfying

$$\|g\|_{\nu,r} = \left(\int_{\mathbb{R}_+^{d+1}} \left|g(x)\right|^r d\varpi_\nu(x)\right)^{1/r} < \infty,\ r \in [1,\infty),$$

$$\|g\|_{\nu,\infty} = \text{ess}\sup_{x\in\mathbb{R}_+^{d+1}} \left|g(x)\right| < \infty,$$

where $d\varpi_\nu(x)$ denote measure on $\mathbb{R}^{d+1}_+ = \mathbb{R}^n \times (0,\infty)$ defined by

$$d\varpi_\nu(x) = \frac{x_{d+1}^{2\nu+1}}{(2\pi)^{\frac{d}{2}} 2^\nu \Gamma^2(\nu+1)} dx.$$

If $g \in L^1_\nu(\mathbb{R}^{d+1}_+)$ is radial function then $\tilde{u}$ defined on $\mathbb{R}_+$ by $g(x) = \tilde{g}(|x|)$, for all $x \in \mathbb{R}^{d+1}_+$, is integrable function with respect to $r^{2\nu+d+1}dr$, and we have the equality

$$a_\nu \int_0^\infty \tilde{g}(r) r^{2\nu+d+1} dr = \int_{\mathbb{R}^{d+1}_+} g(x) d\varpi_\nu(x), \tag{2.3}$$

where a_ν is a constant given by

$$a_\nu = \frac{1}{2^{\beta/2-1}\Gamma(\beta/2)}, \tag{2.4}$$

with

$$\beta = 2\nu + d + 2. \tag{2.5}$$

The Weinstein (Laplace Bessel) Fourier transform is a hybrid integral transformation defined for $g \in L^1_\nu(\mathbb{R}^{d+1}_+)$ by

$$\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \mathcal{F}_\nu(g)(\lambda) = \int_{\mathbb{R}^{d+1}_+} g(x) \mathscr{K}_\nu(x,\lambda) d\varpi_\nu(x).$$

From [7], we list the next properties which are useful in the rest of this paper:

- If $g \in L^1_\nu(\mathbb{R}^{d+1}_+)$, then $\mathcal{F}_\nu(g)$ is continuous on $\mathbb{R}^{d+1}_+$ such that
$$\left\|\mathcal{F}_\nu(g)\right\|_{\nu,\infty} \leq \left\|g\right\|_{\nu,1}. \tag{2.6}$$
- For all $g \in L^2_\nu(\mathbb{R}^{d+1}_+)$, we have
$$\left\|\mathcal{F}_\nu(g)\right\|_{\nu,2} = \left\|g\right\|_{\nu,2}. \tag{2.7}$$

For a function $g \in \mathcal{S}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$ the generalized translation $\tau^\nu_x g$ is defined by the following relation [3]

$$\mathcal{F}_\nu(\tau^\nu_x g)(y) = \Lambda^d_\nu(x,y)\mathcal{F}_\nu(g)(y). \tag{2.8}$$

The Weinstein translation operator satisfies the following properties (see [3]):

Proposition 2.1. *The translation operator τ^ν_x, $x \in \mathbb{R}^{d+1}_+$ satisfies the following properties.*

(1) *For $g \in \mathbb{C}_*(\mathbb{R}^{d+1})$, we have for all $x, y \in \mathbb{R}^{d+1}_+$*
$$\tau^\nu_x g(y) = \tau^\nu_y g(x) \text{ and } \tau^\nu_0 g = g. \tag{2.9}$$

(2) *Let $g \in L^r_\nu(\mathbb{R}^{d+1}_+)$, $1 \leq r \leq \infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $\tau^\nu_x g$ belongs to $L^r_\nu(\mathbb{R}^{d+1}_+)$ and we have*
$$\left\|\tau^\nu_x g\right\|_{\nu,r} \leq \left\|g\right\|_{\nu,r}. \tag{2.10}$$

The generalized convolution product $g * h$ of two functions g and h in $L^1_\nu(\mathbb{R}^{d+1}_+)$ is given by (see [3])

$$g * h(x) = \int_{\mathbb{R}^{d+1}_+} \tau^\nu_x g(-y) h(y) d\varpi_\nu(y). \tag{2.11}$$

