Mathematical Analysis for the Influence of Seasonality on Chikungunya Virus Dynamics

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Abstract. In this article, we discuss a mathematical system modelling Chikungunya virus dynamics in a seasonal environment with general incidence rates. We establish the existence, uniqueness, positivity and boundedness of a periodic orbit. We show that the global dynamics is determined using the basic reproduction number denoted by $\mathcal{R}_0$ and calculated using the spectral radius of a linear integral operator. We show the global stability of the disease free periodic solution if $\mathcal{R}_0 < 1$ and we show also the persistence of the disease if $\mathcal{R}_0 > 1$ where the trajectories converge to a periodic orbit. Finally, we display some numerical examples confirming the theoretical findings.

1. Introduction

Chikungunya is an infectious disease caused by a virus transmitted by mosquitoes. Several species of vector mosquitoes transmit it, including the now famous Aedes albopictus (tiger mosquito) which is slowly invading France. But chik, as we say for short, is for the moment a disease of hot countries, which mainland France is not (at least not yet). This does not prevent us from fearing the development of the disease. Chikungunya infection was first described in 1952, and the virus was identified, during an outbreak in Tanzania. The term "chikungunya" comes from the Makonde language and means "the man who walks bent" in reference to the silhouettes of patients suffering from joint pain. However, the virus has probably existed for several centuries. Chikungunya is transmitted from man to man via mosquitoes of the Aedes genus in particular (tiger mosquito). During a bite, the mosquito takes the virus from an infected person, and during another bite, it transmits it to a healthy person. This is why the best way to fight against the

transmission of chikungunya is to protect yourself individually against mosquito bites (long cloth-ing, skin repellents, mosquito nets), and to slow down their reproduction by destroying the most obvious larval breeding grounds (below pots, waste, gutters). The disease manifests itself after an incubation of 4 to 7 days on average. A high fever (above 38.5°C) appears suddenly, accompanied by headaches, aches or joint pain, which can be intense, mainly affecting the extremities of the limbs (wrists, ankles, phalanges). Other symptoms may also be associated, such as conjunctivitis, rash, nausea. The outcome can quickly be favorable if the patient responds well to symptomatic treatment. However, the disease can also progress into a chronic phase marked by persistent and incapacitating joint pain. There is no cure for the virus. No vaccine has been finalized and each symptom is treated specifically. It is also strongly recommended not to use herbal treatments or substances that have not been prescribed by a doctor.

Several diseases have appeared or reappeared, such as AIDS, Lyme disease, Cholera, Dengue fever or more recently chikungunya. We then speak of emerging or re-emerging diseases. S. Morse is at the origin of this concept of "Emerging infectious diseases" which he introduced and helped to make known during a conference on emerging viruses which he organized e in 1989. The study of emerging diseases is therefore relatively recent (1990s). This phenomenon is nevertheless not new and many factors, such as climate change, the intensification of trade and travel, favor the spread, maintenance or emergence of numerous infectious diseases. The infectious agent, like all living beings, adapts and evolves according to changes in its environment. The chikungunya virus transmitted by mosquitoes of the Aedes genus is no exception to the rule. The peak of the chikungunya epidemic encountered in 2005-2006 on the island of Reunion, having affected a third of the total population of the island, bears witness to this. Indeed, a genetic mutation of the virus would have allowed it to better adapt to the Aedes Albopictus mosquito. The vectorial capacity of the mosquito is thus amplified since this mutated strain of virus is found in the salivary glands of the mosquito only 2 days after infection, compared to 7 days initially. In addition, this mosquito has developed over the years a fairly significant capacity to adapt to non-tropical regions, being found, for example, in Italy and France. Another adaptive capacity of these infectious agents lies in their ability to transmit from one species to another, thus crossing the species barrier. At present, it is not possible to predict the emergence of new events, which may be more or less geographically localized, sporadic or epidemic. Surveillance must therefore be global and constant since a single case can cause an epidemic. This surveillance must be oriented not only towards humans, but also upstream in vector populations in the case of zoonoses such as chikungunya.

Epidemiology is the study of the distribution of diseases in humans and the factors that influence them. In other words, it is the study of epidemics and the factors that could cause them. It aims to understand the causes of diseases and improve their treatment and means of prevention. The contribution of mathematics is then made initially through modeling. This stage of mathematical modeling of the epidemiology of communicable diseases constitutes a real public health tool. It makes it possible to put to the test, without loss of time or expense, the control measures that are

envisaged: preventive measures, isolation of patients, treatments, vaccinations, etc. The model is nevertheless not reality and is not supposed to reproduce it in full. It must reproduce as best as possible the characteristics of the phenomenon studied according to the objectives set for the framework of the study. Modeling then consists of applying mathematical tools to a fragment of reality. It is transforming a need into equations, trying as much as possible to account for the constraints identified. The modeling stage turns out to be the most delicate, the longest and often the most dangerous. Indeed, we must succeed in understanding the real problem in order to try to propose an adapted model. The first attempt proposed only very rarely meets expectations, then several modifications follow, until reaching a model which brings together and translates a maximum of constraints that the real phenomenon must observe. If this step is neglected or omitted, if the constraints are not well posed, we then end up with a mathematical formulation which does not correspond to the problem. The resolution of the mathematical problem then provides a solution not adapted to the concrete problem. Finally, if the problem is well posed, the next step then consists of solving this problem, that is to say, analyzing the model in order to understand, to predict and act. Several models have been proposed to describe the transmission dynamics of vector-borne diseases [1–8]. Since several infectious diseases exhibit seasonal peak periods, then studying the impact of seasonal environment becomes a necessity to more understand such a disease transmission [9–16]. More mathematical models of infectious diseases considering the seasonality were given in [17–26].

In this paper we are interested in the modeling and study of vector disease, applied to the case of chikungunya, transmitted by the Aedes albopictus mosquito in a seasonal environment with general incidence rates. The existence, uniqueness, positivity and boundedness of the periodic solution were established. Later, the global dynamics with respect to the basic reproduction number that will be calculated using the spectral radius of an integral operator was carried out. The global stability of the disease free periodic solution was proved if $\mathcal{R}_0 < 1$, however, the persistence of the disease was proved if $\mathcal{R}_0 > 1$ and the solution converges to a periodic orbit. Finally, we display some numerical examples confirming the theoretical findings.

2. Proposed mathematical model

The considered epidemic model for the CHIKV dynamics in a seasonal environment with general form of incidence rate is given by the following fourth-dimensional ordinary differential equation system:

$$\begin{cases} \dot{X}_i(t) & = \rho_1(t)f_1(X_p(t))X_s(t) + \rho_2(t)f_2(X_i(t))X_s(t) - d(t)X_i(t), \\ \dot{X}_p(t) & = \sigma(t)X_i(t) - d_p(t)X_p(t) - \rho_3(t)f_3(X_p(t))X_a(t), \\ \dot{X}_s(t) & = d(t)S_{in}(t) - d(t)X_s(t) - \rho_1(t)f_1(X_p(t))X_s(t) - \rho_2(t)f_2(X_i(t))X_s(t), \\ \dot{X}_a(t) & = d_a(t)A_{in}(t) + \kappa(t)\rho_3(t)f_3(X_p(t))X_a(t) - d_a(t)X_a(t), \end{cases} \qquad (2.1)$$

with positive initial condition $(X_i^0, X_p^0, X_s^0, X_a^0) \in \mathbb{R}_+^4$.

Parameter	Description
$S_{in}(t)$	Periodic susceptible recruitment rate
$A_{in}(t)$	Periodic antibodies expansion rate
$d(t)$	Periodic cells death rate
$d_p(t)$	Periodic CHIKV death rate
$d_a(t)$	Periodic antibodies death rate
$\sigma(t)$	Periodic production rate of the virus
$\rho_1(t), \rho_2(t)$	Periodic incidence rates
$\rho_3(t)$	Periodic destruction rate of the virus
$\kappa(t)$	Periodic proliferate rate of antibodies

TABLE 1. Meaning of the parameters and variables of (2.1).

