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Fixed-Point Approaches for Multi-Valued Contraction Mappings in a Novel Space With an Application

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Abstract. The vital goals of this manuscript are to combine metric-like spaces with $S-$metric spaces under a control function to obtain a new space called the controlled $S-$metric-like spaces (CSMLSs, for short). Under this name, many fixed-point (FP) results have been obtained for multi-valued mappings (MVMs). In addition, we present several non-trivial examples to back up our statements. The results obtained generalize and unify many results in the same direction. Finally, to support and test the adequacy of the theoretical results, the existence of the solution to the differential inclusion problem (DIP) was studied as a type of application.

1. Introduction

Let $\mho$ be a non-empty set and $\mu : \mho \to \mho$ be a self-mapping. A FP of μ is a solution to an equation $\mu(\vartheta) = \vartheta$. Theorems dealing with the existence and construction of a solution to an operator equation $\mu(\vartheta) = \vartheta$ are considered the most important part of fixed point theory. Theorems concerning the existence and development of a solution to the operator equations $\mu(\vartheta) = \vartheta$ are regarded as the most significant portion of FP theory. This theory is a well-known study subject in nonlinear analysis. Among its essential theorems, Banach's and Brouwer's FP theorems are crucial.

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Banach's FP theorem, in particular, is an important tool in the metric theory of FPs. Its generalizations are crucial in many various domains, including physics, resolving electric circuit equations, etc. [1–5]. Numerous papers have been written to investigate and solve a wide range of practical and theoretical problems using Banach's FP theorem and its generalization; see [6–9].

In 1906, Fréchet [10] proposed the metric space (MS) concept. Since this date, many writers have been interested in exploiting the metric function by introducing various restrictions in order to extend and expand the idea of MSs. For more details, we direct the reader to the papers [11–16].

Czerwik [17] developed the concept of $b-$MS in 1993 as a generalization of MS. To broaden the definition of b-MS, Kamran et al. [18] proposed the concept of an extended $b-$MS in 2017. In 2018, Mlaiki et al. [19] introduced a brand-new type of extended $b-$MS called controlled MS. Later, many authors examined controlled MSs and arrived at various FP findings for single and MVMs; see [20–25], for more information. In 2012, S-MS was investigated by Sedghi et al. [26] as a generalization of G-MS [27] and D-MS [28]. Also, they obtained some FP results for $S-$MSs.

Nadler [29] demonstrated one of the more exciting and well-known generalizations by utilizing the Hausdorff metric, which is defined on a family of closed and bounded subsets of a complete MS. He was the first to propose the concept of multivalued contraction maps. We recall a few common notations and words for the reader's benefit.

Let $(\mho, \omega)$ be a MS (where ω refers to the distance) and $\mho^{cb}, \mho^{cp}$ be a non-empty closed bounded (compact) subsets of $\mho$, respectively. The symmetric functional $Z : \mho^{cb} \times \mho^{cb} \to \mathbb{R}_+ = [0, +\infty)$ described as

$$Z(\phi, \psi) = \max\left\{D\left(\phi, \psi\right), D\left(\psi, \phi\right)\right\},$$

where

$$D\left(\phi, \psi\right) = \sup_{\theta \in \phi} \inf_{\vartheta \in \psi} \omega(\theta, \vartheta), \text{ for all } \phi, \psi \in \mho^{cb},$$

is called the Hausdorff metric. The same author proved a FP theorem for MVMs that satisfy the symmetric contraction condition. Following that, some exciting FP results for MVMs were discovered; see, for example [30–35].

Similar to the previous approach, in this manuscript, we merge metric-like spaces with S-metric spaces using a control function to produce a new space known as CSMLSs. Further, some FP findings for MVMs have been obtained under this space. In addition, we provide several non-trivial examples to support our claims. The results produced generalize and unify several results in the same direction. Finally, as a sort of application, the existence of the solution to the DI was explored to support and test the adequacy of the theoretical results.

2. Preliminaries work

Here, to be thorough, we will now recollect certain fundamental definitions, premises, lemmas, and necessary results from the current literature for our research.

Assume that $\mho$ is a non-empty set with a parameter $\eta > 1$. Let $\zeta : \mho \times \mho \to [0, \infty)$ be a control function and $\omega_\eta : \mho \times \mho \to [0, \infty)$ be a mapping fulfilling the assumptions below for all $\theta, \vartheta, \rho \in \mho$,

(η_1) $\omega_\eta(\theta,\vartheta)=0$ if and only if $\theta=\vartheta$;

(η_2) $\omega_\eta(\theta,\vartheta)=0$ implies $\theta=\vartheta$;

(η_3) $\omega_\eta(\theta,\vartheta)=\omega_\eta(\vartheta,\theta)$;

(η_4) $\omega_\eta(\theta,\vartheta)\leq\omega_\eta(\theta,\rho)+\omega_\eta(\rho,\vartheta)$;

(η_5) $\omega_\eta(\theta,\vartheta)\leq\eta\left(\omega_\eta(\theta,\rho)+\omega_\eta(\rho,\vartheta)\right)$;

(η_6) $\omega_\eta(\theta,\vartheta)\leq\zeta(\theta,\rho)\,\omega_\eta(\theta,\rho)+\zeta(\rho,\vartheta)\,\omega_\eta(\rho,\vartheta)$.

Definition 2.1. *The pair $\left(\mho,\omega_\eta\right)$ is called*

(i) *metric-like space [17] if the axioms $(\eta_2)-(\eta_4)$ are satisfied.*

(ii) *$b-$MS [37] if the axioms (η_1), (η_3) and (η_5) hold.*

(iii) *$b-$metric-like space [37] if the axioms (η_2), (η_3) and (η_5) are true.*

(iv) *controlled MS [19] if the axioms (η_1), (η_3) and (η_6) are fulfilled.*

Definition 2.2. *[38] Assume that $\mho\neq\emptyset$ is a given set and $S:\mho^3\to\mathbb{R}_+$ (where $\mho^3=\mho\times\mho\times\mho$) is a function fulfilling the hypotheses below for all $\theta,\vartheta,\rho,\nu\in\mho$,*

(a) $S(\theta,\vartheta,\rho)=0$ *iff* $\theta=\vartheta=\rho$;

(b) $S(\theta,\vartheta,\rho)\leq S(\theta,\theta,\nu)+S(\vartheta,\vartheta,\nu)+S(\rho,\rho,\nu)$.

Then, $(\mho,S)$ is an $S-$MS and S is called an $S-$metric on $\mho$.

Example 2.1. *[39] Let $\mho=[0,\infty)$ and $\zeta\geq 0$. Define $S:\mho^3\to\mathbb{R}_+$ by*

$$S(\theta,\vartheta,\rho)=\begin{cases}0, & \text{if } \theta=\vartheta=\rho,\\ \max\{\theta,\vartheta,\rho\}-\zeta & \text{otherwise.}\end{cases}$$

Then S is an $S-$MS on $\mho$.

Example 2.2. *[39] Let $\mho=[0,\infty)$. Describe $S:\mho^3\to\mathbb{R}_+$ as*

$$S(\theta,\vartheta,\rho)=\begin{cases}0, & \text{if } \theta=\vartheta=\rho,\\ \theta+\vartheta+2\rho & \text{otherwise.}\end{cases}$$

Then S is an $S-$MS on $\mho$.

Definition 2.3. *[40] Let $\mho\neq\emptyset$ be a given set, $\sigma\geq 1$ and $S_\sigma:\mho^3\to\mathbb{R}_+$ (where $\mho^3=\mho\times\mho\times\mho$) be a function satisfying the axioms below for all $\theta,\vartheta,\rho,\nu\in\mho$,*

(s_1) $S_\sigma(\theta,\vartheta,\rho)=0$ *iff* $\theta=\vartheta=\rho$;

(s_2) $S_\sigma(\theta,\theta,\vartheta)=S_\sigma(\vartheta,\vartheta,\theta)$;

(b) $S_\sigma(\theta,\vartheta,\rho)\leq\sigma\left[S_\sigma(\theta,\theta,\nu)+S_\sigma(\vartheta,\vartheta,\nu)+S_\sigma(\rho,\rho,\nu)\right]$.