2.2. **Continuous Weinstein Gabor Transform.** Along this paper, we will denote by $L^r_\nu(E)$, for all $r \in [1,\infty]$, the space of all measurable functions g on $E = \mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+$ with respect to the following measure

$$d\varpi_\nu(x,y) = d\varpi_\nu(x)d\varpi_\nu(y),$$

and provided with the below norm

$$\|g\|_{L^r_\nu(E)} = \left(\int_E |g(x,y)|^r d\varpi_\nu(x,y)\right)^{1/r} < \infty,\ r \in [1,\infty),$$

$$\|g\|_{L^\infty_\nu(E)} = \operatorname{ess}\sup_{(x,y)\in E} |g(x,y)| < \infty.$$

The modulation of function h in $L^2_\nu(\mathbb{R}^{d+1}_+)$ by $t \in \mathbb{R}^{d+1}_+$, is define as below

$$\mathcal{M}_t h := h_t := \mathcal{F}_\nu \sqrt{\tau^\nu_t(|h|^2)}, \tag{2.12}$$

where τ^ν is the Weinstein translation operator, defined by (2.8). For all $t, y \in \mathbb{R}^{d+1}_+$, we consider the following family of functions $h_{t,y}$:

$$h_{t,y}(x) = \tau^\nu_{-y} h_t, \quad \forall x \in \mathbb{R}^{d+1}_+. \tag{2.13}$$

Let h be a function belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$. The continuous Weinstein Gabor transform is defined for a function g belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$ as below:

$$\mathsf{G}_h(g)(y,t) = \int_{\mathbb{R}^{d+1}_+} g(x)\overline{h_{t,y}(x)}d\varpi_\nu(x). \tag{2.14}$$

The above statement can be alternatively expressed as

$$\mathsf{G}_h(g)(y,t) = g * \overline{\check{h}_t}(y), \tag{2.15}$$

where $\check{h}(s) = h(-s)$.

The continuous Weinstein Gabor transform satisfies the subsequent properties.

Proposition 2.2. *[1]*

(1) *Let g and h be two functions belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$. Then, we have*

$$\|\mathsf{G}_h g\|_{L^\infty_\nu(E)} \le \|g\|_{\nu,2}\|h\|_{\nu,2}. \tag{2.16}$$

(2) *Let g and h be two functions belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$. The continuous Weinstein Gabor transform satisfies the below Plancherel-type formula:*

$$\|\mathsf{G}_h g\|_{L^2_\nu(E)} = \|g\|_{\nu,2}\|h\|_{\nu,2}. \tag{2.17}$$

By Riesz-Thorin's interpolation Theorem, yields the following result.

Proposition 2.3. *Suppose that g and h be two functions belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$ and $r \in [2,\infty]$. Then, we have*

$$\|\mathsf{G}_h g\|_{L^r_\nu(E)} \le \|g\|_{\nu,2}\|h\|_{\nu,2}. \tag{2.18}$$

Proposition 2.4. *[5] Let g and h be two functions belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$. For all $\lambda > 0$ and $(s,t) \in \mathbb{R}^{d+1}_+$, we have*

$$\mathsf{G}_{h_{\frac{1}{\lambda}}}(g_\lambda)(y,t) = \mathsf{G}_h(g)(\frac{y}{\lambda}, \lambda t). \tag{2.19}$$

3. HUP via the κ-Entropy for the Weinstein Gabor Transform

A probability density function ρ defined on the space E is a measurable function on E that is non-negative and satisfies

$$\int_E \rho(y,t) d\varpi_\nu(y,t) = 1.$$

According to Shannon's definition [16], the ν-entropy of a probability density function ρ on E can be defined as

$$\mathcal{E}_\nu(\rho) := -\int_E \ln(\rho(y,t))\rho(y,t) d\varpi_\nu(y,t),$$

whenever the integral on the right hand side is well defined.

