Int. J. Anal. Appl. (2024), 22:175

International Journal of Analysis and Applications

Periodically and Stability Properties of a Higher Order Rational Difference Equation

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Abstract. The aim of this research is to study the local, global, and boundedness of the difference equation

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{p_2 T_{\eta-l_1}}{T_{\eta-m_2}} + ... + \frac{p_s T_{\eta-l_1}}{T_{\eta-m_s}},$$

where $l_1, m_1, m_2, ..., m_s, s$, are positive real numbers. It also studies periodic solutions of special case of this equation. Finally, numerical examples are given to confirm results.

1. Introduction

The main goal of this study is to investigate the poropties of solutions such as boundedness, local stability and global stability of the difference equation

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{p_2 T_{\eta-l_1}}{T_{\eta-m_2}} + ... + \frac{p_s T_{\eta-l_1}}{T_{\eta-m_s}}, \quad \eta = 0, 1, ..., \tag{1.1}$$

such that $l_1, m_1, m_2, ..., m_s, s$, and the initial values T_{-l_1}, T_{-m_1}, T_{-m_2}, T_{-m_s} are arbitrary positive real numbers. In addition, we study periodic solutions for special case of above equations. Numerical examples are given to confirm results.

Many researchers find the study of difference equations interesting and fruitful because it supports the analysis of modeling in various phenomena in life [15]. For example, Elsayed [15] study third and second periodic solution of the difference equation given by

$$T_{\eta+1} = a + \frac{bT_\eta}{T_{\eta-1}} + \frac{bT_{\eta-1}}{T_\eta}.$$

El-Metwally et a studied the global attractivity and the periodic character of some difference equation

$$T_{\eta+1} = \frac{T_{\eta-(2k+1)} + p}{T_{\eta-(2k+1)} + qT_{\eta-2l}}.$$

Received: Mar. 28, 2024.

2020 *Mathematics Subject Classification.* 39A10.

Key words and phrases. difference equations, stability, boundedness, solution of difference equation.

https://doi.org/10.28924/2291-8639-22-2024-175

ISSN: 2291-8639

Camouzis et al. in [3] investigated ,the dynamics of following difference equation

$$T_{\eta+1} = \frac{\alpha + \gamma T_{\eta-1} + \delta T_{\eta-2}}{A + T_{\eta-2}}$$

The global attractivity and local stability of the difference equation

$$T_{\eta+1} = \frac{T_{\eta-1}}{c + dT_{\eta-1}T_{\eta-2}},$$

have investigated by Yang et al. [24].

Khaliq et al. [17] studies the dynamical behavior of solutions of the seventh order difference equation

$$T_{\eta+1} = aT_{\eta-3} + \frac{\alpha T_{n-3}T_{\eta-7}}{\beta T_{\eta-3} + \gamma T_{\eta-7}}.$$

Elabbasy et al. [11] studied the qualitative properties of the difference equation

$$T_{\eta+1} = aT_{\eta} + \frac{\alpha T_{n}}{\beta T_{\eta} + \gamma T_{\eta-1}},$$

Dilip, et al. [7] studied the behavior of solutions of difference equation

$$T_{\eta+1} = a + \alpha T_{\eta-1}\lambda^{-T_n}.$$

Another associated papers on rational difference equations see [1-24].

2. Behavior of the Solutions of Eq. (1.1)

In this secion we investigated the behavior of the solution of Eq. (1.1),

2.1. Local Stability.

In this subsection we investigate the local stability character of the solutions of Eq. (1.1).

Theorem 2.1. *Assume that $p_1 + p_2 + ... + p_s < r$,then the equiliribum point $\bar{T} = r + p_1 + p_2 + ... + p_s$, of Eq. (1.1) is Locally asymptotically stable.*

proof: The equilibrium point of Eq. (1.1) is given by

$$\bar{T} = r + p_1 + p_2 + ... + p_s. \tag{2.1}$$

Define a function $g : (0, \infty) \to (0, \infty)$ as

$$g(x_1, y_1, y_2, ..., y_s) = r + \frac{p_1 x_1}{y_1} + \frac{p_2 x_1}{y_2} + ... + \frac{p_s x_1}{y_s}.$$

Hence we obtain,

$$\frac{\partial g}{\partial x_1}(x_1, y_1, y_2, ..., y_s) = \frac{p_1}{y_1} + \frac{p_2}{y_2} + ... + \frac{p_s}{y_s},$$
$$\frac{\partial g}{\partial y_1}(x_1, y_1, y_2, ..., y_s) = -\frac{p_1 x_1}{y_1^2},$$
$$\frac{\partial g}{\partial y_2}(x_1, y_1, y_2, ..., y_s) = -\frac{p_2 x_2}{y_2^2}, ...,$$
$$, ...,$$
$$\frac{\partial g}{\partial y_s}(x_1, y_1, y_2, ..., y_s)) = -\frac{p_s x_s}{y_s^2}.$$