Then, $(\mho,S_\sigma)$ is an $S_\sigma-$MS and S_σ is called an $S_\sigma-$metric on $\mho$.

Example 2.3. *[40] Let $\mho\neq\emptyset$ be a given set with card $(\mho)\geq 5$ and $\mho=\mho_1\times\mho_2$ be a partition of $\mho$ such that card $(\mho_1)\geq 4$. Assume that $\sigma\geq 1$ and for all $\theta,\vartheta,\rho\in\mho$,*

$$S_\sigma(\theta,\vartheta,\rho)=\begin{cases}0, & \text{if } \theta=\vartheta=\rho,\\ 3\sigma, & \text{if } (\theta,\vartheta,\rho)\in\mho_1^3,\\ 1 & \text{if } (\theta,\vartheta,\rho)\notin\mho_1^3.\end{cases}$$

Then S_σ is an $S_\sigma-$metric on $\mho$.

Definition 2.4. *[39] Let $(\mho, S)$ be an $S-$MS. Describe the function $S_Z : \left(\mho^{cb}\right)^3 \to \mathbb{R}_+$ as*

$$S_Z\left(\wp_1, \wp_2, \wp_3\right) = Z_s\left(\wp_1, \wp_3\right) + Z_s\left(\wp_2, \wp_3\right),$$

where $Z_s\left(\wp_1, \wp_2\right) = \max\left\{z_s(\wp_1, \wp_2), z_s(\wp_2, \wp_1)\right\}$, $z_s(\wp_1, \wp_2) = \sup\left\{S\left(b, b, \wp_2\right) : b \in \wp_1\right\}$ and $S\left(b, b, \wp_2\right) = \inf\left\{S\left(b, b, c\right) : c \in \wp_2\right\}$. Further, S_Z is called the Hausdorff $S-$metric on $\mho^{cb}$ induced by S.

3. A CSMLS and topological properties

This part is concerned with introducing the idea of the CSMLS and studying its topological properties in terms of convergence, Cauchy sequences, and some examples, in addition to presenting some results for FPs under this new distance. Here, we suggest the control function $\zeta : \mho^3 \to [1, \infty)$.

Definition 3.1. *Let $\mho \neq \emptyset$ be any set and $S : \mho^3 \to \mathbb{R}_+$ and $\zeta : \mho^3 \to [1, \infty)$ be functions such that for all $\theta, \vartheta, \rho, \nu \in \mho$,*

(CS 1) $S\left(\theta, \vartheta, \rho\right) = 0$ *implies* $\theta = \vartheta = \rho$;

(CS 2) $S\left(\theta, \vartheta, \rho\right) \leq \zeta\left(\theta, \theta, \nu\right) S\left(\theta, \theta, \nu\right) + \zeta\left(\vartheta, \vartheta, \nu\right) S\left(\vartheta, \vartheta, \nu\right) + \zeta\left(\rho, \rho, \nu\right) S\left(\rho, \rho, \nu\right).$

Then, $(\mho, S, \zeta)$ is a CSMLS.

Remark 3.1. *It should be noted that the class of CSMLS is larger than the class of $S-$metric-like space. Further, every $S-$metric-like space is CSMLS with $\zeta\left(., ., .\right) = 1$.*

Example 3.1. *Consider $\mho = [0, 1]$ and describe the functions $S : \mho^3 \to [0, +\infty)$ and $\zeta : \mho^3 \to [1, +\infty)$ as $S\left(\theta, \vartheta, \rho\right) = \left|\theta - \rho\right|^2 + \left|\vartheta - \rho\right|^2$, and*

$$\zeta\left(\theta, \vartheta, \rho\right) = \begin{cases} \max\left\{\frac{1}{\theta}, \frac{1}{\vartheta}, \frac{1}{\rho}\right\}, & \text{if } \theta \neq 0,\ \vartheta \neq 0,\ \rho \neq 0, \\ 1, & \text{otherwise}, \end{cases}$$

receptively. Then, a trio $(\mho, S, \zeta)$ is a CSMLS.

Verifications. *We realize the following cases:*

(i) *If $\theta = \vartheta = \rho = 0$, or $\theta = \vartheta = \rho \neq 0$, the investigation is trivial.*

(ii) *If $\theta \neq 0$, $\vartheta \neq 0$, $\rho \neq 0$ with $\theta > \nu$, $\vartheta > \nu$ and $\rho > \nu$, we have*

$$\begin{aligned} S\left(\theta, \vartheta, \rho\right) &= \left|\theta - \rho\right|^2 + \left|\vartheta - \rho\right|^2 = \left|\theta - \nu + \nu - \rho\right|^2 + \left|\vartheta - \nu + \nu - \rho\right|^2 \\ &= \left|\theta - \nu\right|^2 + \left|\nu - \rho\right|^2 + 2\left(\theta - \nu\right)\left(\nu - \rho\right) + \left|\vartheta - \nu\right|^2 + \left|\nu - \rho\right|^2 + 2\left(\vartheta - \nu\right)\left(\nu - \rho\right) \\ &\leq \left|\theta - \nu\right|^2 + \left|\nu - \rho\right|^2 + 2\left|\theta - \nu\right|\left|\nu - \rho\right| + \left|\vartheta - \nu\right|^2 + \left|\nu - \rho\right|^2 + 2\left|\vartheta - \nu\right|\left|\nu - \rho\right| \\ &= \left(\left|\theta - \nu\right|^2 + \left|\vartheta - \nu\right|^2 + \left|\rho - \nu\right|^2\right) + \left(\left|\nu - \rho\right|^2 + \left|\nu - \rho\right|^2\right) \\ &\quad + 2\left|\nu - \rho\right|\left(\left|\theta - \nu\right| + \left|\vartheta - \nu\right|\right) \\ &\leq \left(\left|\theta - \nu\right|^2 + \left|\vartheta - \nu\right|^2 + \left|\rho - \nu\right|^2\right) + \left(\left|\rho - \rho\right|^2 + \left|\rho - \rho\right|^2\right) \end{aligned}$$

$$
\begin{aligned}
&\quad +2\left|\rho-\rho\right|\left(|\theta-\nu|+|\vartheta-\nu|\right)\\
&= \left(|\theta-\nu|^2+|\vartheta-\nu|^2+\left|\rho-\nu\right|^2\right)\\
&\leq \frac{2}{v}\left(|\theta-\nu|^2+|\vartheta-\nu|^2+\left|\rho-\nu\right|^2\right)\\
&= \max\left\{\frac{1}{\theta},\frac{1}{\theta},\frac{1}{\nu}\right\}\left(2|\theta-\nu|^2\right)+\max\left\{\frac{1}{\vartheta},\frac{1}{\vartheta},\frac{1}{\nu}\right\}\left(2|\vartheta-\nu|^2\right)\\
&\quad +\max\left\{\frac{1}{\rho},\frac{1}{\rho},\frac{1}{\nu}\right\}\left(2\left|\rho-\nu\right|^2\right)\\
&= \zeta(\theta,\theta,\nu)\,S(\theta,\theta,\nu)+\zeta(\vartheta,\vartheta,\nu)\,S(\vartheta,\vartheta,\nu)+\zeta(\rho,\rho,\nu)\,S(\rho,\rho,\nu).
\end{aligned}
$$