The main objective of this section is to examine the localization of the ν-entropy of the Weinstein Gabor transform within the space E. In fact, we provide the following result.

Proposition 3.1. *Let g and h be two functions in $L^2_\nu(\mathbb{R}^{d+1}_+)$ such that g is nonzero function. Then we have*

$$\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) \geq -2\ln\left(\left\|g\right\|_{\nu,2} \|h\|_{\nu,2}\right) \left\|g\right\|^2_{\nu,2} \|h\|^2_{\nu,2}. \tag{3.1}$$

Proof. We suppose that $\|g\|_{\nu,2} = \|h\|_{\nu,2} = 1$, then according to inequality (2.16), we obtain that

$$\forall (y,t) \in E, \quad \left|\mathsf{G}_h(g)(y,t)\right| \leq \left\|g\right\|_{\nu,2} \|h\|_{\nu,2} = 1.$$

It is obviously that $\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) \geq 0$. If the entropy $\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) = \infty$, then the inequality (3.1) holds. Now, we assume that the entropy $\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) < \infty$. Suppose that g and h be two functions in $L^2_\nu(\mathbb{R}^{d+1}_+)$ such that g is nonzero function and we put

$$u = \frac{g}{\left\|g\right\|_{\nu,2}} \quad \text{and} \quad v = \frac{h}{\|h\|_{\nu,2}}.$$

Therefore, u and v are in $L^2_\nu(\mathbb{R}^{d+1}_+)$ and $\|u\|_{\nu,2} = \|v\|_{\nu,2} = 1$, in consequence

$$\mathcal{E}_\nu(|\mathsf{G}_v(u)|^2) \geq 0.$$

Nevertheless,

$$\mathsf{G}_v(u) = \frac{1}{\left\|g\right\|_{\nu,2} \|h\|_{\nu,2}} \mathsf{G}_h(g).$$

It follows that

$$\mathcal{E}_\nu(|\mathsf{G}_v(u)|^2) = \frac{1}{\left\|g\right\|^2_{\nu,2} \|h\|^2_{\nu,2}} \mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) + 2\ln\left(\left\|g\right\|_{\nu,2} \|h\|_{\nu,2}\right).$$

Ultimately, it follows that

$$\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) \geq -2\ln\left(\left\|g\right\|_{\nu,2} \|h\|_{\nu,2}\right) \left\|g\right\|^2_{\nu,2} \|h\|^2_{\nu,2}.$$

□

By utilizing the previous ν-entropy associated with the Weinstein Gabor transform, it is possible to derive an alternative form of the Heisenberg uncertainty principle specifically tailored for G_h.

Theorem 3.1. *For any two positive real numbers, denoted as r and s, there exists a positive constant $K_{r,s}(\nu)$ such that for every g and h in $L^2_\nu(\mathbb{R}^{d+1}_+)$, the following inequality holds:*

$$\|g\|^2_{\nu,2}\|h\|^2_{\nu,2} \leq \frac{1}{K_{r,s}(\nu)}\left(\int_E |y|^r \left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t)\right)^{\frac{s}{r+s}} \times\left(\int_E |t|^s \left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t)\right)^{\frac{r}{r+s}} \tag{3.2}$$

where

$$K_{r,s}(\nu) = \frac{\beta}{r^{\frac{s}{r+s}}s^{\frac{r}{r+s}}}e^{\Phi(r,s)}, \tag{3.3}$$

with

$$\Phi(r,s) = rs\frac{\ln\left(\frac{rs}{a^2_\nu\Gamma(\beta/r)\Gamma(\beta/s)}\right)}{\beta(r+s)} - 1, \tag{3.4}$$

here a_ν is given by the identity (2.4).