It follows that

$$\frac{\partial g}{\partial x_1}(\bar{T}, \bar{T}, ..., \bar{T}) = \frac{p_1 + p_2 + ... + p_s}{\bar{T}} = -a_1,$$
$$\frac{\partial g}{\partial y_1}(\bar{T}, \bar{T}, ..., \bar{T}) = -\frac{p_1}{\bar{T}} = -b_1,$$
$$\frac{\partial g}{\partial y_2}(\bar{T}, \bar{T}, ..., \bar{T}) = -\frac{p_2}{\bar{T}} = -b_2, ...,$$
$$, ...,$$
$$\frac{\partial g}{\partial y_s}(\bar{T}, \bar{T}, ..., \bar{T}) = -\frac{p_s}{\bar{T}} = -b_s.$$

Therefore, the linearized equation becomes

$$S_{\eta+1} = a_1 S_{\eta-l_1} + b_1 S_{\eta-m_1} + b_2 S_{\eta-m_2} + ... + b_s S_{\eta-m_s},$$

using Theorem A, we get that the equiliribum point is asympototically stable if

$$|a_1| + |b_1| + |b_1| + ... + |b_1| < 1,$$

and hence

$$p_1 + p_2 + ... + p_s < r,$$

which means the prove is complete.

2.2. **Global Attractor.** In this subsection we investigate the global attractivity character of solutions of Eq. (1.1)

Theorem 2.2. *The equiliribum point of Eq. (1.1) is global Attractor if* $r \neq p_1 + p_2 + ... + p_s$.

proof: Let a, b are real number and define $f : [a, b]^{s+1} \to [a, b]$ a function $f(x_1, y_1, y_2, ..., y_s) = r + \frac{p_1 x_1}{y_1} + \frac{p_2 x_1}{y_2} + ... + \frac{p_s x_2}{y_s}$. Clearly, the function f is increasing in x_1 and decreasing in $y_1, y_2, ..., y_s$, hence

$$N = f(N, n, n, ..., n) \qquad \text{and} \qquad n = f(n, N, ..., N).$$

Hence we get

$$N = r + \frac{p_1 N}{n} + \frac{p_2 N}{n} + ... + \frac{p_s N}{n},$$
$$n = r + \frac{p_1 n}{N} + \frac{p_2 n}{N} + ... + \frac{p_s n}{N},$$

or

$$Nn = rn + \frac{p_1 Nn}{n} + \frac{p_2 Nn}{n} + ... + \frac{p_s Nn}{n},$$
$$nN = rN + \frac{p_1 nN}{N} + \frac{p_2 nN}{N} + ... + \frac{p_s nN}{N},$$

subtracting these two equations, we get

$$0 = (N - n)\left[r - p_1 - p_2 - ... - p_s\right].$$

Under the conditions $r \neq p_1 + p_2 + ... + p_s$, we obtain

$$N = n,$$

we obtain by therom (B) that he equiliribum point $\bar{T}$ of Eq.(1.1) is global Attractor.

2.3. **Boundness of solutions.** In this subsection we study the boundedness of solutions of Eq. (1.1).

Theorem 2.3. *Every solution of Eq. (1.1) is bounded and prsists if* $r > p_1 + p_2 + ... + p_s$,

Proof: Suppose $\{T_n\}_{n=-L}^{\infty}$ be solution of Eq. (1.1). It follows from Eq. (1.1) that

$$T_{n+1} = r + \frac{p_1 T_{n-l_1}}{T_{n-m_1}} + \frac{p_2 T_{n-l_1}}{T_{n-m2}} + + ... + \frac{p_s T_{n-l_1}}{T_{n-m_s}} > r, \tag{2.2}$$

thus

$$T_{n+1} > r, \qquad \text{for} \quad n \geq 0.$$

Also, it follows from Eq. (1.1) that

$$T_{n+1} \leq r + \frac{p_1 T_{n-l_1}}{r} + \frac{p_2 T_{n-l_1}}{r} + ... + \frac{p_s T_{n-l_1}}{r},$$

using Comparisons Theroms, we get

$$\lim_{n\to\infty} subT_n \leq \frac{r^2}{(r - p_1 - p_2 ... - p_s)}.$$

Therefore $\{T_n\}_{n=-L}^{\infty}$ is bounded and persists.