For the rest of the paper, we will use the following assumptions:

assumption 2.1.   (1) $f_i, i = 1, 2, 3$ *are increasing, non-negative* $C^1(\mathbb{R}_+)$ *concave functions such that*
$f_1(0) = f_2(0) = f_3(0) = f_4(0) = 0$.
(2) $S_{in}(t)$, $A_{in}(t)$, $d(t)$, $d_p(t)$, $d_a(t)$, $\sigma(t)$, $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ *and* $\kappa(t)$ *are non-negative continuous bounded T-periodic functions.*
(3) $d_a(t) \leq d_p(t)$, $\forall t \geq 0$.

Lemma 2.1. *The incidence rates* $f_i, i = 1, 2, 3$ *satisfy* $f_i'(x)x \leq f_i(x) \leq f_i'(0)x$, $\forall x > 0$ *for* $i = 1, 2, 3$.

Proof. Let $x, x_1 \in \mathbb{R}_+$, and the function $g_1(x) = f(x) - xf_i'(x)$. Since $f_i'(x) \geq 0$ (f_i is increasing function) and $f_i''(x) \leq 0$ (f_i is concave) then $g_1'(x) = -xf_i''(x) \geq 0$ and $g_1(x) \geq g_1(0) = 0$. Therefore, $f_i(x) \geq xf_i'(x)$. Similarly, let $g_2(x) = f_i(x) - xf_i'(0)$ then $g_2'(x) = f_i'(x) - f_i'(0) \leq 0$ once f_i is a concave function. Thus $g_2(x) \leq g_2(0) = 0$ and $f_i(x) \leq xf_i'(0)$ for $i = 1, 2, 3$. $\square$

Let us consider the system (2.1) and prove that there exists a unique bounded positive T-periodic solution for (2.1). For a continuous, positive T-periodic function $g(t)$, we set $g^u = \max\limits_{t \in [0,T)} g(t)$ and $g^l = \min\limits_{t \in [0,T)} g(t)$.

2.1. **Preliminary.** Let $(\mathbb{R}^m, \mathbb{R}^m_+)$ to be the ordered m-dimensional Euclidean space associated with norm $\|\cdot\|$. For $X_1, X_2 \in \mathbb{R}^m$, we denote by $X_1 \geq X_2$ if $X_1 - X_2 \in \mathbb{R}^m_+$. We denote by $X_1 > X_2$ if $X_1 - X_2 \in \mathbb{R}^m_+ \backslash \{0\}$. We denote by $X_1 \gg X_2$ if $X_1 - X_2 \in \text{Int}(\mathbb{R}^m_+)$. Consider a T-periodic $m \times m$ matrix function denoted by $C(t)$ which is continuous, irreducible and cooperative. Let us denote by $\phi_C(t)$ the fundamental matrix, solution of the following system

$$\dot{x}(t) = C(t)x(t). \tag{2.2}$$

Let us denote the spectral radius of the matrix $\phi_C(T)$ by $r(\phi_C(T))$. Therefore, all entries of $\phi_C(t)$ are positive for each $t > 0$. Let apply the theorem of Perron-Frobenius to deduce that $r(\phi_C(T))$ is

the principal eigenvalue of $\phi_C(T)$ (simple and admits an eigenvector $y^* \gg 0$). For the rest of the paper, the following lemma will be useful.

Lemma 2.2. [27]. *There exists a positive T-periodic function $y(t)$ such that $x(t) = y(t)e^{kt}$ will be a solution of system (2.2) where $k = \dfrac{1}{T}\ln(r(\phi_C(T)))$.*

Let us start by proving the existence (and uniqueness) of the disease free periodic trajectory of model (2.1). Let us consider the following subsystem

$$
\begin{aligned}
\dot{X}_s(t) &= d(t)S_{in}(t) - d(t)X_s(t), \\
\dot{X}_a(t) &= d_a(t)A_{in}(t) - d_a(t)X_a(t),
\end{aligned}
\tag{2.3}
$$

with initial condition $(X_s^0, X_a^0) \in \mathbb{R}_+^2$. (2.3) admits a unique T-periodic solution $(X_s^*(t), X_a*(t))$ with $X_s^*(t) > 0$ and $X_a^*(t) > 0$ which is globally attractive in $\mathbb{R}_+^2$ and hence, system (2.1) has a unique disease free periodic solution $(0, 0, X_s^*(t), X_a^*(t))$.

Let us introduce the following result.

Proposition 2.1. $\Omega^u = \left\{ (X_i, X_p, X_s, X_a) \in \mathbb{R}_+^4 \ / \ X_s + X_i \le S_{in}^u ; \kappa^l X_p + X_a \le A_{in}^u + \dfrac{\sigma^u k^u S_{in}^u}{d_a^l} \right\}$ *is a positively invariant, compact and attractor set for model (2.1). Furthermore, we have*

$$
\begin{aligned}
\lim_{t \to \infty} X_s(t) + X_i(t) - X_s^*(t) &= 0, \\
\lim_{t \to \infty} \kappa^l X_p(t) + X_a(t) - X_a^*(t) &= 0.
\end{aligned}
\tag{2.4}
$$

Proof. From (2.1), we have

$$
\begin{aligned}
\dot{X}_s(t) + \dot{X}_i(t) \ &= \ d(t)S_{in}(t) - d(t)(X_s(t) + X_i(t)) \\
&\le \ d(t)(S_{in}^u - X_s(t) - X_i(t)) \\
&\le \ 0, \ \text{if} \ X_s(t) + X_i(t) \ge S_{in}^u,
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa^l \dot{X}_p(t) + \dot{X}_a(t) \ &\le \ \kappa(t)\dot{X}_p(t) + \dot{X}_a(t) \\
&= \ \kappa(t)\sigma(t)X_i(t) - \kappa(t)d_p(t)X_p(t) + d_a(t)A_{in}(t) - d_a(t)X_a(t) \\
&\le \ \kappa(t)\sigma(t)X_i(t) + d_a(t)\Big[A_{in}^u - (\kappa(t)X_p(t) + X_a(t))\Big] \\
&\le \ \kappa^u\sigma^u S_{in}^u + d_a^l\Big[A_{in}^u - (\kappa(t)X_p(t) + X_a(t))\Big] \ \text{if} \ X_s(t) + X_i(t) \le S_{in}^u \\
&= \ d_a^l\left[A_{in}^u + \dfrac{\kappa^u\sigma^u S_{in}^u}{d_a^l} - (\kappa(t)X_p(t) + X_a(t))\right] \\
&\le \ 0, \ \text{if} \ \kappa^l X_p(t) + X_a(t) \ge A_{in}^u + \dfrac{\kappa^u\sigma^u S_{in}^u}{d_a^l},
\end{aligned}
\tag{2.5}
$$

which implies that Ω^u is a forward invariant compact absorbing set for (2.1). Let $N_1(t) = X_s(t) + X_i(t)$ and $N_2(t) = \kappa(t)X_p(t) + X_a(t)$ be the sub-population sizes at time t. Next, let $y_1(t) = N_1(t) - X_s^*(t), t \ge 0$. Then, it follows $\dot{y}_1(t) = -d(t)y_1(t)$, which implies that $\lim_{t \to \infty} y_1(t) = \lim_{t \to \infty}(N_1(t) - X_s^*(t)) = 0$. Similarly, let $y_2(t) = N_2(t) - X_a^*(t), t \ge 0$. Then, it follows $\dot{y}_2(t) = -d(t)y_2(t)$, which implies that $\lim_{t \to \infty} y_2(t) = \lim_{t \to \infty}(N_2(t) - X_a^*(t)) = 0$. $\qquad\square$

Next, in subsection 2.2, we define $\mathcal{R}_0$, the basic reproduction number and we will prove that the disease free periodic trajectory $(0, 0, X_s^*(t), X_a^*(t))$ is globally asymptotically stable (and therefore, the disease dies out) once $\mathcal{R}_0 < 1$. Then, in subsection 2.11, we will prove that $X_i(t)$ and $X_p(t)$ are uniform persistence (and then the disease persists) once $\mathcal{R}_0 > 1$. Therefore, we deduce that $\mathcal{R}_0$ is the threshold parameter between the uniform persistence and the extinction of the disease.