Proof. Suppose that $\|g\|_{\nu,2} = \|h\|_{\nu,2} = 1$. For all positive real numbers z, r, s, we put the function $\Psi^z_{r,s}$ defined on E as below:

$$\Psi^z_{r,s}(y,t) = \frac{rse^{-\frac{|y|^2+|t|^2}{z}}}{a^2_\nu\Gamma(\beta/r)\Gamma(\beta/s)z^{\frac{\beta(r+s)}{rs}}}.$$

With a straightforward application of calculus, it becomes evident that

$$\int_E \Psi^z_{r,s}(y,t)d\varpi_\nu(y,t) = 1.$$

We deduce that the measure $dm^{z,\nu}_{r,s}(y,t) = \Psi^z_{r,s}(y,t)d\varpi_\nu(y,t)$ is a probability measure on the space E. According to the convexity of the function $w(t) = t\ln(t)$ over $(0,\infty)$ and by using Jensen's inequality, we obtain

$$\int_E \frac{\left|\mathsf{G}_h(g)(y,t)\right|^2}{\Psi^z_{r,s}(y,t)}\ln\left(\frac{\left|\mathsf{G}_h(g)(y,t)\right|^2}{\Psi^z_{r,s}(y,t)}\right)dm^{z,\nu}_{r,s}(y,t) \geq 0.$$

which implies, in terms of ν-entropy, that for any positive real numbers z, r, s, we have the following inequality

$$\mathcal{E}_\nu(|\mathsf{G}_h(g)|^2) + \ln\left(\frac{rs}{a^2_\nu\Gamma(\beta/r)\Gamma(\beta/s)}\right)\|g\|^2_{\nu,2}\|h\|^2_{\nu,2} \leq \ln\left(z^{\frac{\beta(r+s)}{rs}}\right)\|g\|^2_{\nu,2}\|h\|^2_{\nu,2} + \frac{1}{z}\int_E(|y|^r+|t|^s)|\mathsf{G}_h(g)(y,t)|^2 d\varpi_\nu(y,t).$$

Hence, according to Proposition 3.1, we obtain

$$z\left[\ln\left(\frac{rs}{a_\nu^2\Gamma(\beta/r)\Gamma(\beta/s)}\right)-\ln\left(z^{\frac{\beta(r+s)}{rs}}\right)\right]\left\|\mathsf{G}_h g\right\|_{L^2_\nu(E)}$$
$$\leq \int_E (|y|^r+|t|^s)|\mathsf{G}_h(g)(y,t)|^2 d\varpi_\nu(y,t).$$

Nevertheless, below expression

$$z\left[\ln\left(\frac{rs}{a_\nu^2\Gamma(\beta/r)\Gamma(\beta/s)}\right)-\ln\left(z^{\frac{\beta(r+s)}{rs}}\right)\right]\left\|\mathsf{G}_h g\right\|_{L^2_\nu(E)},$$

reaches its maximum value at $z_0 = e^{\Phi(r,s)}$. Therefore, we have

$$C_{r,s}(\nu)\left\|g\right\|_{\nu,2}^2 \|h\|_{\nu,2}^2 \leq \int_E (|y|^r+|t|^s)|\mathsf{G}_h(g)(y,t)|^2 d\varpi_\nu(y,t),$$

where

$$C_{r,s}(\nu)=\frac{\beta(r+s)}{rs}e^{\Phi(r,s)}.$$

Thus, for all g and h in $L^2_\nu(\mathbb{R}^{d+1}_+)$, we have

$$\left\|g\right\|_{\nu,2}^2 \|h\|_{\nu,2}^2 \leq \frac{1}{C_{r,s}(\nu)}\int_E |y|^r\left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t) \times \int_E |t|^s\left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t). \tag{3.5}$$

In other hand, we have for all $\lambda>0$, the dilates functions g_λ and $h_{\frac{1}{\lambda}}$ belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$. Moreover, $h_{\frac{1}{\lambda}}$ is a nonzero function. Now, according to relation (3.5), we get

$$\left\|g_\lambda\right\|_{\nu,2}^2 \left\|h_{\frac{1}{\lambda}}\right\|_{\nu,2}^2 \leq \frac{1}{C_{r,s}(\nu)}\int_E |y|^r\left|\mathsf{G}_{h_{\frac{1}{\lambda}}}(g_\lambda)(y,t)\right|^2 d\varpi_\nu(y,t) \times \int_E |t|^s\left|\mathsf{G}_{h_{\frac{1}{\lambda}}}(g_\lambda)(y,t)\right|^2 d\varpi_\nu(y,t).$$