3. Periodic two Solution of Eq. (1.1)

In this section, we investigate the periodic two solutions of special cases of Eq. (1.1). We states theorem that gives us necessary and sufficient conditions of the following equation

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{p_2 T_{\eta-l_1}}{T_{\eta-m_2}}, \qquad \eta = 0, 1, ..., \tag{3.1}$$

where $T_{\eta-2l} = ... = T_{\eta-2} = T_\eta = u$, and $T_{\eta-(2l+1)} = ... = T_{\eta-3} = T_{\eta-1} = v, L = 2l+2$ has a prime period solution of periodic two.

Theorem 3.1. *Assume that l_1, m_2, odd and m_1 even, then Eq. (3.1) has a periodic solution of prime periodic two if and only if $r = p_1 - p_2$, where $c = \frac{u}{v}, u = p_1(1+c), v = p_1\left(\frac{c+1}{c}\right)$ and $c \in R/\{0, \pm 1\}$ such that $u, v, u, v, ...$ is a periodic solution of Eq. (3.1).*

Proof: From Eq. (3.1), we obtain

$$u = r + \frac{p_1 u}{v} + \frac{p_2 u}{u}, \text{and}$$

$$v = r + \frac{p_1 v}{u} + \frac{p_2 v}{v}.$$

Since $c = \frac{u}{v} \neq 0, \pm 1$, hence

$$u = r + p_1 c + p_2, \text{and} \tag{3.2}$$

$$v = r + \frac{p_1}{c} + p_2. \tag{3.3}$$

Then, it follows

$$u - vc = r(1-c) + p_1(c-1) - p_2(1-c) = 0.$$

Since $c \neq 1$, we conclude

$$r = p_1 - p_2,$$

which is the condition of this theorem holds.

Furthermore, we rewrite Eqs. (3.3) and Eq. (3.2) as follows

$$\begin{aligned} u &= p_1 - p_2 + p_1 c + p_2, \\ &= p_1(1+c) \\ v &= p_1 - p_2 + \frac{p_1}{c} + p_2. \\ &= p_1\left(\frac{c+1}{c}\right), \end{aligned} \tag{3.4}$$

and therefore, u, v distinct real numbers. Let $T_{\eta-2l} = ... = T_{\eta-2} = T_\eta = u$, and $T_{\eta-(2l+1)} = ... = T_{\eta-3} = T_{\eta-1} = v$. Acooording Eq. (3.1), we staste

$$T_1 = u, T_2 = v.$$

$$\begin{aligned} T_1 &= r + \frac{p_1 v}{u} + \frac{p_2 v}{v} \\ &= p_1 - p_2 + \frac{p_1}{c} + p_2 \\ &= v, \\ T_2 &= r + \frac{p_1 u}{v} + \frac{p_2 u}{u} \\ &= p_1 - p_2 + p_1 c + p_2, \\ &= u, \end{aligned}$$

Hence simmlar T_1, T_2, we get $T_{2\eta+1} = v, T_{2\eta} = u, \quad \text{for} \;\; \eta \geq 0$, therefore the proof is completed.

Theorem 3.2. *Assume that l_1, m_2, even and m_1 odd, then Eq. (3.1) has a periodic solution of prime periodic two if and only if $r = -p_1\left(\frac{c^2+c+1}{c}\right) - p_2$, where $c = \frac{u}{v}, u = -p_1(1+c), v = -p_1\left(\frac{c+1}{c\nabla}\right)$ and $c \in R/\ \{0, \pm 1\}$ such that $u, v, u, v, \ldots$ is a periodic solution of Eq. (3.1).*

Proof: From Eq. (3.1), we obtain

$$v = r + \frac{p_1 u}{v} + \frac{p_2 u}{u}, \text{and}$$
$$u = r + \frac{p_1 v}{u} + \frac{p_2 v}{v}.$$

Since $c = \frac{u}{v} \neq 0, \pm 1$, hence

$$v = r + p_1 c + p_2, \text{and}$$
$$u = r + \frac{p_1}{c} + p_2.$$

Then, it follows

$$\begin{aligned} u - vc &= r(1-c) + p_1(\frac{1}{c} - c^2) + p_2(1-c) \\ &= r(1-c) + p_1\left(\frac{1-c^3}{c}\right) + p_2(1-c) \\ &= 0. \end{aligned}$$

Since $c \neq 1$, we conclude

$$r = -p_1\left(\frac{c^2+c+1}{c}\right) - p_2,$$

which is the condition of this theorem holds.