(iii) *if $\theta \neq 0$, $\vartheta \neq 0$, $\rho \neq 0$ with $\theta < \nu$, $\vartheta < \nu$ and $\rho < \nu$, we get*

$$
\begin{aligned}
&\quad \zeta(\theta,\theta,\nu)\,S(\theta,\theta,\nu)+\zeta(\vartheta,\vartheta,\nu)\,S(\vartheta,\vartheta,\nu)+\zeta(\rho,\rho,\nu)\,S(\rho,\rho,\nu)\\
&= \max\left\{\frac{1}{\theta},\frac{1}{\theta},\frac{1}{\nu}\right\}\left(2|\theta-\nu|^2\right)+\max\left\{\frac{1}{\vartheta},\frac{1}{\vartheta},\frac{1}{\nu}\right\}\left(2|\vartheta-\nu|^2\right)\\
&\quad +\max\left\{\frac{1}{\rho},\frac{1}{\rho},\frac{1}{\nu}\right\}\left(2\left|\rho-\nu\right|^2\right)\\
&= 2\left(\frac{1}{\theta}|\theta-\nu|^2+\frac{1}{\vartheta}|\vartheta-\nu|^2+\frac{1}{v}\left|\rho-\nu\right|^2\right)\\
&\geq \frac{2}{v}\left(|\theta-\nu|^2+|\vartheta-\nu|^2+\left|\rho-\nu\right|^2\right)\ \ (\textit{since } \theta<\nu,\ \vartheta<v \textit{ and } \rho<v)\\
&\geq |\theta-\nu|^2+\left|\rho-\nu\right|^2+|\vartheta-\nu|^2\\
&= |\nu-\theta|^2+\left|\nu-\rho\right|^2+|\nu-\vartheta|^2\\
&\geq \left|\rho-\theta\right|^2+\left|\rho-\rho\right|^2+\left|\rho-\vartheta\right|^2\ \ (\textit{since } \rho<\nu)\\
&= \left|\theta-\rho\right|^2+\left|\vartheta-\rho\right|^2=S(\theta,\vartheta,\rho).
\end{aligned}
$$

(iv) *if $\theta, \vartheta, \rho \in \mho - \{0\}$, we have $\zeta(.,.,.) = 1$ and Condition (CS 2) is met directly.*

From the above cases we deduce that $(\mho, S, \zeta)$ is a CSMLS.

In the context of CSMLS $(\mho, S, \zeta)$, we define a topology μ_S on $\mho$ whose base is the family of open-balls defined by $\mu_S = \{G \subset \mho : G \text{ is a union of open balls}\}$.

Example 3.2. *Let $\mho = [0, +\infty)$ with the $S-$metric*

$$
S(\theta,\vartheta,\rho)=\begin{cases} 0, & \textit{if } \theta=\vartheta=\rho,\\ \max\left\{|\theta|,|\vartheta|,\left|\rho\right|\right\} & \textit{otherwise,}\end{cases}
$$

for $\zeta > 0$. Also, assume the control function

$$
\zeta(\theta,\vartheta,\rho)=\begin{cases} \max\left\{\frac{1}{\theta},\frac{1}{\vartheta},\frac{1}{\rho}\right\}, & \textit{if } \theta\neq 0,\ \vartheta\neq 0,\ \rho\neq 0,\\ 1, & \textit{otherwise,}\end{cases}
$$

Then for $r \in \mho$ and $\lambda > 0$, we have

$$B_S(\lambda, r) = \begin{cases} [0, r) & \text{if } \lambda < r, \\ \{\lambda\}, & \text{if } \lambda \geq r, \end{cases}$$

where $B_S(\lambda, r)$ refers to an open ball with radius r and center λ and it is described as

$$B_S(\lambda, r) = \{\rho \in \mho : S(\rho, \rho, \lambda) < r\}, \text{ for } r > 0.$$

Definition 3.2. *Let $(\mho, S, \zeta)$ be a CSMLS, then S is continuous if $S(\theta_u, \vartheta_u, \rho_u) \to S(\theta, \vartheta, \rho)$, whenever $\theta_u \to \theta$, $\vartheta_u \to \vartheta$ and $\rho_u \to \rho$, as $u \to \infty$, but the converse is not true.*

The example below illustrates the above definition.

Example 3.3. *Let $\mho = [0, +\infty)$, define $S : \mho^3 \to \mathbb{R}_+$ and $\zeta : \mho^3 \to [1, +\infty)$ by*

$$S(\theta, \vartheta, \rho) = \begin{cases} 1, & \text{if } (\theta, \vartheta, \rho) = (1, 2, 3), \\ |\theta - \rho|^2 + |\vartheta - \rho|^2 & \text{otherwise,} \end{cases}$$

and

$$\zeta(\theta, \vartheta, \rho) = \begin{cases} \max\left\{\frac{1}{\theta}, \frac{1}{\vartheta}, \frac{1}{\rho}\right\}, & \text{if } \theta \neq 0,\ \vartheta \neq 0,\ \rho \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

According to Example 3.1, if $(\theta, \vartheta, \rho) \neq (1, 2, 3)$, for $\nu \in \mho$, then a trio $(\mho, S, \zeta)$ is a CSMLS. Now, if $(\theta, \vartheta, \rho) = (1, 2, 3)$, then $S(\theta, \vartheta, \rho) = 1$ and for $\nu \in (1, 2, 3)$, we have $\zeta(\theta, \theta, \nu) = 1$, $S(\vartheta, \vartheta, \nu) = 1$, $S(\rho, \rho, \nu) = 1$, and

$$\begin{aligned} S(\theta, \vartheta, \rho) &= 1 \\ &\leq \max\left\{\frac{1}{\theta}, \frac{1}{\vartheta}, \frac{1}{\rho}, \frac{1}{\nu}\right\} \\ &\leq \max\left\{\frac{1}{\theta}, \frac{1}{\theta}, \frac{1}{v}\right\} + \max\left\{\frac{1}{\vartheta}, \frac{1}{\vartheta}, \frac{1}{\nu}\right\} + \max\left\{\frac{1}{\rho}, \frac{1}{\rho}, \frac{1}{\nu}\right\} \\ &= \zeta(\theta, \theta, \nu) S(\theta, \theta, \nu) + \zeta(\vartheta, \vartheta, \nu) S(\vartheta, \vartheta, \nu) + \zeta(\rho, \rho, \nu) S(\rho, \rho, \nu). \end{aligned}$$

Therefore, $(\mho, S, \zeta)$ is a CSMLS. Since $\lim_{u\to\infty} S(\theta_u, \vartheta_u, \rho_u) = S(\theta, \vartheta, \rho)$, provided that $\lim_{u\to\infty} \theta_u = \theta$, $\lim_{u\to\infty} \vartheta_u = \vartheta$ and $\lim_{u\to\infty} \rho_u = \rho$, then take $\theta_u = 1 + \frac{1}{u}$, $\vartheta_u = 2 + \frac{2}{u}$ and $\rho_u = 3 + \frac{3}{u}$, we have $\theta_u \to 1$, $\vartheta_u \to 2$, and $\rho_u \to 3$. But

$$5 = \lim_{u\to\infty} S(\theta_u, \vartheta_u, \rho_u) \neq S(1, 2, 3) = 1.$$

Therefore, S is not continuous.

Definition 3.3. *Let $(\mho, S, \zeta)$ be a CSMLS. Then, the assertions below are true:*

(i) *A sequence $\{\theta_u\} \subset \mho$ is convergent to $\theta \in \mho$, if $\lim_{u\to+\infty} S(\theta_u, \theta_u, \theta) = S(\theta, \theta, \theta)$ and we can write $\theta_u \to \theta$ or $\lim_{u\to+\infty} \theta_u = \theta$.*

(ii) *We say that a sequence $\{\theta_u\} \subset \mho$ is a Cauchy sequence, if $\lim_{u,j\to+\infty} S\left(\theta_u, \theta_u, \theta_j\right)$ exists and is finite.*

(iii) *A trio* $(\mho, S, \zeta)$ *is complete, if every Cauchy sequence in* $\mho$*, there exists* $\theta \in \mho$ *such that* $\lim_{u\to+\infty} S\left(\theta_u, \theta_u, \theta_j\right) = S(\theta, \theta, \theta) = \lim_{u\to+\infty} S(\theta_u, \theta_u, \theta)$.

The proof of the following lemma is obvious:

Lemma 3.1. *Let* $(\mho, S, \zeta)$ *be a CSMLS and* $\{\theta_u\}$ *be a sequence in* $\mho$*. Then, for all* $\rho, \theta \in \mho$*, we get*

($\heartsuit_1$) *If* $S(\theta, \theta, \rho) = 0$*, then* $S(\theta, \theta, \theta) = S(\rho, \rho, \rho) = 0$;

($\heartsuit_2$) *If* $\lim_{u\to+\infty} S(\theta_u, \theta_u, \theta_{u+1}) = 0$*, then* $\lim_{u\to+\infty} S(\theta_u, \theta_u, \theta_u) = \lim_{u\to+\infty} S(\theta_{u+1}, \theta_{u+1}, \theta_{u+1}) = 0$;

($\heartsuit_3$) *If* $\theta \neq \rho$*, then* $S(\theta, \theta, \rho) > 0$.