Furthermore, we have

$$\left\|g_\lambda\right\|_{\nu,2}^2=\left\|g\right\|_{\nu,2}^2, \quad \text{and} \quad \|h_{\frac{1}{\lambda}}\|_{\nu,2}^2=\|h\|_{\nu,2}^2.$$

Thus, according to relation (2.19), we obtain

$$\left\|g\right\|_{\nu,2}^2 \|h\|_{\nu,2}^2 \leq \frac{1}{C_{r,s}(\nu)}\lambda^{-r}\int_E |y|^r\left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t) \times\lambda^s\int_E |t|^s\left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t).$$

Specifically, the inequality is valid at the critical point

$$\lambda=\left(\frac{r\int_E |y|^r\left|\mathsf{G}_h(g)(y,t)\right|^2 d\varpi_\nu(y,t)}{s\int_E |t|^s\left|\mathsf{G}_h(g)(y,t)\right|^2 , d\varpi_\nu(y,t)}\right)^{\frac{1}{r+s}}.$$

This implies that

$$\left\|g\right\|_{\nu,2}^{2}\|h\|_{\nu,2}^{2}\leq\frac{1}{K_{r,s}(\nu)}\left(\int_{E}|y|^{r}\left|\mathsf{G}_{h}(g)(y,t)\right|^{2}d\varpi_{\nu}(y,t)\right)^{\frac{s}{r+s}}\times\left(\int_{E}|t|^{s}\left|\mathsf{G}_{h}(g)(y,t)\right|^{2}d\varpi_{\nu}(y,t)\right)^{\frac{r}{r+s}},$$

where

$$K_{r,s}(\nu)=C_{r,s}(\nu)\frac{r^{\frac{r}{r+s}}s^{\frac{s}{r+s}}}{r+s}=\frac{\beta}{r^{\frac{s}{r+s}}s^{\frac{r}{r+s}}}e^{\Phi(r,s)}.$$

□

4. L^p HUP for the Weinstein Gabor Transform

Let $\lambda>0$. We consider the function h_λ defined on the space E by

$$h_\lambda=e^{-\lambda|(y,t)|^2},\quad\forall(y,t)\in E.$$

By performing straightforward calculations, it is simple to demonstrate the following lemma.

Lemma 4.1. *Let $\lambda>0$ and $1\leq s<\infty$. Then, there exists a positive constant K, such that we have*

$$\|h_\lambda\|_{L^s_\nu(E)}\leq K\lambda^{\frac{-\beta}{s}}.$$

Lemma 4.2. *Let h be a function in $L^2_\nu(E)$, $r\in(1,2]$ and $a\in(0,\beta/2r')$ where r' denote the conjugate of r. There is a positive constant, denoted as K, such that for every function g in $L^r_\nu(\mathbb{R}^{d+1}_+)$ and $\lambda>0$, we have*

$$\|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}\leq K\|h\|_{\nu,2}\,\lambda^{-2a}\left(\left\||y|^a g\right\|_{\nu,2}+\left\||y|^a g\right\|_{\nu,2r}\right)\tag{4.1}$$

Proof. If the term $\left\||y|^a g\right\|_{\nu,2}+\left\||y|^a g\right\|_{\nu,2r}$ is infinite, then the inequality (4.1) holds. Now, suppose that

$$\left\||y|^a g\right\|_{\nu,2}+\left\||y|^a g\right\|_{\nu,2r}<\infty.$$

Let $\varepsilon>0$, we put $g_\varepsilon=g\chi_{B(0,\varepsilon)}$ and $g^\varepsilon=g-g^\varepsilon$. Then, taking account to below inequality

$$|g^\varepsilon(y)|\leq\varepsilon^{-a}\left||y|^a g(y)\right|,$$

and according to the relation (2.18), we obtain

$$\begin{aligned}\|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g\chi_{B^c(0,\varepsilon)})\|_{L^{r'}_\nu(E)}&\leq\|e^{-\lambda|(y,t)|^2}\|_{L^\infty_\nu(E)}\|\mathsf{G}_h(g\chi_{B^c(0,\varepsilon)})\|_{L^{r'}_\nu(E)}\\&\leq\|h\|_{\nu,2}\left\|g\chi_{B^c(0,\varepsilon)}\right\|_{\nu,2}\\&\leq\|h\|_{\nu,2}\,\varepsilon^{-a}\left\||y|^a g\right\|_{\nu,2}.\end{aligned}$$