Furthermore, we rewrite Eqs. (3.3) and Eq. (3.2) as follows

$$\begin{aligned} u &= -p_1\left(\frac{c^2+c+1}{c}\right) - p_2 + \frac{p_1}{c} + p_2, \\ &= -p_1(1+c), \\ v &= -p_1\left(\frac{c^2+c+1}{c}\right) - p_2 + p_1 c + p_2 \end{aligned}$$

$$= \quad -p_1\left(\frac{c+1}{c}\right).,\tag{3.5}$$

and therefore, u, v distinct real numbers. Let $T_{\eta-2l} = ... = T_{\eta-2} = T_\eta = u$, and $T_{\eta-(2l+1)} = ... = T_{\eta-3} = T_{\eta-1} = v$. Acooording Eq. (3.1), we staste

$$T_1 = v, T_2 = u.$$

$$\begin{aligned}
T_1 &= r + p_1 c + p_2 \\
&= -p_1\left(\frac{c^2+c+1}{c}\right) - p_2 + p_1 c + p_2 \\
&= v, \\
T_2 &= r + \frac{p_1}{c} + p_2 \\
&= -p_1\left(\frac{c^2+c+1}{c}\right) - p_2 + \frac{p_1}{c} + p_2, \\
&= u,
\end{aligned}$$

Hence simmlar T_1, T_2, we get $T_{2\eta+1} = v, T_{2\eta} = u, \quad \text{for} \quad \eta \geq 0$, therefore the proof is completed.

4. Numerical results:

Example 4.1. *For confirming the results of subsection (2.1), we consider difference equation*

$$T_{\eta+1} = 10 + \frac{T_{\eta-2}}{T_{\eta-3}} + \frac{4T_{\eta-2}}{T_{\eta-1}} + \frac{2T_{\eta-2}}{T_\eta},\tag{4.1}$$

with the initial conditions $T_{-3} = 17.5,\ T_{-2} = 16.5, T_{-1} = 17.2$ *and* $T_0 = 16.8$, *where the equilibrium point is* $\bar{T} = 17$. *(See Fig. 1).*

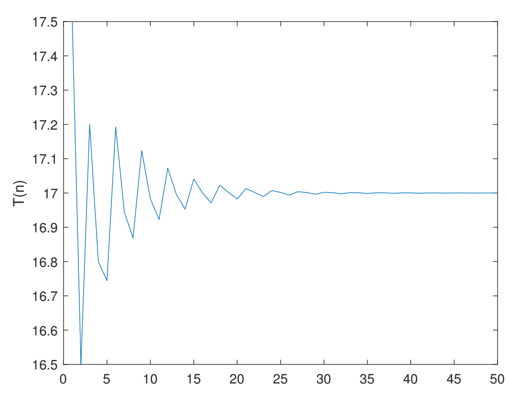

Figure 1. The figure shows the local stability of $\bar{T} = 17$ in Eq. (4.1).

Example 4.2. *For confirming the results of subsection (2.1), we consider difference equation*

$$T_{\eta+1} = 12 + \frac{T_{\eta-1}}{T_{\eta-3}} + \frac{4T_{\eta-1}}{T_{\eta-2}}, \tag{4.2}$$

with the initial conditions $T_{-3} = 19.5,\ T_{-2} = 16.5, T_{-1} = 19.1$ *and* $T_0 = 16.9$, *where the equilibrium point is* $\bar{T} = 17$. *(See Fig. 2).*

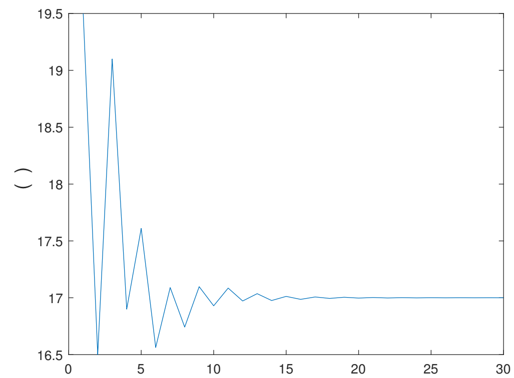

Figure 2. The figure shows the local stability of $\bar{T} = 7$ in Eq. (4.2).

Example 4.3. *For confirming the results of this subsection (2.2), we consider numerical example for Eq. (4.1) with the initial conditions*
IC1: $T_{-3} = 18, T_{-2} = 15, T_{-1} = 19, T_0 = 14,$
IC2: $T_{-3} = 20, T_{-2} = 13, T_{-1} = 21, T_0 = 16,$
IC3: $T_{-3} = 25, T_{-2} = 8, T_{-1} = 26, T_0 = 11,$
IC4: $T_{-3} = 30, T_{-2} = 3, T_{-1} = 29, T_0 = 5.$
(See Fig. 3).