4. Results on fixed points

We begin this section with the following definition:

Definition 4.1. *Assume that* $(\mho, S, \zeta)$ *is a CSMLS and* $\Im : \mho \to \mho^{cb}$ *is a MVM. We say that* $\Im$ *is a contraction, if there is a constant* $\mu \in (0,1)$ *such that*

$$S_Z(\Im(\theta), \Im(\theta), \Im(\rho)) \leq \mu S(\theta, \theta, \rho). \tag{4.1}$$

for all $\theta, \rho \in \mho$.

Theorem 4.1. *Let* $(\mho, S, \zeta)$ *be a CSMLS and* $\Im : \mho \to \mho^{cb}$ *be a MVM. If* $\Im$ *is a contraction and for* $\theta_0 \in \mho$*, there exists a sequence* $\theta_u = \Im(\theta_{u-1})$*,* $u \in \mathbb{N}$ *such that*

$$\sup_{v\geq 1} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} \leq \frac{1}{2\mu}, \tag{4.2}$$

and $\lim_{j\to+\infty} \zeta(\theta_u, \theta_u, \theta_{u+1})$ *exists. Then,* $\Im$ *owns a unique FP.*

Proof. Assume that $\theta \in \mho$ and select $\theta_1 \in \Im(\theta_0)$. Then for $\mu \in (0,1)$, there is $\theta_2 \in \Im(\theta_1)$ such that

$$S(\theta_2, \theta_2, \theta_1) \leq S_Z(\Im(\theta_1), \Im(\theta_1), \Im(\theta_0)) + \mu \leq \mu S(\theta_1, \theta_1, \theta_0) + \mu.$$

Analogously, there is $\theta_3 \in \Im(\theta_2)$ fulfilling

$$\begin{aligned}
S(\theta_3, \theta_3, \theta_2) &\leq S_Z(\Im(\theta_2), \Im(\theta_2), \Im(\theta_1)) + \mu \\
&\leq \mu S(\theta_2, \theta_2, \theta_1) + \mu \\
&\leq \mu(\mu S(\theta_1, \theta_1, \theta_0) + \mu) + \mu \\
&\leq \mu^2 S(\theta_1, \theta_1, \theta_0) + \mu^2 + \mu \\
&\leq \mu^2 S(\theta_1, \theta_1, \theta_0) + \mu + \mu \\
&= \mu^2 S(\theta_1, \theta_1, \theta_0) + 2\mu.
\end{aligned}$$

Repeating the same approach, we find that $\theta_{u+1} \in \Im(\theta_u)$ and $\mu^u \in (0,1)$ such that

$$S(\theta_{u+1}, \theta_{u+1}, \theta_u) \leq S_Z(\Im(\theta_u), \Im(\theta_u), \Im(\theta_{u-1})) + \mu^u \leq \mu^u S(\theta_1, \theta_1, \theta_0) + u\mu.$$

Next, we claim that the sequence $\{\theta_u\}_{u\in\mathbb{N}}$ is a Cauchy sequence. Let $v > u$ and by Axiom (CS 2), one has

$$\begin{aligned} S(\theta_v,\theta_v,\theta_u) &\leq \zeta(\theta_v,\theta_v,\theta_{u+1})\, S(\theta_v,\theta_v,\theta_{u+1}) + \zeta(\theta_v,\theta_v,\theta_{u+1})\, S(\theta_v,\theta_v,\theta_{u+1}) \\ &\quad +\zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}) \\ &= 2\zeta(\theta_v,\theta_v,\theta_{u+1})\, S(\theta_v,\theta_v,\theta_{u+1}) + \zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}). \end{aligned}$$

Again, applying Axiom (CS 2) on $S(\theta_v,\theta_v,\theta_{u+1})$, we can write

$$\begin{aligned} S(\theta_v,\theta_v,\theta_u) &\leq 2\zeta(\theta_v,\theta_v,\theta_{u+1}) \begin{pmatrix} 2\zeta(\theta_v,\theta_v,\theta_{u+2})\, S(\theta_v,\theta_v,\theta_{u+2}) \\ +\zeta(\theta_{u+1},\theta_{u+1},\theta_{u+2})\, S(\theta_{u+1},\theta_{u+1},\theta_{u+2}) \end{pmatrix} \\ &\quad +\zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}) \\ &= \zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}) \\ &\quad +2\zeta(\theta_v,\theta_v,\theta_{u+1})\, \zeta(\theta_{u+1},\theta_{u+1},\theta_{u+2})\, S(\theta_{u+1},\theta_{u+1},\theta_{u+2}) \\ &\quad +2^2\zeta(\theta_v,\theta_v,\theta_{u+1})\, \zeta(\theta_v,\theta_v,\theta_{u+2})\, S(\theta_v,\theta_v,\theta_{u+2}). \end{aligned}$$

Using Axiom (CS 2) in the same way as before, we get

$$\begin{aligned} S(\theta_v,\theta_v,\theta_u) &\leq \zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}) \\ &\quad + \sum_{j=u+1}^{v-2} 2^{j-u} \prod_{w=n+1}^{j} \zeta(\theta_v,\theta_v,\theta_w)\, \zeta\left(\theta_j,\theta_j,\theta_{j+1}\right) S\left(\theta_j,\theta_j,\theta_{j+1}\right) \\ &\quad +2^{v-u-1} \prod_{y=n+1}^{v-1} \zeta\left(\theta_v,\theta_v,\theta_y\right) S(\theta_{v-1},\theta_{v-1},\theta_v) \\ &\leq \zeta(\theta_u,\theta_u,\theta_{u+1})\, S(\theta_u,\theta_u,\theta_{u+1}) \\ &\quad + \sum_{j=u+1}^{v-1} 2^{j-u} \prod_{w=n+1}^{j} \zeta(\theta_v,\theta_v,\theta_w)\, \zeta\left(\theta_j,\theta_j,\theta_{j+1}\right) S\left(\theta_j,\theta_j,\theta_{j+1}\right). \end{aligned}$$

Now, by ratio test, the series

$$S_v = \sum_{j=u}^{v-1} 2^{j-u} \prod_{w=n}^{j} \zeta(\theta_v,\theta_v,\theta_w)\, \zeta\left(\theta_j,\theta_j,\theta_{j+1}\right) \left(\mu^j S(\theta_0,\theta_0,\theta_1) + 2\mu\right),$$

converges if

$$\sup_{v\geq j} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1},\theta_{j+1},\theta_{j+2}\right) \zeta\left(\theta_v,\theta_v,\theta_{j+1}\right)}{\zeta\left(\theta_j,\theta_j,\theta_{j+1}\right)} \leq \frac{1}{2\mu}.$$

Passing $v,u \to +\infty$, we obtain that

$$\lim_{v,u\to+\infty} S(\theta_v,\theta_v,\theta_u) = 0.$$

Hence, $\lim_{u,j\to+\infty} S\left(\theta_u,\theta_u,\theta_j\right)$ exists and is finite. Therefore $\{\theta_u\}_{u\in\mathbb{N}}$ is a Cauchy sequence in $\mho$. Since $\mho$ is complete, then the sequence θ_u in $\mho$ converges to $\theta\in\mho$ such that

$$\lim_{u\to+\infty} S\left(\theta_u,\theta_u,\theta\right) = S\left(\theta,\theta,\theta\right) = \lim_{v,u\to+\infty} S\left(\theta_u,\theta_u,\theta_v\right) = 0.$$

Applying the contraction condition (4.1), the MVM $\Im$ is continuous and since $\theta_u \in \Im\left(\theta_{u-1}\right)$, we have $\theta\in\Im\left(\theta\right)$, that is θ is a FP of $\Im$. The uniqueness follows immediately by the condition (4.1). □

The example below support Theorem 4.1.