However, according to relation (2.16) and Hölder inequality, we get

$$\begin{aligned}\|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g\chi_{B^c(0,\varepsilon)})\|_{L^{r'}_\nu(E)}&\leq\|e^{-\lambda|(y,t)|^2}\|_{L^{r'}_\nu(E)}\|\mathsf{G}_h(g\chi_{B^c(0,\varepsilon)})\|_{L^\infty_\nu(E)}\\&\leq\|h\|_{\nu,2}\|e^{-\lambda|(y,t)|^2}\|_{L^{r'}_\nu(E)}\left\|g\chi_{B^c(0,\varepsilon)}\right\|_{\nu,2}\\&\leq\|h\|_{\nu,2}\|e^{-\lambda|(y,t)|^2}\|_{L^{r'}_\nu(E)}\left\||y|^{-a}\chi_{B^c(0,\varepsilon)}\right\|_{\nu,2r'}\left\||y|^a g\right\|_{\nu,2r}.\end{aligned}$$

By a simple calculus give that there exists a positive constant K, such that

$$\left\| |y|^{-a}\chi_{B^c(0,\varepsilon)} \right\|_{\nu,2r'} = K\varepsilon^{-a+\frac{\beta}{2r'}}.$$

Therefore,

$$\begin{aligned} \|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} &\le \|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g_\varepsilon)\|_{L^{r'}_\nu(E)}\|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g^\varepsilon)\|_{L^{r'}_\nu(E)} \\ &\le K\varepsilon^{-a}\|h\|_{\nu,2}\left(\left\||y|^a g\right\|_{\nu,2} + \varepsilon^{\frac{\beta}{2r'}}\|h_\lambda\|_{L^{r'}_\nu(E)}\left\||y|^a g\right\|_{\nu,2r}\right). \end{aligned}$$

In the end, we obtain the desired result by choosing $\varepsilon = \lambda^2$. □

Theorem 4.1. *Let h be a function in $L^2_\nu(E)$, $r \in (1,2]$, $a \in (0, \beta/2r')$ where r' denote the conjugate of r and $b > 0$. There is a positive constant, denoted as K, such that for every function g in $L^r_\nu(\mathbb{R}^{d+1}_+)$, we have*

$$\|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \le K\|h\|_{\nu,2}^{\frac{b}{a+b}}\left(\left\||y|^a g\right\|_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right)^{\frac{b}{a+b}}\||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}^{\frac{a}{a+b}}. \tag{4.2}$$

Proof. Let $r \in (1,2]$ and $a \in (0, \beta/2r')$. Suppose that $b \le 1/2$. According to previous lemma, we have for all $\lambda > 0$

$$\begin{aligned} \|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} &\le \|e^{-\lambda|(y,t)|^2}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} + \|(1-e^{-\lambda|(y,t)|^2})\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \\ &\le K\|h\|_{\nu,2}\lambda^{-2a}\left(\left\||y|^a g\right\|_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right) \\ &+ \|(1-e^{-\lambda|(y,t)|^2})\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}. \end{aligned}$$

On the other hand, we have

$$\|(1-e^{-\lambda|(y,t)|^2})\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} = \lambda^{2b}\|(\lambda|(y,t)|^2)^{-2b}(1-e^{-\lambda|(y,t)|^2})|(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}.$$

Furthermore, the function $w \to (1-e^{-w})w^{-2b}$ is bounded for b □0 and $b \le 1/2$. Then, we get

$$\|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \le K\|h\|_{\nu,2}\lambda^{-2a}\left(\left\||y|^a g\right\|_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right) + \lambda^{2b}\||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}.$$

By optimizing with respect to λ in the above inequality, we derive the result (4.3) for all $r \in (1,2]$ and $a \in (0, \beta/2r')$.