2.2. Disease Free Periodic Solution .

To define the basic reproduction number of model (2.1), by using [16], we define $X = \begin{pmatrix} X_i \\ X_p \\ X_s \\ X_a \end{pmatrix}, \mathcal{F}(t, X) = \begin{pmatrix} \rho_1(t) f_1(X_p(t)) X_s(t) + \rho_2(t) f_2(X_i(t)) X_s(t) \\ \sigma(t) X_i(t) \\ 0 \\ 0 \end{pmatrix},$

$\mathcal{V}^-(t, X) = \begin{pmatrix} d(t) X_i(t) \\ d_p(t) X_p(t) + \rho_3(t) f_3(X_p(t)) X_a(t) \\ d(t) X_s(t) + \rho_1(t) f_1(X_p(t)) X_s(t) + \rho_2(t) f_2(X_i(t)) X_s(t) \\ d_a(t) X_a(t) \end{pmatrix}$ and

$\mathcal{V}^+(t, X) = \begin{pmatrix} 0 \\ 0 \\ d(t) S_{in}(t) \\ d_a(t) A_{in}(t) + \kappa(t) \rho_3(t) f_3(X_p(t)) X_a(t) \end{pmatrix}.$

Let us check the conditions (A1)-(A7) in [16, Section 1]. The system(2.1) can be written as follows:

$$\dot{X} = \mathcal{F}(t, X) - \mathcal{V}(t, X) = \mathcal{F}(t, X) - \mathcal{V}^-(t, X) + \mathcal{V}^+(t, X). \tag{2.6}$$

Conditions (A1)-(A5) are satisfied and the system (2.6) admits a disease free periodic solution $X^*(t) = \begin{pmatrix} 0 \\ 0 \\ X_s^*(t) \\ X_a^*(t) \end{pmatrix}.$ Let $f(t, X(t)) = \mathcal{F}(t, X) - \mathcal{V}^-(t, X) + \mathcal{V}^+(t, X)$ and $M(t) = \left(\dfrac{\partial f_i(t, X^*(t))}{\partial X_j} \right)_{3 \leq i, j \leq 4}$ where $f_i(t, X(t))$ and I are the i-th component of $f(t, X(t))$ and X, respectively. By an easy calculus, we get $M(t) = \begin{pmatrix} -d(t) & 0 \\ 0 & -d_a(t) \end{pmatrix}$ and then $r(\phi_M(T)) < 1$. Therefore $X^*(t)$ is linearly asymptotically stable in the subspace $\Gamma_s = \left\{ (0, 0, S, A) \in R_+^4 \right\}$. Thus, the condition (A6) in [16, Section 1] is satisfied.

Now, let us define $\mathbf{F}(t)$ and $\mathbf{V}(t)$ to be two by two matrices given by $\mathbf{F}(t) = \left(\dfrac{\partial \mathcal{F}_i(t, X^*(t))}{\partial X_j} \right)_{1 \leq i, j \leq 2}$ and $\mathbf{V}(t) = \left(\dfrac{\partial \mathcal{V}_i(t, X^*(t))}{\partial X_j} \right)_{1 \leq i, j \leq 2}$ where $\mathcal{F}_i(t, X)$ and $\mathcal{V}_i(t, X)$ are the i-th component of $\mathcal{F}(t, X)$ and $\mathcal{V}(t, X)$, respectively. By an easy calculus, we obtain from system (2.6)

$$\mathbf{F}(t) = \begin{pmatrix} \rho_2(t) f_2'(0) X_s^*(t) & \rho_1(t) f_1'(0) X_s^*(t) \\ \sigma(t) & 0 \end{pmatrix}, \mathbf{V}(t) = \begin{pmatrix} d(t) & 0 \\ 0 & d_p(t) + \rho_3(t) f_3'(0) X_a^*(t) \end{pmatrix}.$$

Consider $Z(t_1, t_2)$ to be the two by two matrix solution of the system $\dfrac{d}{dt} Z(t_1, t_2) = -\mathbf{V}(t_1)Z(t_1, t_2)$ for any $t_1 \geq t_2$, with $Z(t_1, t_1) = I$, the two by two identity matrix. Thus, condition (A7) was satisfied.

Let us define C_T to be the ordered Banach space of T-periodic functions defined on $\mathbb{R} \mapsto \mathbb{R}^2$, associated to the maximum norm $\|.\|_\infty$ and the positive cone $C_T^+ = \{\psi \in C_T : \psi(s) \geq 0, \text{ for any } s \in \mathbb{R}\}$. Define the linear operator $K : C_T \to C_T$ by

$$(K\psi)(s) = \int_0^\infty Z(s, s-w)\mathbf{F}(s-w)\psi(s-w)dw, \quad \forall s \in \mathbb{R}, \psi \in C_T \tag{2.7}$$

Let us now define the basic reproduction number, $\mathcal{R}_0$, of model (2.1) by $\mathcal{R}_0 = r(K)$.

Therefore, we conclude the local asymptotic stability of the disease free periodic solution $\mathcal{E}_0(t) = (0, 0, X_s^*(t), X_a^*(t))$ for (2.1) as follows.

Theorem 2.1. *[16, Theorem 2.2].*

- $\mathcal{R}_0 < 1 \Leftrightarrow r(\phi_{F-V}(T)) < 1.$
- $\mathcal{R}_0 = 1 \Leftrightarrow r(\phi_{F-V}(T)) = 1.$
- $\mathcal{R}_0 > 1 \Leftrightarrow r(\phi_{F-V}(T)) > 1.$

Therefore, $\mathcal{E}_0(t)$ is unstable if $\mathcal{R}_0 > 1$ and it is asymptotically stable if $\mathcal{R}_0 < 1$.

Theorem 2.2. $\mathcal{E}_0(t)$ *is globally asymptotically stable if $\mathcal{R}_0 < 1$. It is unstable if $\mathcal{R}_0 > 1$.*

Proof. Using the Theorem 2.1, we have $\mathcal{E}_0(t)$ is locally stable once $\mathcal{R}_0 < 1$ and it is unstable once $\mathcal{R}_0 > 1$. Therefore, it remains to prove the global attractivity of $\mathcal{E}_0(t)$ when $\mathcal{R}_0 < 1$.