Example 4.1. *Assume that $\mho=[0,1]$. Describe the functions $S:\mho^3\to\mathbb{R}_+$ and $\zeta:\mho^3\to[1,+\infty)$ by*

$$S\left(\theta,\vartheta,\rho\right) = \frac{\left|\theta-\rho\right|^2+\left|\vartheta-\rho\right|^2}{2}$$

and

$$\zeta\left(\theta,\vartheta,\rho\right) = 1+|\theta|.$$

for all $\theta,\vartheta,\rho\in\mho$. The axiom (CS 1) holds directly. To check the axiom (CS 2), for all $\nu\in\mho$, we have

$$\begin{aligned}
& \zeta\left(\theta,\theta,\nu\right)S\left(\theta,\theta,\nu\right)+\zeta\left(\vartheta,\vartheta,\nu\right)S\left(\vartheta,\vartheta,\nu\right)+\zeta\left(\rho,\rho,\nu\right)S\left(\rho,\rho,\nu\right)\\
= & \left(1+|\theta|\right)\left(|\theta-\nu|^2\right)+\left(1+|\vartheta|\right)\left(|\vartheta-\nu|^2\right)+\left(1+\left|\rho\right|\right)\left(\left|\rho-\nu\right|^2\right)\\
= & \left(1+|\theta|\right)\left(\left|\theta-\rho+\rho-\nu\right|^2\right)+\left(1+|\vartheta|\right)\left(\left|\vartheta-\rho+\rho-\nu\right|^2\right)+\left(1+\left|\rho\right|\right)\left(\left|\rho-\nu\right|^2\right)\\
\geq & \max\{1+|\theta|,1+|\vartheta|,1+\left|\rho\right|\}\left(\left(\left|\theta-\rho+\rho-\nu\right|^2\right)+\left(\left|\vartheta-\rho+\rho-\nu\right|^2\right)+\left(\left|\rho-\nu\right|^2\right)\right)\\
\geq & \left(\left|\theta-\rho\right|^2+\left|\rho-v\right|^2\right)+\left(\left|\vartheta-\rho\right|^2+\left|\rho-v\right|\right)^2+\left|\rho-\nu\right|^2\\
= & 3\left|\rho-\nu\right|^2+\left(\left|\theta-\rho\right|^2+\left|\vartheta-\rho\right|^2\right)\\
\geq & \left|\theta-\rho\right|^2+\left|\vartheta-\rho\right|^2\\
\geq & \frac{\left|\theta-\rho\right|^2+\left|\vartheta-\rho\right|^2}{2} = S\left(\theta,\vartheta,\rho\right).
\end{aligned}$$

Hence, $(\mho,S,\zeta)$ is a CSMLS. Define the MVM $\Im:\mho\to\mho^{cb}$ by $\Im\left(\theta\right)=\left\{\frac{\theta}{3},0\right\}$. Now, we examine the condition (4.1). Using Definition 2.4, for $\theta,\vartheta,\rho\in\mho$, we get

$$\begin{aligned}
S_Z\left(\Im\left(\theta\right),\Im\left(\theta\right),\Im\left(\rho\right)\right) &= Z_s\left(\Im\left(\theta\right),\Im\left(\rho\right)\right)+Z_s\left(\Im\left(\theta\right),\Im\left(\rho\right)\right)\\
&= 2Z_s\left(\Im\left(\theta\right),\Im\left(\rho\right)\right)\\
&= 2\max\left\{z_s\left(\Im\left(\theta\right),\Im\left(\rho\right)\right),z_s\left(\Im\left(\rho\right),\Im\left(\theta\right)\right)\right\},
\end{aligned}$$

where

$$\begin{aligned} z_s\left(\mathfrak{I}(\theta), \mathfrak{I}(\rho)\right) &= \sup_{\theta^* \in \mathfrak{I}(\theta)} \inf_{\rho^* \in \mathfrak{I}(\rho)} S\left(\theta^*, \theta^*, \rho^*\right) \\ &= \sup_{\theta^* \in \mathfrak{I}(\theta)} \inf_{\rho^* \in \mathfrak{I}(\rho)} \frac{\left|\theta^* - \rho^*\right|^2 + \left|\theta^* - \rho^*\right|^2}{2} \\ &= \sup_{\theta^* \in \mathfrak{I}(\theta)} \inf \left\{ \left|\theta^* - \frac{\rho}{3}\right|^2 \right\}. \end{aligned}$$

If $\theta^* = \frac{\theta}{3} \in \mathfrak{I}(\theta)$ *then* $\inf\left\{\left|\frac{\theta}{3} - \frac{\rho}{3}\right|^2\right\} = 0$. *If* $\theta^* = 0 \in \mathfrak{I}(\theta)$, *then* $\inf\left\{\left|\frac{\theta}{3} - \frac{\rho}{3}\right|^2\right\} = 0$.

Analogously, $z_s\left(\mathfrak{I}(\rho), \mathfrak{I}(\theta)\right) = 0$. *Hence,*

$$S_Z\left(\mathfrak{I}(\theta), \mathfrak{I}(\theta), \mathfrak{I}(\rho)\right) = 2.\max\{0,0\} = 0 \leq \mu S(\theta, \theta, \rho).$$

Thus, the contractive condition (4.1) is fulfilled with any $\mu \in (0,1)$. *Taking* $\mu = \frac{2}{9} < 1$ *and the sequence* $\theta_j = \left(1 + \frac{1}{j}\right)$, *we have* $\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right) = \left(2 + \frac{1}{|j+1|}\right)$, $\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right) = \left(2 + \frac{1}{|v|}\right)$ *and* $\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right) = \left(2 + \frac{1}{|j|}\right)$.

Further,

$$\sup_{v\geq 1} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} = 2 < \frac{9}{4} = \frac{1}{2\mu}.$$

Therefore, all requirements of Theorem 4.1 are satisfied. Hence, $\mathfrak{I}$ *owns a unique FP which is* $0 \in \mho$.

Corollary 4.1. *Let* $(\mho, S, \zeta)$ *be a CSMLS and* $\mathfrak{I} : \mho \to \mho^{cb}$ *be a contraction MVM on* $\mho$ *with* $\mu \in \left(0, \frac{1}{2}\right)$. *Then,* $\mathfrak{I}$ *possesses a unique FP.*

Proof. We reach the required result by taking $\zeta(\theta, \vartheta, \rho) = 1$ and using the same methods as in the preceding theorem. □

Theorem 4.2. *Let* $(\mho, S, \zeta)$ *be a CSMLS and* $\mathfrak{I} : \mho \to \mho^{cb}$ *be a MVM such that*

$$S_Z\left(\mathfrak{I}(\theta), \mathfrak{I}(\theta), \mathfrak{I}(\rho)\right) \leq \gamma S\left(\theta, \theta, \mathfrak{I}(\theta)\right) + \delta S\left(\rho, \rho, \mathfrak{I}(\rho)\right), \tag{4.3}$$

where $\gamma, \delta \in (0,1)$ *with* $\gamma + \delta < 1$. *Moreover, for* $\theta_0 \in \mho$, *there exists a sequence* $\theta_u = \mathfrak{I}(\theta_{u-1})$, $u \in \mathbb{N}$ *such that*

$$\sup_{v\geq 1} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} \leq \frac{1-\delta}{2\gamma},$$

and $\lim_{u\to+\infty} \zeta\left(\theta_u, \theta_u, \theta_{u+1}\right)$ *exists. Then,* $\mathfrak{I}$ *has a unique FP.*