Now, we suppose that $b > 1/2$ and let $b' \ge 1/2$. For all $w \ge 0$ and $b' < b$, we have

$$w^{4b'} \le 1 + w^{4b}.$$

By choosing $w = \frac{|(y,t)|}{\sigma}$, we have

$$\frac{|(y,t)|}{\sigma}^{4b'} \le 1 + \frac{|(y,t)|}{\sigma}^{4b}, \quad \forall \sigma > 0.$$

Therefore,

$$\||(y,t)|^{4b'}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \le \sigma^{4b'}\|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} + \sigma^{4(b'-b)}\||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}.$$

By optimizing with respect to σ in the above inequality, we get

$$\||(y,t)|^{4b'}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \le \|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}^{\frac{b-b'}{b}}\||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}^{\frac{b'}{b}}.$$

According to (4.3) for b', we obtain the result for $b > 1/2$. □

Using the above theorem for $r = 2$, and according Plancherel formula for the continuous Weinstein Gabor transform (2.17), we obtain the following result.

Corollary 4.1. *Let h be a nonzero function belongs to $L^2_\nu(E)$, $a \in (0, \beta/2r')$ and $b > 0$. There is a positive constant, denoted as K, such that for every function g in $L^2_\nu(\mathbb{R}^{d+1}_+)$, we have*

$$\|\mathsf{G}_h(g)\|_{L^2_\nu(E)} \leq K\,\|h\|^{\frac{a}{a+b}}_{\nu,2}\left(\left\||y|^a g\right\|_{\nu,2} + \left\||y|^a g\right\|_{\nu,4}\right)^{\frac{b}{a+b}} \||(y,t)|^{4b}\mathsf{G}_h(g)\|^{\frac{a}{a+b}}_{L^2_\nu(E)}. \tag{4.3}$$

Definition 4.1. (1) *Let X be a measurable set of $\mathbb{R}^{d+1}_+$. We define a projection operator P_X as follows*

$$P_X g(t) = \begin{cases} g(t), & \text{if } t \in X \\ 0, & \text{if } t \notin X. \end{cases}$$

For $0 \leq \varepsilon_X < 1$, we say that g is concentrated on the subset X in $L^2_\nu(\mathbb{R}^{d+1}_+)$-norm, if

$$\left\|g - P_X g\right\|_{\nu,2} \leq \varepsilon_X \left\|g\right\|_{\nu,2}.$$

(2) *Let h be a function belongs to $L^2_\nu(E)$ and Ω be a measurable subset of E. We define a projection operator S_Ω as below*

$$S_\Omega(g) = (\mathsf{G}_h)^{-1}\left(P_X(\mathsf{G}_h(g))\right). \tag{4.4}$$

Let $0 \leq \varepsilon_\Omega < 1$. We say that G_h is ε_Ω-concentrated on the subset E in $L^{r'}_\nu(E)$-norm for $1 \leq r \leq 2$, if we have

$$\|\mathsf{G}_h(g) - \mathsf{G}_h(S_\Omega(g))\|_{L^{r'}_\nu(E)} \leq \varepsilon_\Omega \|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}.$$

Definition 4.2. *Let h be a function belongs to $L^2_\nu(E)$, $b > 0$, g be a function belongs to $L^r_\nu(\mathbb{R}^{d+1}_+)$ with $1 \leq r \leq 2$, Ω be a measurable subset of E and $0 \leq \varepsilon_\Omega < 1$. We say that $|(y,t)|^{4b}\mathsf{G}_h(g)$ is ε_Ω-concentrated on the subset E in $L^{r'}_\nu(E)$-norm, if there exists a function f vanishing outside Ω such that*

$$\||(y,t)|^{4b}\mathsf{G}_h(g) - f\|_{L^{r'}_\nu(E)} \leq \varepsilon_\Omega \||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}. \tag{4.5}$$

According to the definition of projection operator S_Ω, we have the following result.