Consider the case where $\mathcal{R}_0 < 1$. Using the limit (2.4) in Proposition 2.1, for any $\sigma_1 > 0$, there exists $T_1 > 0$ satisfying $X_s(t) + X_i(t) \leq X_s^*(t) + \sigma_1$ and $\kappa(t)X_p(t) + X_a(t) \leq X_a^*(t) + \sigma_1$. Then $X_s(t) \leq X_s^*(t) + \sigma_1$ and $X_a(t) \leq X_a^*(t) + \sigma_1$ and we deduce that

$$\begin{cases} \dot{X}_i(t) & \leq \quad \rho_1(t)f_1(X_p(t))(X_s^*(t) + \sigma_1) + \rho_2(t)f_2(X_i(t))(X_s^*(t) + \sigma_1) - d(t)X_i(t), \\ \dot{X}_p(t) & \leq \quad \sigma(t)X_i(t) - d_p(t)X_p(t) \end{cases} \tag{2.8}$$

for $t > T_1$. Let $M_2(t)$ to be the following 2×2 matrix function

$$M_2(t) = \begin{pmatrix} \rho_2(t)f_2'(0) & \rho_1(t)f_1'(0) \\ \sigma(t) & 0 \end{pmatrix}. \tag{2.9}$$

By Theorem 2.1, we have $r(\varphi_{F-V}(T)) < 1$. Let us choose $\sigma_1 > 0$ such that $r(\varphi_{F-V+\sigma_1 M_2}(T)) < 1$. Consider the system hereafter system

$$\begin{cases} \dot{\bar{X}}_i(t) & \leq \quad \rho_1(t)f_1(\bar{X}_p(t))(X_s^*(t) + \sigma_1) + \rho_2(t)f_2(\bar{X}_i(t))(X_s^*(t) + \sigma_1) - d(t)\bar{X}_i(t), \\ \dot{\bar{X}}_p(t) & = \quad \sigma(t)\bar{X}_i(t) - d_p(t)\bar{X}_p(t). \end{cases} \tag{2.10}$$

Applying Lemma 2.2 and using the standard comparison principle, we deduce that there exists a positive T-periodic function $y_1(t)$ satisfying $x(t) \leq y_1(t)e^{k_1 t}$ where $x(t) = \begin{pmatrix} X_i(t) \\ X_p(t) \end{pmatrix}$ and $k_1 = \dfrac{1}{T}\ln\left(r(\varphi_{F-V+\sigma_1 M_2}(T)\right) < 0$. Thus, $\lim\limits_{t\to\infty} X_i(t) = 0$ and $\lim\limits_{t\to\infty} X_p(t) = 0$. Furthermore, we have

$\lim\limits_{t\to\infty} X_s(t) - X_s^*(t) = \lim\limits_{t\to\infty} N_1(t) - X_i(t) - X_s^*(t) = 0$ and $\lim\limits_{t\to\infty} X_a(t) - X_a^*(t) = \lim\limits_{t\to\infty} N_2(t) - k(t)X_p(t) - X_a^*(t) = 0$. Then, we deduce that the disease free periodic solution $\mathcal{E}_0(t)$ is globally attractive which complete the proof.   □

For the following subsection, we consider only the case where $\mathcal{R}_0 > 1$.

2.3. Endemic Periodic Solution .

$$\begin{cases} \dot{X}_i(t) & = \rho_1(t)f_1(X_p(t))X_s(t) + \rho_2(t)f_2(X_i(t))X_s(t) - d(t)X_i(t), \\ \dot{X}_p(t) & = \sigma(t)X_i(t) - d_p(t)X_p(t) - \rho_3(t)f_3(X_p(t))X_a(t), \\ \dot{X}_s(t) & = d(t)S_{in}(t) - d(t)X_s(t) - \rho_1(t)f_1(X_p(t))X_s(t) - \rho_2(t)f_2(X_i(t))X_s(t), \\ \dot{X}_a(t) & = d_a(t)A_{in}(t) + \kappa(t)\rho_3(t)f_3(X_p(t))X_a(t) - d_a(t)X_a(t), \end{cases} \qquad (2.11)$$

From Proposition 2.1, system (2.1) admits a positively invariant compact set Ω^u.

Let us define the function $Q : \mathbb{R}^4_+ \to \mathbb{R}^4_+$ to be the Poincaré map associated to system (2.1) such that $X_0 \mapsto u(T, X^0)$, where $u(t, X^0)$ is the unique solution of the system (2.1) with the initial condition $u(0, X^0) = X^0 \in \mathbb{R}^4_+$.

Let us define

$$\Gamma = \left\{ (X_i, X_p, X_s, X_a) \in \mathbb{R}^4_+ \right\}, \ \Gamma_0 = Int(\mathbb{R}^4_+) \text{ and } \partial\Gamma_0 = \Gamma \setminus \Gamma_0.$$

Note that from Proposition 2.1, both Γ and Γ_0 are positively invariant. Q is point dissipative. Define

$$M_\partial = \left\{ (X_i^0, X_p^0, X_s^0, X_a^0) \in \partial\Gamma_0 : Q^n(X_i^0, X_p^0, X_s^0, X_a^0) \in \partial\Gamma_0, \text{ for any } n \geq 0 \right\}.$$

In order to apply the theory of uniform persistence detailed in Zhao [28] (also in [27, Theorem 2.3]), we prove that

$$M_\partial = \{(0, 0, X_s, X_a), \ X_s \geq 0, X_a \geq 0\}. \qquad (2.12)$$

Note that $M_\partial \supseteq \{(0, 0, X_s, X_a), \ X_s \geq 0, X_a \geq 0\}$. To show that $M_\partial \setminus \{(0, 0, X_s, X_a), \ X_s \geq 0, X_a \geq 0\} = \emptyset$. Let us consider $(X_i^0, X_p^0, X_s^0, X_a^0) \in M_\partial \setminus \{(0, 0, X_s, X_a), \ X_s \geq 0, X_a \geq 0\}$. If $X_p^0 = 0$ and $0 < X_i^0$, thus $X_i(t) > 0$ for any $t > 0$. Then, it holds that $\dot{X}_p(t)_{|t=0} = \sigma(0)X_i^0 > 0$. If $X_p^0 > 0$ and $X_i^0 = 0$, then $X_p(t) > 0$ and $X_s(t) > 0$ for any $t > 0$. Therefore, for any $t > 0$, we have

$$X_i(t) = \left[X_i^0 + \int_0^t \left[\rho_1 f_1(X_p(\omega) + \rho_2 f_2(X_i(\omega)) \right] X_s(\omega) e^{\int_0^\omega d(u)du} d\omega \right] e^{-\int_0^t d(u)du} > 0$$

for all $t > 0$. This means that $(X_s(t), X_i(t), X_p(t), X_a(t)) \notin \partial\Gamma_0$ for $0 < t \ll 1$. Therefore, Γ_0 is positively invariant from which we deduce (2.12). Using the previous discussion, we deduce that there exists one fixed point $(0, 0, X_s^*(0), X_a^*(0))$ of Q in M_∂. We deduce, therefore, the uniform persistence of the disease as follows.

Theorem 2.3. *Consider the case $\mathcal{R}_0 > 1$. (2.1) admits at least one positive periodic trajectory and $\exists \ \gamma > 0$ satisfying $\forall \ (X_i^0, X_p^0, X_s^0, X_a^0) \in \mathbb{R}_+ \times Int(\mathbb{R}^2_+) \times \mathbb{R}_+$,*

$$\liminf_{t\to\infty} X_i(t) \geq \gamma > 0.$$

Proof. Let us start by proving that Q is uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$, which will prove that the trajectory of the system (2.1) is uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$ using [28, Theorem 3.1.1]. Recall that using Theorem 2.1, we obtain $r(\varphi_{F-V}(T)) > 1$. Therefore, $\exists\, \eta > 0$ small enough and satisfying $r(\varphi_{F-V-\eta M_2}(T)) > 1$. Let us consider the following perturbed equation

$$\begin{cases} \dot{X}_s^\alpha(t) &=& d(t)S_{in}(t) - d(t)X_s^\alpha(t) - (\rho_1(t)f_1(\alpha) + \rho_2(t)f_2(\alpha))X_s^\alpha(t), \\ \dot{X}_a^\alpha(t) &=& d_a(t)A_{in}(t) + \kappa(t)\rho_3(t)f_3(\alpha)X_a^\alpha(t) - d_a(t)X_a^\alpha(t). \end{cases} \tag{2.13}$$