Proof. Continuing along the same path as Theorem 4.1, there is a sequence $\{\theta_u\}_{u\in\mathbb{N}}$ in $\mho$ and $\mu^u \in (0,1)$ so that

$$\begin{aligned} S\left(\theta_u, \theta_u, \theta_{u+1}\right) &\leq S_Z\left(\mathfrak{I}(\theta_{u-1}), \mathfrak{I}(\theta_{u-1}), \mathfrak{I}(\theta_u)\right) + \mu^u \\ &\leq \gamma S\left(\theta_{u-1}, \theta_{u-1}, \mathfrak{I}(\theta_{u-1})\right) + \delta S\left(\theta_u, \theta_u, \mathfrak{I}(\theta_u)\right) + \mu^u \\ &\leq \gamma S\left(\theta_{u-1}, \theta_{u-1}, \theta_u)\right) + \delta S\left(\theta_u, \theta_u, \theta_{u+1})\right) + \mu^u, \end{aligned}$$

which yields

$$\begin{aligned} S\left(\theta_u,\theta_u,\theta_{u+1}\right) &\leq \frac{\gamma}{1-\delta}S\left(\theta_{u-1},\theta_{u-1},\theta_u\right)) + \frac{\mu^u}{1-\delta} \\ &\leq \frac{\gamma^2}{(1-\delta)^2}S\left(\theta_{u-2},\theta_{u-2},\theta_{u-1}\right)) + \frac{\mu^u}{1-\delta} + \frac{\gamma\mu^{u-1}}{(1-\delta)^2}. \end{aligned}$$

Proceeding in this manner, we arrive at

$$\begin{aligned} S\left(\theta_u,\theta_u,\theta_{u+1}\right) &\leq \frac{\gamma^u}{(1-\delta)^u}S\left(\theta_0,\theta_0,\theta_1\right)) + \left(\frac{\mu^u}{1-\delta} + \frac{\gamma\mu^{u-1}}{(1-\delta)^2} + \cdots + \frac{\gamma^{u-1}\mu}{(1-\delta)^u}\right) \\ &\leq \frac{\gamma^u}{(1-\delta)^u}S\left(\theta_0,\theta_0,\theta_1\right)) + \sum_{j=0}^{u-1}\frac{\gamma^j\mu^{u-j}}{(1-\delta)^{j+1}}. \end{aligned}$$

Letting $u \to +\infty$, we have

$$\lim_{u\to+\infty} S\left(\theta_u,\theta_u,\theta_{u+1}\right) = 0.$$

In the same way as in the proof of Theorem 4.1, for $v,u \in \mathbb{N}$ with $v > u$, we find that

$$\sum_{j=u+1}^{v-1} 2^{j-u} \prod_{w=n+1}^{j} \zeta\left(\theta_v,\theta_v,\theta_w\right)\zeta\left(\theta_j,\theta_j,\theta_{j+1}\right)S\left(\theta_j,\theta_j,\theta_{j+1}\right) \to 0, \text{ as } u,v \to +\infty,$$

provided that

$$\sup_{v\geq 1}\lim_{j\to+\infty}\frac{\zeta\left(\theta_{j+1},\theta_{j+1},\theta_{j+2}\right)\zeta\left(\theta_v,\theta_v,\theta_{j+1}\right)}{\zeta\left(\theta_j,\theta_j,\theta_{j+1}\right)} \leq \frac{1-\delta}{2\gamma}.$$

Thus,

$$\lim_{v,u\to+\infty} S\left(\theta_v,\theta_v,\theta_u\right) = 0.$$

Hence, $\lim_{u,j\to+\infty} S\left(\theta_u,\theta_u,\theta_j\right)$ exists and is finite. Therefore $\{\theta_u\}_{u\in\mathbb{N}}$ is a Cauchy sequence in $\mho$. The completeness of $\mho$ implies that there is a convergent sequence θ_u in $\mho$ to $\theta \in \mho$ so that

$$\lim_{u\to+\infty} S\left(\theta_u,\theta_u,\theta\right) = S\left(\theta,\theta,\theta\right) = \lim_{v,u\to+\infty} S\left(\theta_u,\theta_u,\theta_v\right) = 0.$$

Applying the contraction (4.3), the MVM $\mathfrak{I}$ is continuous and since $\theta_u \in \mathfrak{I}\left(\theta_{u-1}\right)$, we have $\theta \in \mathfrak{I}\left(\theta\right)$, that is θ is a FP of $\mathfrak{I}$. The uniqueness follows immediately by condition (4.3). □

Example 4.2. *Let $\mho = \{1,2,3\}$. Describe the functions $S : \mho^3 \to \mathbb{R}_+$ and $\zeta : \mho^3 \to [1,+\infty)$ as*

$$S\left(\theta,\vartheta,\rho\right) = \frac{\left|\theta-\rho\right|^2 + \left|\vartheta-\rho\right|^2}{2}$$

and

$$\zeta\left(\theta,\vartheta,\rho\right) = 1 + \frac{|\theta+\vartheta|}{2},$$

respectively, for all $\theta, \vartheta, \rho \in \mho$. Clearly, the axiom (CS 1) is trivial. To check the axiom (CS 2), for all $\nu \in \mho$, we have

$$\begin{aligned}
& \zeta(\theta,\theta,\nu) S(\theta,\theta,\nu) + \zeta(\vartheta,\vartheta,\nu) S(\vartheta,\vartheta,\nu) + \zeta(\rho,\rho,\nu) S(\rho,\rho,\nu) \\
= \; & (1+|\theta|)\left(|\theta-\nu|^2\right) + (1+|\vartheta|)\left(|\vartheta-\nu|^2\right) + \left(1+\left|\rho\right|\right)\left(\left|\rho-\nu\right|^2\right) \\
= \; & (1+|\theta|)\left(\left|\theta-\rho+\rho-\nu\right|^2\right) + (1+|\vartheta|)\left(\left|\vartheta-\rho+\rho-\nu\right|^2\right) + \left(1+\left|\rho\right|\right)\left(\left|\rho-\nu\right|^2\right) \\
\geq \; & \max\{1+|\theta|, 1+|\vartheta|, 1+\left|\rho\right|\}\left(\left(\left|\theta-\rho+\rho-\nu\right|^2\right) + \left(\left|\vartheta-\rho+\rho-\nu\right|^2\right) + \left(\left|\rho-\nu\right|^2\right)\right) \\
\geq \; & \left(\left|\theta-\rho\right|^2 + \left|\rho-v\right|^2\right) + \left(\left|\vartheta-\rho\right|^2 + \left|\rho-v\right|\right)^2 + \left|\rho-\nu\right|^2 \\
= \; & 3\left|\rho-\nu\right|^2 + \left(\left|\theta-\rho\right|^2 + \left|\vartheta-\rho\right|^2\right) \\
\geq \; & \left|\theta-\rho\right|^2 + \left|\vartheta-\rho\right|^2 \\
\geq \; & \frac{\left|\theta-\rho\right|^2 + \left|\vartheta-\rho\right|^2}{2} = S(\theta,\vartheta,\rho).
\end{aligned}$$

Hence, $(\mho, S, \zeta)$ is a CSMLS. Define the MVM $\Im : \mho \to \mho^{cb}$ by

$$\Im(\theta) = \begin{cases} \{1,3\}, & \text{if } \theta \in \{1,2\}, \\ 2, & \text{if } \theta = 3. \end{cases}$$

Now, we check the contraction (4.1). Using Definition 2.4, for $\theta, \vartheta, \rho \in \mho$, we have

$$\begin{aligned}
S_Z(\Im(\theta), \Im(\theta), \Im(\rho)) &= Z_s(\Im(\theta), \Im(\rho)) + Z_s(\Im(\theta), \Im(\rho)) \\
&= 2Z_s(\Im(\theta), \Im(\rho)) \\
&= 2\max\{z_s(\Im(\theta), \Im(\rho)), z_s(\Im(\rho), \Im(\theta))\},
\end{aligned}$$

where

$$\begin{aligned}
z_s(\Im(\theta), \Im(\rho)) &= \sup_{\theta^* \in \Im(\theta)} \inf_{\rho^* \in \Im(\rho)} S(\theta^*, \theta^*, \rho^*) \\
&= \sup_{\theta^* \in \Im(\theta)} \inf_{\rho^* \in \Im(\rho)} \frac{\left|\theta^*-\rho^*\right|^2 + \left|\theta^*-\rho^*\right|^2}{2} \\
&= \sup_{\theta^* \in \Im(\theta)} \inf_{\rho^* \in \Im(\rho)} \left\{\left|\theta^*-\rho^*\right|^2\right\}. \\
&= \sup_{\theta^* \in \Im\theta} \inf\left\{\left|\theta^*-\rho\right|^2\right\}
\end{aligned}$$

If $\theta \in \{1,2\}$, then $\theta^* \in \{1,3\}$ and $\inf\left\{|\theta^*-1|^2, |\theta^*-2|^2, |\theta^*-3|^2\right\} = 0$. If $\theta = 3$, then $\theta^* = 2$ and $\inf\left\{|\theta^*-1|^2, |\theta^*-2|^2, |\theta^*-3|^2\right\} = 0$.