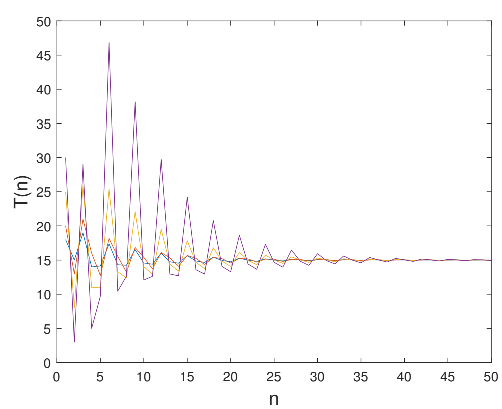

Figure 3. The figure shows the global stability of $\bar{T} = 17$ in Eq. (4.1).

Example 4.4. *For confirming the results of this subsection (2.2), we consider numerical example for Eq. (4.2)with the initial conditions IC1-IC4. (See Fig. 4).*

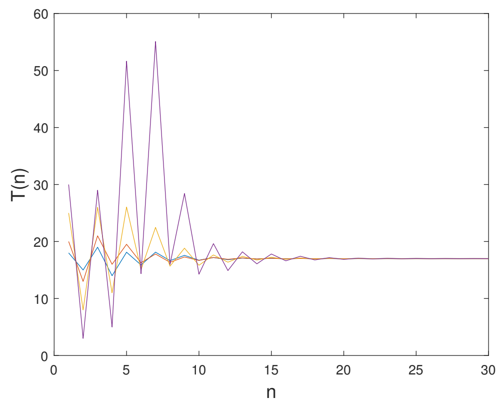

FIGURE 4. The figure shows the global stability of $\bar{T} = 17.$in Eq. (4.2).

Example 4.5. *For confirming the results of Theorem 4, we consider difference equation*

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-5}}{T_{\eta-2}} + \frac{p_2 T_{\eta-5}}{T_{\eta-3}}, \tag{4.3}$$

where $p_1 = 8, p_2 = 4, r = 4, c = 3,$*with the initial condition* $T_{-5} = 32,\ T_{-4} = 10.6667, T_{-3} = 32,\ T_{-2} = 10.6667, T_{-1} = 32$ *and* $T_0 = 10.6667$*. (See Fig. 5).*

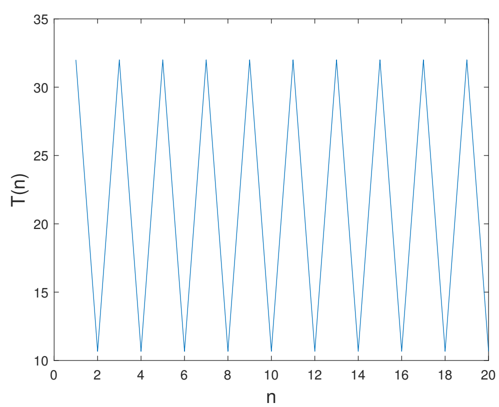

FIGURE 5. The figure shows Eq. (4.3) has period two solutions where r and initial condition satisfies the condition of Theorem 4.

Example 4.6. *For confirming the results of Theorem 5, we consider difference equation*

$$T_{\eta+1} = r + \frac{p_1 T_{\eta}}{T_{\eta-3}} + \frac{p_2 T_{\eta}}{T_{\eta-2}}, \tag{4.4}$$

where $p_1 = 2, p_2 = 4, r = -12.6667, c = 3,$*with the initial condition* $T_{-3} = -8,\ T_{-2} = -2.6667, T_{-1} = -8$ *and* $T_0 = -2.6667$*. (See Fig. 6).*

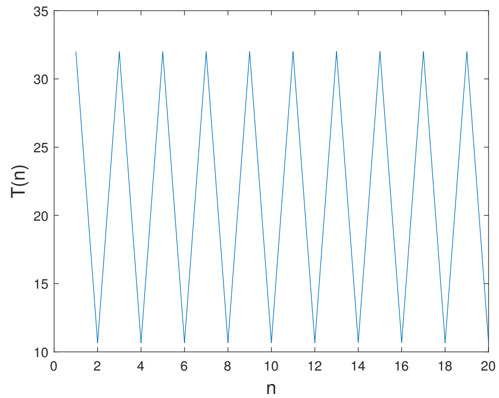

FIGURE 6. The figure shows Eq. (4.3) has period two solutions where r and initial condition satisfies the condition of Theorem 5.

Availability of Data: The paper includes the information used to verify the study's finding.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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