Lemma 4.3. *$|(y,t)|^{4b}\mathsf{G}_h$ is ε_Ω-concentrated on the subset E in $L^{r'}_\nu(E)$-norm, if we have*

$$\||(y,t)|^{4b}\mathsf{G}_h(g) - |(y,t)|^{4b}\mathsf{G}_h(S_\Omega(g))\|_{L^{r'}_\nu(E)} \leq \varepsilon_\Omega \||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)}. \tag{4.6}$$

Corollary 4.2. *Let h be a function belongs to $L^2_\nu(E)$, $b > 0$, g be a function belongs to $L^r_\nu(\mathbb{R}^{d+1}_+)$ with $1 < r \leq 2$, Ω be a measurable subset of E and $0 \leq \varepsilon_\Omega < 1$. If $|(y,t)|^{4b}\mathsf{G}_h$ is ε_Ω-concentrated on the subset E in $L^{r'}_\nu(E)$-norm, then we have for all $a \in (0, \beta/2r')$*

$$\begin{aligned}(1-\varepsilon_\Omega)^{\frac{a}{a+b}}\,\|\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \leq & K\,\|h\|^{\frac{b}{a+b}}_{\nu,2}\left(\left\||y|^a g\right\|^{\frac{b}{a+b}}_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right)^{\frac{b}{a+b}} \\ & \||(y,t)|^{4b}\mathsf{G}_h(S_\Omega(g))\|^{\frac{a}{a+b}}_{L^{r'}_\nu(E)}.\end{aligned}$$

Proof. Let h be a function belongs to $L^2_\nu(E)$, $b > 0$, g be a function belongs to $L^r_\nu(\mathbb{R}^{d+1}_+)$ with $1 < r \le 2$, Ω be a measurable subset of E and $0 \le \varepsilon_\Omega < 1$. Since $|(y,t)|^{4b}\mathsf{G}_h$ is ε_Ω-concentrated on the subset E in $L^{r'}_\nu(E)$-norm, then we have

$$\||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} \le \varepsilon_\Omega \||(y,t)|^{4b}\mathsf{G}_h(g)\|_{L^{r'}_\nu(E)} + \||(y,t)|^{4b}\mathsf{G}_h(S_\Omega(g))\|_{L^{r'}_\nu(E)}.$$

Therefore,

$$(1-\varepsilon_\Omega)^{\frac{a}{a+b}} \|\mathsf{G}_h(g)\|^{\frac{a}{a+b}}_{L^{r'}_\nu(E)} \le \||(y,t)|^{4b}\mathsf{G}_h(S_\Omega(g))\|^{\frac{a}{a+b}}_{L^{r'}_\nu(E)}.$$

Multiply the last inequality by the following term

$$K\|h\|^{\frac{b}{a+b}}_{\nu,2}\left(\left\||y|^a g\right\|^{\frac{b}{a+b}}_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right)^{\frac{b}{a+b}},$$

and according to Theorem 4.1, we get the desired result. □

Corollary 4.3. *Let h be a nonzero function belongs to $L^2_\nu(E)$, $b > 0$, g be a function belongs to $L^2_\nu(\mathbb{R}^{d+1}_+)$, Ω be a measurable subset of E and $0 \le \varepsilon_\Omega < 1$. If $|(y,t)|^{4b}\mathsf{G}_h$ is ε_Ω-concentrated on the subset E in $L^2_\nu(E)$-norm, then we have for all $a \in (0, \beta/4)$*

$$(1-\varepsilon_\Omega)^{\frac{a}{a+b}} \|g\|_{\nu,2} \le K\|h\|^{\frac{a}{a+b}}_{\nu,2}\left(\left\||y|^a g\right\|^{\frac{b}{a+b}}_{\nu,2} + \left\||y|^a g\right\|_{\nu,2r}\right)^{\frac{b}{a+b}} \||(y,t)|^{4b}\mathsf{G}_h(S_\Omega(g))\|^{\frac{a}{a+b}}_{L^2_\nu(E)}.$$

Proof. We follow the same steps as the previous corollary and according to Corollary 4.1, we obtain the desired result. □

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