Consequently,

$$z_s(\Im(\theta), \Im(\rho)) = 0.$$

Analogously, $z_s(\Im\rho, \Im\theta) = 0$. *Hence,*

$$S_Z(\Im(\theta), \Im(\theta), \Im(\rho)) = 2\max\{0,0\} = 0 \leq \gamma S(\theta,\theta,\Im(\theta)) + \delta S(\rho,\rho,\Im(\rho)).$$

for any $\gamma, \delta \in (0,1)$. *Thus, the contractive condition (4.1) holds. Taking* $\delta = \frac{1}{3}$, $\gamma = \frac{1}{10}$ *and the sequence* $\theta_j = \left(2 + \frac{2}{j}\right)$, *we have* $\delta + \gamma < 1$, $\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right) = \left(3 + \frac{2}{|j+2|}\right)$, $\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right) = \left(3 + \frac{2}{|v|}\right)$ *and* $\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right) = \left(3 + \frac{2}{|j|}\right)$.

Further,

$$\sup_{v\geq 1} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} = 3 < \frac{10}{3} = \frac{1-\delta}{2\gamma}.$$

Therefore, all requirements of Theorem 4.2 are fulfilled. Hence, $\Im$ *has a unique FP which is* $1 \in \mho$.

Corollary 4.2. *Let* $(\mho, S, \zeta)$ *be a CSMLS and* $\Im : \mho \to \mho^{cb}$ *be a MVM such that*

$$S_Z(\Im(\theta), \Im(\theta), \Im(\rho)) \leq \gamma\left(S(\theta,\theta,\Im(\theta)) + S(\rho,\rho,\Im(\rho))\right),$$

where $\gamma \in \left(0, \frac{1}{2}\right)$. *Moreover, for* $\theta_0 \in \mho$, *there exists a sequence* $\theta_u = \Im(\theta_{u-1})$, $u \in \mathbb{N}$ *such that*

$$\sup_{v\geq 1} \lim_{j\to+\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} \leq \frac{1-\gamma}{2\gamma},$$

and $\lim_{j\to+\infty} \zeta(\theta_u, \theta_u, \theta_{u+1})$ *exists. Then,* $\Im$ *has a unique FP.*

Proof. We reach the required result by taking $\delta = \gamma$ in Theorem 4.2. □

5. Existence of solution to the DIP

In this part, we apply the theoretical results, specifically Theorem 4.1 to discuss the existence of solution to the following DIP:

$$\text{Find } \theta \in C(V) \text{ such that } \theta'(\tau) \in \Phi(\tau, \theta(\tau)), \text{ for all } \tau \in V, \tag{5.1}$$

where $V = [0, c]$ with $c > 0$, $\mho = C(V)$ is the set of all continuous functions on $\mathbb{R}$, and $\Phi : V \times \mathbb{R} \to 2^{\mathbb{R}}$ is a MVM. Describe the functions $S : \mho^3 \to \mathbb{R}_+$ and $\zeta : \mho^3 \to [1, +\infty)$ as

$$S(\theta, \vartheta, \rho) = \frac{\sup_{\tau\in V}\left|\theta(\tau) - \rho(\tau)\right|^2 + \sup_{\tau\in V}\left|\vartheta(\tau) - \rho(\tau)\right|^2}{2}$$

and

$$\zeta(\theta, \vartheta, \rho) = 2,$$

respectively, for all $\theta, \vartheta, \rho \in \mho$. Because the metric established above is identical to the supremum metric, we can say that $(\mho, S, \zeta)$ is a CSMLS. Define the set of Lebesgue integrable (LI) functions in $\Phi(., \theta(.))$ by $S_\Phi(\theta) = \left\{z \in L^1(V, \mathbb{R}) : z(\tau) \in \Phi(\tau, \theta(\tau))\right\}$ for all $\tau \in V$.

Problem (5.1) will be considered under the following assumptions:

(A$_1$) for all $\theta \in \mho$, $S_\Phi(\theta) \neq \emptyset$;

(A$_2$) for $(\tau, \theta) \in V \times \mho$, $\Phi(\tau, \theta(\tau))$ is closed;

(A$_3$) $\Phi(., \theta(.))$ is bounded on V, for all $\theta \in \mho$;

(A$_4$) for a sequence $\{z_u\} \in S_\Phi(\theta)$, there is a subsequence $\{z_{u_j}\}$ of $\{z_u\}$ such that $z_{u_j} \to z \in L^1(V, \mathbb{R})$, as $j \to +\infty$. Also,
$$\int_0^\tau z_{u_j}(r)dr \to \int_0^\tau z(r)dr, \text{ as } j \to +\infty,$$
for all $\theta \in \mho$ and all $\tau \in V$;

(A$_5$) for all $\tau \in V$ and for $\theta, \vartheta \in \mho$, there exist a function $\varkappa(\tau) \in L^1(V, \mathbb{R})$ such that
$$\sup_{\tau \in V} \int_0^\tau |\varkappa(r)|^2 dr \leq \frac{1}{9} \text{ and } 0 \leq \sup_{\tau \in V} |z_\theta - z_\vartheta|^2 \leq |\varkappa(\tau)|^2 |\theta(\tau) - \vartheta(\tau)|^2,$$
where $z_\theta \in S_\Phi(\theta)$, $z_\vartheta \in S_\Phi(\vartheta)$.

Definition 5.1. *We say that $\theta \in \mho$ is a solution to the DIP (5.1), if there exists $z \in S_\Phi(\theta)$ such that $\theta'(\tau) = z(\tau)$, $\tau \in V$.*

Our main theorem in this part is as follows:

Theorem 5.1. *Under the assumptions $(A_1) - (A_5)$, the DIP (5.1) has a unique solution.*

Proof. Define the MVM $\Im$ on $\mho$ by
$$\Im(\theta) = \left\{ \ell \in \mho : \ell(\tau) = \int_0^\tau z(r)dr, \ \tau \in V, \ z \in S_\Phi(\theta) \right\}.$$
Assumption (A_1) illustrate that the MVM $\Im$ is non-empty and well-defined. If $\theta \in \Im(\theta)$, then $\theta(\tau) = \int_0^\tau z(r)dr$, this yields $\theta'(\tau) = z(\tau)$, for $\tau \in V$.

Now, we claim that the MVM $\Im$ fulfills the axioms of Theorem 4.1. To prove that $\Im(\theta)$ is a closed and bounded subset of $\mho$, assume that $\theta \in \mho$ is a fixed and $\{z_u\}$ is a sequence in $\Im(\theta)$ so that $z_u \to z \in \mho$, as $u \to +\infty$. Then $\{z_u\} \in S_\Phi(\theta)$, this leads to
$$\ell_u(\tau) = \int_0^\tau z_u(r)dr, \ \tau \in V.$$
By Assumption (A_4), there is a subsequence $\{z_{u_j}\}$ of $\{z_u\}$ such that $z_{u_j} \to z \in L^1(V, \mathbb{R})$, as $j \to +\infty$. Also,
$$\int_0^\tau z_{u_j}(r)dr \to \int_0^\tau z(r)dr, \text{ as } j \to +\infty,$$
for all $\theta \in \mho$ and all $\tau \in V$. By Assumption (A_2), for $(\tau, \theta) \in V \times \mho$, $\Phi(\tau, \theta(\tau))$ is closed and $z(\theta) \in \Phi(\tau, \theta(\tau))$, we have $z \in S_\Phi(\theta)$. It is clear that
$$\ell(\tau) = \lim_{u \to +\infty} \ell_u(\tau) = \lim_{u \to +\infty} \int_0^\tau z_u(r)dr = \lim_{j \to +\infty} \int_0^\tau z_{u_j}(r)dr = \int_0^\tau z(r)dr,$$

hence $z \in \Im(\theta)$, that is $\Im$ is closed. Form the assumptions (A_3), we have $\Phi(.,\theta(.))$ is bounded on V, for all $\theta \in \mho$ and there exists $v > 0$ such that $\left|z(\tau)\right| \leq v$, for $z \in S_{\Phi}(\theta)$. Since $z(\tau) \in \Phi(\tau, \theta(\tau))$, then $z \in \Im(\theta)$ and

$$\sup_{\tau \in V}\left|\ell(\tau)\right| \leq \sup_{\tau \in V}\int_0^\tau \left|z(r)\right| dr \leq vc.$$