The function Q associated to the perturbed system (2.13) has a unique positive fixed point $(\bar{X}_s^{\alpha^0}, \bar{X}_a^{\alpha^0})$ that it is globally attractive in $\mathbb{R}_+^2$. Applying the implicit function theorem to deduce that $(\bar{X}_s^{\alpha^0}, \bar{X}_a^{\alpha^0})$ is continuous respecting to α. Therefore, we can chose $\alpha > 0$ small enough and satisfying $\bar{X}_s^\alpha(t) > \bar{X}_s(t) - \eta$, and $\bar{X}_a^\alpha(t) > \bar{X}_a(t) - \eta$, $\forall\, t > 0$. Let $M_1 = (0, 0, \bar{X}_s^0, \bar{X}_a^0)$. Since the trajectory is continuous respecting to the initial condition, $\exists\, \alpha^*$ satisfying $(X_i^0, X_p^0, X_s^0, X_a^0) \in \Gamma_0$ with $\|(X_i^0, X_p^0, X_s^0, X_a^0) - u(t, M_1)\| \leq \alpha^*$, it holds that,

$$\|u(t, (X_i^0, X_p^0, X_s^0, X_a^0)) - u(t, M_1)\| < \alpha \text{ for } 0 \leq t \leq T.$$

We prove by contradiction that

$$\limsup_{n \to \infty} d(Q^n(X_i^0, X_p^0, X_s^0, X_a^0), M_1) \geq \alpha^* \, \forall\, (X_i^0, X_p^0, X_s^0, X_a^0) \in \Gamma_0. \tag{2.14}$$

Suppose that $\limsup_{n \to \infty} d(Q^n(X_i^0, X_p^0, X_s^0, X_a^0), M_1) < \alpha^*$ for some $(X_i^0, X_p^0, X_s^0, X_a^0) \in \Gamma_0$. We can assume that $d(Q^n(X_i^0, X_p^0, X_s^0, X_a^0), M_1) < \alpha^* \, \forall\, n > 0$. Therefore

$$\|u(t, Q^n(X_i^0, X_p^0, X_s^0, X_a^0)) - u(t, M_1)\| < \alpha \, \forall\, n > 0 \text{ and } 0 \leq t \leq T.$$

For all $t \geq 0$, let $t = nT + t_1$, with $t_1 \in [0, T)$ and $n = [\frac{t}{T}]$ (greatest integer $\leq \frac{t}{T}$). Then, we get

$$\|u(t, (X_i^0, X_p^0, X_s^0, X_a^0)) - u(t, M_1)\| = \|u(t_1, Q^n(X_i^0, X_p^0, X_s^0, X_a^0)) - u(t_1, M_1)\| < \alpha \text{ for all } t \geq 0.$$

Set $(X_s(t), X_i(t), X_p(t), X_a(t)) = u(t, (X_i^0, X_p^0, X_s^0, X_a^0))$. Therefore $0 \leq X_i(t), X_p(t) \leq \alpha, t \geq 0$ and

$$\begin{cases} \dot{X}_s(t) &\geq& d(t)S_{in}(t) - d(t)X_s(t) - (\rho_1(t)f_1(\alpha) + \rho_2(t)f_2(\alpha))X_s(t), \\ \dot{X}_a(t) &=& d_a(t)A_{in}(t) + \kappa(t)\rho_3(t)f_3(\alpha)X_a(t) - d_a(t)X_a(t). \end{cases} \tag{2.15}$$

The fixed point $\bar{S}_\alpha^0$ of the function Q associated to the perturbed system (2.13) is globally attractive such that $\bar{X}_s^\alpha(t) > \bar{X}_s(t) - \eta$, and $\bar{X}_a^\alpha(t) > \bar{X}_a(t) - \eta$, , then $\exists\, T_2 > 0$ large enough and satisfying $X_s(t) > \bar{X}_s(t) - \eta$ and $X_a(t) > \bar{X}_a(t) - \eta$ for $t > T_2$. Therefore, for $t > T_2$

$$\begin{cases} \dot{X}_i(t) &\geq& \rho_1(t)f_1(X_p(t))X_s(t) + \rho_2(t)f_2(X_i(t))X_s(t) - d(t)X_i(t), \\ \dot{X}_p(t) &\geq& \sigma(t)X_i(t) - d_p(t)X_p(t) - \rho_3(t)f_3(X_p(t))X_a(t). \end{cases} \tag{2.16}$$

Note that we have $r(\varphi_{F-V-\eta M_2}(T)) > 1$. Applying Lemma 2.2 and the comparison principle, there exists a positive T-periodic trajectory $y_2(t)$ satisfying $J(t) \geq e^{k_2 t}y_2(t)$ with $k_2 = \frac{1}{T}\ln r\left(\varphi_{F-V-\eta M_2}(T)\right) > 0$, which implies that $\lim_{t \to \infty} X_i(t) = \infty$ which is impossible since the trajectories are bounded. Therefore, the inequality (2.14) is satisfied and Q is weakly uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$. By applying Proposition 2.1, Q has a global attractor. We deduce

that $M_1 = (0, 0, \bar{X}_s^0, \bar{X}_a^0)$ is an isolated invariant set inside X and $W^s(M_1) \cap \Gamma_0 = \emptyset$. All trajectory inside M_∂ converges to M_1 which is acyclic in M_∂. Applying [28, Theorem 1.3.1 and Remark 1.3.1], we deduce that Q is uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$. Furthermore, using [28, Theorem 1.3.6], Q admits a fixed point $(\tilde{X}_i^0, \tilde{X}_p^0, \tilde{X}_s^0, \tilde{X}_a^0) \in \Gamma_0$. Note that $(\tilde{X}_i^0, \tilde{X}_p^0, \tilde{X}_s^0, \tilde{X}_a^0) \in R_+ \times Int(R_+^2) \times R_+$. We prove also by contradiction that $\tilde{X}_s^0 > 0$. Assume that $\tilde{X}_s^0 = 0$. Using the first equation of the system (2.1), $\tilde{X}_s(t)$ verifies

$$\dot{\tilde{X}}_s(t) \geq d(t)S_{in}(t) - d(t)\tilde{X}_s(t) - \left[\rho_1(t)f_1(\tilde{X}_p(t)) + \rho_2(t)f_2(\tilde{X}_i(t))\right]\tilde{X}_s(t), \tag{2.17}$$

with $\tilde{X}_s^0 = \tilde{X}_s(pT) = 0, p = 1, 2, 3, \cdots$. Applying Proposition 2.1, $\forall \sigma_3 > 0$, there exists $T_3 > 0$ large enough and satisfying $\tilde{X}_i(t) \leq S_{in}^u + \sigma_3$ and $\tilde{X}_p(t) \leq \dfrac{A_{in}^u}{k^l} + \dfrac{\sigma^u k^u S_{in}^u}{k^l d_a^l} + \sigma_3$ for $t > T_3$. Then, by Lemma 2.1, we obtain

$$\dot{\tilde{X}}_s(t) \geq d(t)S_{in}(t) - \left[\rho_1(t)f_1\left(\frac{A_{in}^u}{k^l} + \frac{\sigma^u k^u S_{in}^u}{k^l d_a^l} + \sigma_3\right) + \rho_2(t)f_2(S_{in}^u + \sigma_3) + d(t)\right]\tilde{X}_s(t), \text{ for } t \geq T_3$$