This proves that $\Im$ is bounded. Thus, we can write $\Im : \mho \to \mho^{cb}$. Finally, we prove that $\Im$ is a contraction. Indeed, using Assumption (A_5), for $\theta, \rho \in \mho$, we have

$$S_Z\left(\Im(\theta), \Im(\theta), \Im(\rho)\right) \leq 2\max\left\{z_s\left(\Im(\theta), \Im(\rho)\right), z_s\left(\Im(\rho), \Im(\theta)\right)\right\}, \tag{5.2}$$

where

$$\begin{aligned}
z_s\left(\Im(\theta), \Im(\rho)\right) &= \sup_{\ell_\theta \in \Im(\theta)} \inf_{\ell_\rho \in \Im(\rho)} S\left(\ell_\theta, \ell_\theta, \ell_\rho\right) \\
&= \sup_{\ell_\theta \in \Im(\theta)} \inf_{\ell_\rho \in \Im(\rho)} \sup_{\tau \in V}\left|\int_0^\tau z_\theta(r)dr - \int_0^\tau z_\rho(r)dr\right|^2 \\
&\leq \sup_{\ell_\theta \in \Im(\theta)} \inf_{\ell_\rho \in \Im(\rho)} \sup_{\tau \in V}\int_0^\tau \left|\varkappa(r)\right|^2 \left|\theta(r) - \rho(r)\right|^2 dr \\
&\leq \sup_{\ell_\theta \in \Im(\theta)} \inf_{\ell_\rho \in \Im(\rho)} \sup_{\tau \in V} S(\theta, \theta, \rho)\int_0^\tau \left|\varkappa(r)\right|^2 dr \\
&\leq \frac{1}{9}S(\theta, \theta, \rho).
\end{aligned}$$

Similarly, we can write

$$z_s\left(\Im(\rho), \Im(\theta)\right) \leq \frac{1}{9}S(\theta, \theta, \rho).$$

Hence, the inequality (5.2) reduces to

$$\begin{aligned}
S_Z\left(\Im(\theta), \Im(\theta), \Im(\rho)\right) &\leq 2\max\left\{z_s\left(\Im(\theta), \Im(\rho)\right), z_s\left(\Im(\rho), \Im(\theta)\right)\right\} \\
&\leq 2 \times \frac{1}{9}S(\theta, \theta, \rho) \\
&= \frac{2}{9}S(\theta, \theta, \rho).
\end{aligned}$$

Then, $\Im$ is a contraction with $\mu = \frac{2}{9} < 1$. Further, since $\zeta(\theta, \vartheta, \rho) = 2$, we have

$$\sup_{v \geq 1}\lim_{j \to +\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} = 2 < \frac{1}{2\mu}.$$

Therefore, all requirements of Theorem 4.1 are fulfilled. Then $\Im$ has a unique FP, which is a unique solution to the DIP (5.1). □

In order to support Theorem 5.1, we present the following example:

Example 5.1. *Let $\theta'(\tau) = \frac{\theta}{9}$ and $\Phi(\tau, \theta(\tau)) = \left\{0, \frac{\theta}{9}\right\}$, for $\tau \in V = [0,1]$ and the functions $S : \mho^3 \to \mathbb{R}_+$ and $\zeta : \mho^3 \to [1, +\infty)$ be defined as in the above section. Now we shall verify the hypotheses $(A_1) - (A_5)$ of Theorem 5.1.*

(A_1) Since $z(\tau) = \frac{\theta(\tau)}{9}$ is a LI selection of $\Phi(., \theta(.))$, then for all $\theta \in \mho$, $S_\Phi(\theta) \neq \emptyset$.

(A_2) for $(\tau, \theta) \in V \times \mho$, it is clear that $\Phi(\tau, \theta(\tau)) = \left\{0, \frac{\theta}{7}\right\}$ is closed.

(A_3) for all $\theta \in \mho$, $\Phi(., \theta(.))$ is bounded on $[0,1]$.

(A_4) for a sequence $\{\ell_u\} \in S_\Phi(\theta)$, $u \in \mathbb{N}$, there is a subsequence $\{\ell_{u_j}\}$ of $\{\ell_u(\tau)\} = \left\{\frac{\theta_u(\tau)}{9}\right\}$ such that $\ell_{u_j} \to \ell = \left\{\frac{\theta(\tau)}{9}\right\} \in L^1(V, \mathbb{R})$, as $j \to +\infty$. Also,

$$\int_0^\tau \ell_{u_j}(r)dr \to \int_0^\tau \ell(r)dr, \text{ as } j \to +\infty,$$

for every fixed $\theta \in \mho$ and all $\tau \in V$.

(A_5) Take $\varkappa = \frac{1}{3} \in L^1(V, \mathbb{R})$ for all $\tau \in V$ with

$$\sup_{\tau \in V} \int_0^\tau \left|\varkappa(r)\right|^2 dr \leq \frac{1}{9} \text{ and } 0 \leq |z_\theta - z_\vartheta|^2 \leq \frac{1}{9}\left|\theta(\tau) - \vartheta(\tau)\right|^2,$$

where $z_\theta = \frac{\theta(\tau)}{9} \in S_\Phi(\theta)$, and $z_\vartheta = \frac{\vartheta(\tau)}{9} \in S_\Phi(\vartheta)$. Consider $\mu = \frac{1}{9} < 1$, then

$$\sup_{v \geq 1} \lim_{j \to +\infty} \frac{\zeta\left(\theta_{j+1}, \theta_{j+1}, \theta_{j+2}\right)\zeta\left(\theta_v, \theta_v, \theta_{j+1}\right)}{\zeta\left(\theta_j, \theta_j, \theta_{j+1}\right)} = 2 < \frac{1}{2\mu}$$

Hence, all assumptions of Theorem 5.1 are fulfilled. Therefore, for some constant a, the suggested DIP has a solution $\theta(\tau) = ae^{9\tau}$.

6. Conclusion

The concept of an ordinary differential equation (ODE) is generalized by the idea of a DI. As a result, DIs can be explored for all problems that are typically covered in the theory of ODEs, such as the existence and continuation of solutions, reliance on initial conditions and parameters, etc. Due to the fact that a DI typically has a large number of solutions beginning at a given point, new kinds of issues emerge, including the need to select solutions with specific attributes and look into the topological characteristics of the set of solutions. In particular, when studying the dynamics of economic, social, and biological macro-systems, differential inclusions are a valuable tool for analyzing a wide range of dynamical processes that are represented by equations with a discontinuous or multivalued right-hand side. They are also highly helpful in demonstrating control theory existence theorems. So, in this manuscript, we used a control function to combine metric-like spaces and $S-$metric spaces to create CSMLSs. Furthermore, several FP results for MVMs were obtained in this domain. Furthermore, we presented some non-trivial examples to back up our statements. The findings generalize and unify multiple findings in the same direction.

Ultimately, as a form of application, the existence of the solution to a DIP was investigated to support and test the theoretical results.

7. Abbreviation

- CSMLSs controlled $S-$metric-like space
- FP fixed-point
- MVM multi-valued mapping
- DIP differential inclusion problem
- MS metric space
- LI Lebesgue integrable
- ODE ordinary differential equation
- DI differential inclusion

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