There exists $\bar{p}$ large enough and satisfying $pT > T_3$ for all $p > \bar{p}$. Applying the comparison principle, we deduce

$$\tilde{X}_s(pT) = \left[\tilde{X}_s^0 + \int_0^{pT} d(\omega)S_{in}(\omega)e^{\int_0^\omega \left[\rho_1(u)f_1\left(\frac{A_{in}^u}{k^l} + \frac{\sigma^u k^u S_{in}^u}{k^l d_a^l} + \sigma_3\right) + \rho_2(u)f_2(S_{in}^u + \sigma_3) + d(u)\right]du} d\omega\right] \times$$
$$e^{-\int_0^{pT} \left[\rho_1(u)f_1\left(\frac{A_{in}^u}{k^l} + \frac{\sigma^u k^u S_{in}^u}{k^l d_a^l} + \sigma_3\right) + \rho_2(u)f_2(S_{in}^u + \sigma_3) + d(u)\right]du} > 0$$

for any $p > \bar{p}$ which is impossible. Therefore, $\tilde{X}_s^0 > 0$ and $(\tilde{X}_i^0, \tilde{X}_p^0, \tilde{X}_s^0, \tilde{X}_a^0)$ is a positive T-periodic trajectory of the system (2.1). □

3. Numerical simulations

For all our numerical results, we will apply a nonlinear Monod-type functions to describe the incidence rates which satisfy the assumption 2.1 $f_i(X) = \dfrac{X}{k_i + X}$ for $i = 1, 2, 3$. Here k_i for $i = 1, 2, 3$ are non-negative constants. The periodic functions are defined as

$$\begin{cases}
d(t) &= m^0(1 + m^1 \cos(n\pi(t + \phi))), \\
d_p(t) &= d_p^0(1 + d_p^1 \cos(n\pi(t + \phi))), \\
d_a(t) &= d_a^0(1 + d_a^1 \cos(n\pi(t + \phi))), \\
\sigma(t) &= \sigma^0(1 + \sigma^1 \cos(n\pi(t + \phi))), \\
S_{in}(t) &= S_{in}^0(1 + S_{in}^1 \cos(n\pi(t + \phi))), \\
\rho_1(t) &= \rho_1^0(1 + \rho_1^1 \cos(n\pi(t + \phi))), \\
\rho_2(t) &= \rho_2^0(1 + \rho_2^1 \cos(n\pi(t + \phi))), \\
\rho_3(t) &= \rho_3^0(1 + \rho_3^1 \cos(n\pi(t + \phi))), \\
\kappa(t) &= \kappa^0(1 + \kappa^1 \cos(n\pi(t + \phi))),
\end{cases} \tag{3.1}$$

with $|d^1|$, $|d_p^1|$, $|d_a^1|$, $|\sigma^1|$, $|S_{in}^1|$, $|\rho_1^1|$, $|\rho_2^1|$, $|\rho_3^1|$ and $|\kappa^1|$ are the frequency of seasonal cycles and ϕ is the phase shift. The values of d^0, d_p^0, d_a^0, σ^0, S_{in}^0, ρ_1^0, ρ_2^0, ρ_3^0 and κ^0 are given in Table 2. However, the values of d^1, d_p^1, d_a^1, σ^1, S_{in}^1, ρ_1^1, ρ_2^1, ρ_3^1 and κ^1 are given in Table 3.

Parameter	d^0	d_p^0	d_a^0	σ^0	S_{in}^0	A_{in}^0	ρ_1^0	ρ_2^0	ρ_3^0	κ^0	ϕ	n
Value	0.8	1.2	0.9	0.8	6	4	1.25	1.15	1	0.6	4	4

TABLE 2. Used values for d^0, d_p^0, d_a^0, σ^0, S_{in}^0, A_{in}^0, ρ_1^0, ρ_2^0, ρ_3^0, κ^0, ϕ and n.

Parameter	d^1	d_p^1	d_a^1	σ^1	S_{in}^1	A_{in}^1	ρ_1^1	ρ_2^1	ρ_3^1	κ^1
Value	0.5	0.3	0.2	−0.3	−0.2	0.2	0.2	0.3	0.4	−0.4

TABLE 3. Used values for d^1, d_p^1, d_a^1, σ^1, S_{in}^1, A_{in}^1, ρ_1^1, ρ_2^1, ρ_3^1 and κ^1.

We consider three situations. The first case deals with the case of fixed environment where we try to confirm the global stability of either $\mathcal{E}_0$ or $\mathcal{E}^*$ according to the basic reproduction number values. The second case deals with the case where only the contact rates are periodic. The third case deals with the case where all parameters are periodic functions.

3.1. **Case of fixed environment.** In the first step, we consider the case of fixed environment for the system (2.1) and all parameters are assumed to be constants. The system (2.1) takes the following form:

$$
\begin{cases}
\dot{X}_i(t) = \dfrac{\rho_1^0 X_p(t) X_s(t)}{k_1 + X_p(t)} + \dfrac{\rho_2^0 X_i(t) X_s(t)}{k_2 + X_i(t)} - d^0 X_i(t), \\[2mm]
\dot{X}_p(t) = \sigma^0 X_i(t) - d_p^0 X_p(t) - \dfrac{\rho_3^0 X_p(t) X_a(t)}{k_3 + X_p(t)}, \\[2mm]
\dot{X}_s(t) = d^0 S_{in}^0 - d^0 X_s(t) - \dfrac{\rho_1^0 X_p(t) X_s(t)}{k_1 + X_p(t)} - \dfrac{\rho_2^0 X_i(t) X_s(t)}{k_2 + X_i(t)}, \\[2mm]
\dot{X}_a(t) = d_a^0 A_{in}^0 + \kappa^0 \dfrac{\rho_3^0 X_p(t) X_a(t)}{k_3 + X_p(t)} - d_a^0 X_a(t),
\end{cases}
\tag{3.2}
$$

with positive initial condition $(X_i^0, X_p^0, X_s^0, X_a^0) \in \mathbb{R}_+^4$.

The basic reproduction number was approximated by using the next generation matrix method [29–31]. See [2, 32–36] for other applications. Let us consider the matrices $F = \begin{pmatrix} \rho_2^0 f_2'(0) S_{in}^0 & \rho_1^0 f_1'(0) S_{in}^0 \\ 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} d^0 & 0 \\ -\sigma^0 & \rho_3^0 f_3'(0) A_{in}^0 + d_p^0 \end{pmatrix}$. The determinant of V is given by $\det(V) = d^0 (\rho_3^0 f_3'(0) A_{in}^0 + d_p^0) > 0$; thus, $V^{-1} = \begin{pmatrix} \dfrac{1}{d^0} & 0 \\ \dfrac{\sigma^0}{d^0 (\rho_3^0 f_3'(0) A_{in}^0 + d_p^0)} & \dfrac{1}{\rho_3^0 f_3'(0) A_{in}^0 + d_p^0} \end{pmatrix}$ and

the next-generation matrix is given by

$$FV^{-1} = \begin{pmatrix} \dfrac{\rho_2^0 f_2'(0)S_{in}^0}{d^0} + \dfrac{\sigma^0 \rho_1^0 f_1'(0)S_{in}^0}{d^0(\rho_3^0 f_3'(0)A_{in}^0 + d_p^0)} & \dfrac{\rho_1^0 f_1'(0)S_{in}^0}{\rho_3^0 f_3'(0)A_{in}^0 + d_p^0} \\ 0 & 0 \end{pmatrix}. \text{ Therefore, the basic reproduction}$$

number is given by:

$$\mathcal{R}_0 = \frac{\rho_2^0 f_2'(0)S_{in}^0}{d^0} + \frac{\sigma^0 \rho_1^0 f_1'(0)S_{in}^0}{d^0(\rho_3^0 f_3'(0)A_{in}^0 + d_p^0)} = \frac{\rho_2^0 f_2'(0)(\rho_3^0 f_3'(0)A_{in}^0 + d_p^0) + \sigma^0 \rho_1^0 f_1'(0)}{d^0(\rho_3^0 f_3'(0)A_{in}^0 + d_p^0)} S_{in}^0.$$
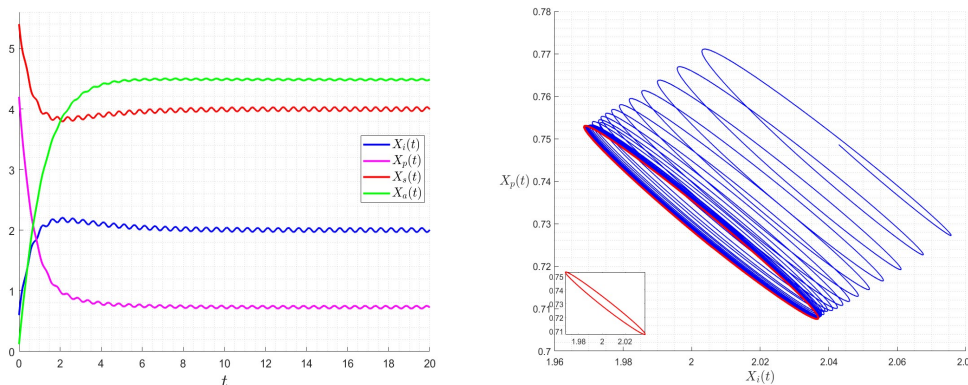
Let us display some numerical tests of system (3.2).
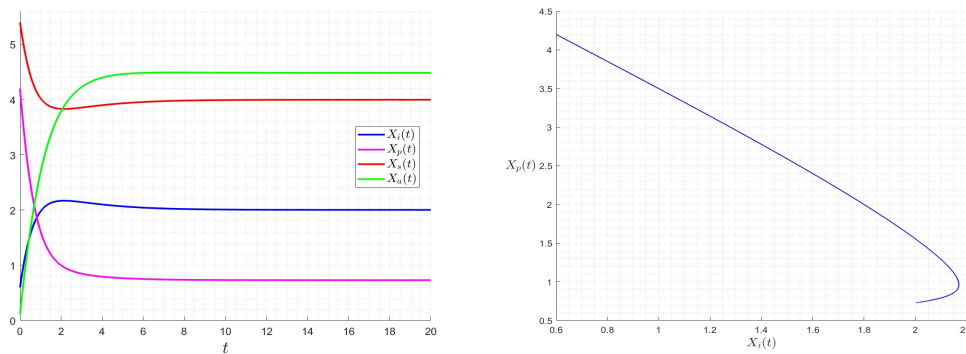
FIGURE 1. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 6.7$, $k_2 = 7$ and $k_3 = 4.9$ then $\mathcal{R}_0 \approx 1.79 > 1$.
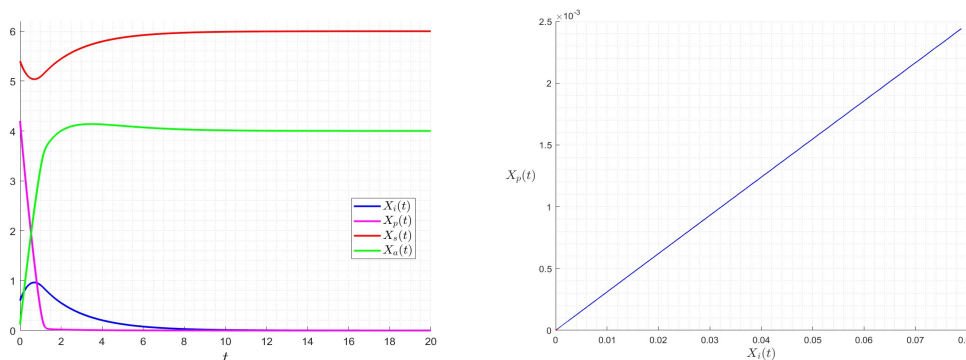
FIGURE 2. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 15$, $k_2 = 24$ and $k_3 = 0.2$ then $\mathcal{R}_0 \approx 0.38 < 1$.

In Figure 1, the approximated trajectory of dynamics (3.2) converges asymptotically to $\mathcal{E}^*$ if $\mathcal{R}_0 > 1$. However, in Figure 2, the approximated trajectory of the dynamics (3.2) converges to the trivial steady state $\mathcal{E}_0$, confirming the fact that $\mathcal{E}_0$ is globally asymptotically stable if $\mathcal{R}_0 \leq 1$.

3.2. **Case of seasonal contact rates.** In the second step, we consider the case of periodic contact rates for the system (2.1) and the system (2.1) takes the following form:

$$\begin{cases} \dot{X}_i(t) &= \dfrac{\rho_1(t)X_p(t)X_s(t)}{k_1 + X_p(t)} + \dfrac{\rho_2(t)X_i(t)X_s(t)}{k_2 + X_i(t)} - d^0 X_i(t), \\[2mm] \dot{X}_p(t) &= \sigma^0 X_i(t) - d_p^0 X_p(t) - \dfrac{\rho_3(t)X_p(t)X_a(t)}{k_3 + X_p(t)}, \\[2mm] \dot{X}_s(t) &= d^0 S_{in}^0 - d^0 X_s(t) - \dfrac{\rho_1(t)X_p(t)X_s(t)}{k_1 + X_p(t)} - \dfrac{\rho_2(t)X_i(t)X_s(t)}{k_2 + X_i(t)}, \\[2mm] \dot{X}_a(t) &= d_a^0 A_{in}^0 + \kappa^0 \dfrac{\rho_3(t)X_p(t)X_a(t)}{k_3 + X_p(t)} - d_a^0 X_a(t), \end{cases} \tag{3.3}$$

with positive initial condition $(X_i^0, X_p^0, X_s^0, X_a^0) \in \mathbb{R}_+^4$. The basic reproduction number $\mathcal{R}_0$ was calculated through the time-averaged system.

FIGURE 3. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 6.7$, $k_2 = 7$ and $k_3 = 4.9$ then $\mathcal{R}_0 \approx 1.79 > 1$.
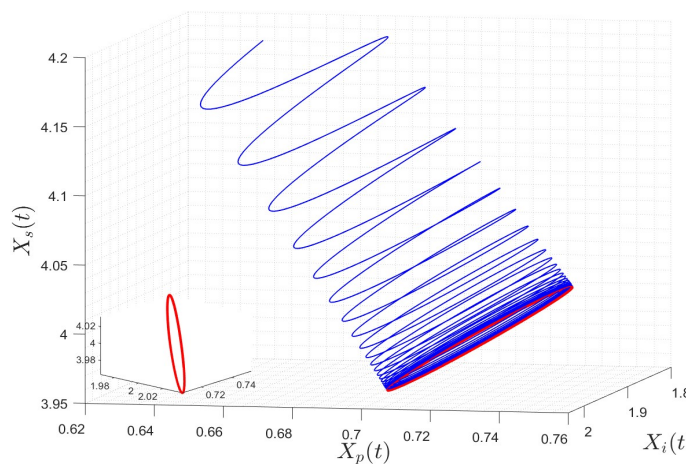

FIGURE 4. Zoom on the trajectory of dynamics (2.1) for $k_1 = 6.7$, $k_2 = 7$ and $k_3 = 4.9$ then $\mathcal{R}_0 \approx 1.79 > 1$.
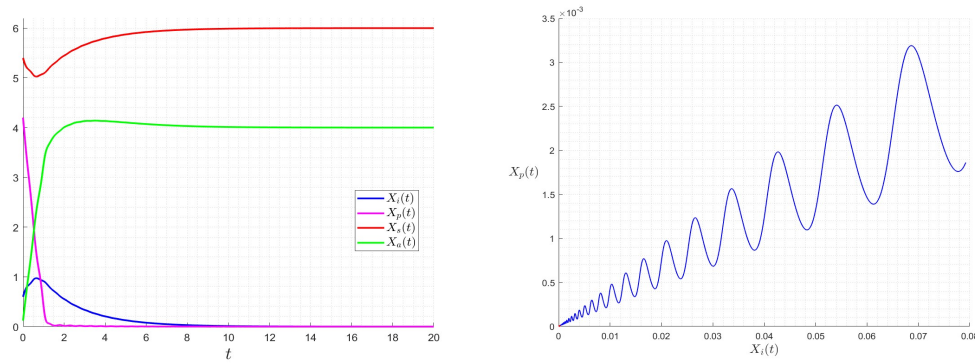
FIGURE 5. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 15$, $k_2 = 24$ and $k_3 = 0.2$ then $\mathcal{R}_0 \approx 0.38 < 1$.

In Figure 3, the approximated trajectory of dynamics (3.3) converges asymptotically to a periodic orbit reflecting the persistence of the disease. In Figure 4, we display a zoom of the periodic orbit for $\mathcal{R}_0 > 1$. In Figure 5, the approximated trajectory of dynamics (3.3) converges to the disease-free solution if $\mathcal{R}_0 < 1$.

3.3. **Case of full seasonal environment.** In the third step, we consider the case of totally seasonal environment and the system (2.1) takes the following form:

$$
\begin{cases}
\dot{X}_i(t) &= \dfrac{\rho_1(t)X_p(t)X_s(t)}{k_1 + X_p(t)} + \dfrac{\rho_2(t)X_i(t)X_s(t)}{k_2 + X_i(t)} - d(t)X_i(t), \\[2mm]
\dot{X}_p(t) &= \sigma(t)X_i(t) - d_p(t)X_p(t) - \dfrac{\rho_3(t)X_p(t)X_a(t)}{k_3 + X_p(t)}, \\[2mm]
\dot{X}_s(t) &= d(t)S_{in}(t) - d(t)X_s(t) - \dfrac{\rho_1(t)X_p(t)X_s(t)}{k_1 + X_p(t)} - \dfrac{\rho_2(t)X_i(t)X_s(t)}{k_2 + X_i(t)}, \\[2mm]
\dot{X}_a(t) &= d_a(t)A_{in}(t) + \kappa(t)\dfrac{\rho_3(t)X_p(t)X_a(t)}{k_3 + X_p(t)} - d_a(t)X_a(t),
\end{cases}
\tag{3.4}
$$

with positive initial condition $(X_i^0, X_p^0, X_s^0, X_a^0) \in \mathbb{R}_+^4$. The basic reproduction number $\mathcal{R}_0$ was calculated through the time-averaged system.
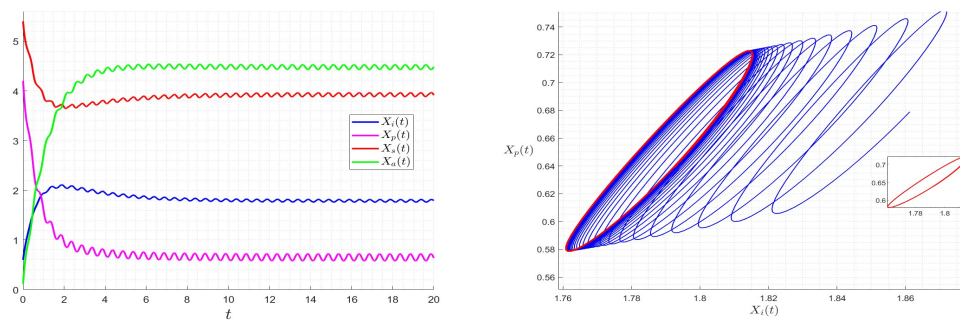

FIGURE 6. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 6.7$, $k_2 = 7$ and $k_3 = 4.9$ then $\mathcal{R}_0 \approx 1.79 > 1$.
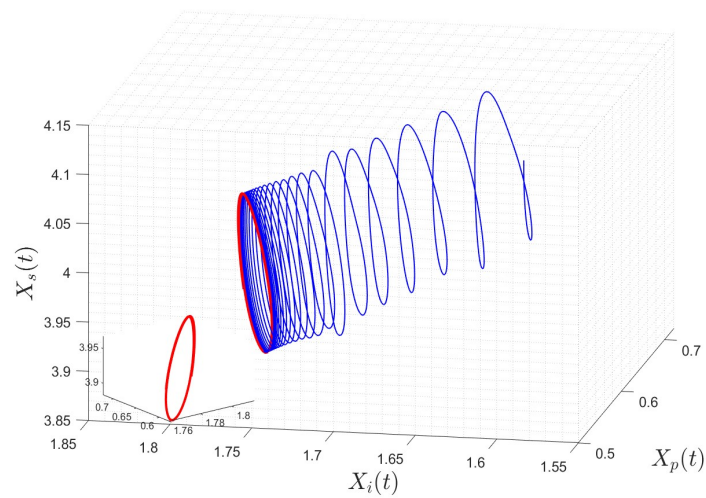
FIGURE 7. Zoom on the trajectory of dynamics (2.1) for $k_1 = 6.7$, $k_2 = 7$ and $k_3 = 4.9$ then $\mathcal{R}_0 \approx 1.79 > 1$.
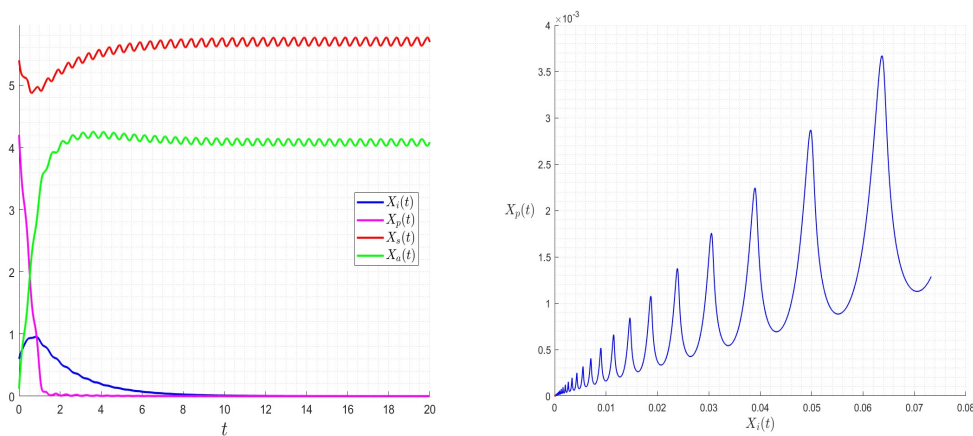

FIGURE 8. Behaviour of the trajectory of dynamics (2.1) for $k_1 = 15$, $k_2 = 24$ and $k_3 = 0.2$ then $\mathcal{R}_0 \approx 0.38 < 1$.

In Figure 6, the calculated trajectory of dynamics (3.4) converges asymptotically to a periodic orbit reflecting the persistence of the disease for $\mathcal{R}_0 > 1$. In Figure 7, we display a zoom of the periodic orbit for $\mathcal{R}_0 > 1$. In Figure 8, the calculated trajectory of dynamics (3.4) converges to the disease-free periodic solution $\mathcal{E}_0(t) = (0, 0, X_s^*(t), X_a^*(t))$ if $\mathcal{R}_0 \leq 1$.

4. CONCLUSION

In this paper, we extended the CHIKV epidemic mathematical models studied in [1–3, 19] by considering the impact of seasonality. We defined the basic reproduction number, $\mathcal{R}_0$ trough a linear integral operator. We proved that for $\mathcal{R}_0$ smaller than the unity, all solutions converge to

a disease-free periodic solution, however, the disease persists for $\mathcal{R}_0$ greater than the unity. We confirmed the theoretical findings by using several numerical examples, including the case of fixed environment, the case of only periodic contact rates and the case of full seasonal environment. For the case of fixed environment, the trajectories converge to one of the steady states of (2.1). However, for the case where at least one of the parameters is periodic, the solutions converge to a periodic orbit according to theorems 2.2 and 2.3.

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