Int. J. Anal. Appl. (2024), 22:230

International Journal of Analysis and Applications

Boundary Layers of Time-Dependent Convection-Diffusion Equations in a Square

Abir Sboui*, Abdelhalim Hasnaoui

Mathematics Department, College of Sciences and Arts, Northern Border University, Rafha 91911*, Saudi Arabia*

**Corresponding author:* abir.sboui@nbu.edu.sa, abirsboui@yahoo.fr

Abstract. The aim of this paper is to study the asymptotic behavior for a class of time-dependent convection-diffusion problems in a square $= (0,1) \times (0,1)$, which is a simplified model of the Oseen equations. By considering this problem in a square, we theoretically treat the case where parabolic and ordinary boundary layers are present. We construct correctors which absorb the singularities of the limit solution; this allows to obtain an approximation of the viscous solution up to the boundary. The expression of the correctors is giving explicitly and the uniform validity of the approximate solution is then proved.

1. Introduction

The behavior of the solutions of Navier-Stokes equations at vanishing viscosity (i.e. large Reynolds numbers) is an outstanding open problem both in fluid mechanics and in mathematical analysis. The study of boundary layers is of great physical and engineering importance. For a historical literature about boundary layers in the fluid mechanics field, we refer to ([2], [3], [7], [8], [4], [16], and [17]). There are quite a few partial results in the convergence of the Navier-Stokes equation to the Euler equations. For this, we though that we could learn much more from simpler equations, (see e.g. [15], [5], [6], [11], [20], and [21]). We consider in this article the boundary layers associated with a class of a time-dependent convection-diffusion problems. We can consider this problem as a simplified model of the Oseen equations, namely the Navier-Stokes equations linearized around a fixed velocity flow. In [19], the authors discussed this problem in a channel in space dimension two and only parabolic boundary layers are observed. Here, we treat the same problem considered in [19] but in a square. We note that multiple boundary layers appear: the ordinary layers with thickness of size ε, and parabolic layers with thickness of size $\sqrt{\varepsilon}$. In fact, some restrictions (compatibility conditions) will be assumed as we will see later. Indeed,

Received: Apr. 1, 2024.

2020 *Mathematics Subject Classification.* 35C20, 35B25, 35B45, 35K05.

Key words and phrases. boundary layers; correctors; singular problems; convection–diffusion equations.

https://doi.org/10.28924/2291-8639-22-2024-230

ISSN: 2291-8639

in the most general case (square with no restriction), several other incontinences occur which have to be accounted for by still other boundary layers. The convergence to the corresponding inviscid equations as $\varepsilon \to 0$ occur only in the weak sense (L^2 norm); indeed because of the loss of some boundary conditions, the convergence cannot occur in the strong sense up to the boundary. Our objective is to introduce a correcting term to absorb these singularities and then obtain an approximation of $\boldsymbol{u}^\varepsilon$, as $\varepsilon \to 0$, up to the boundary.

2. Time-dependent convection-diffusion equations

Our primary goal in this section is to study the asymptotic behavior of the following system

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + u^\varepsilon_x = f, \qquad \text{in } (0,T)\times\Omega, \\ u^\varepsilon = 0, \qquad \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon_{|_{t=0}} = u_0, \end{cases} \tag{2.1}$$

as ε approaches zero. The domain under consideration is the square $\Omega = (0,1)\times(0,1)$, and all the functions u_0 and $f = f(t,x,y)$ are supposed to be as smooth as necessary in $(0,T)\times\Omega$.

We observe that $(2.1)_1$ can be derived from the Oseen equations by simply dropping the pressure term and omitting the divergence free (mass conservation) constraint. The existence of a regular solution for the system (2.1) is obtained by classical methods: For $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists a unique solution u^ε of the system (2.1) such that

$$u^\varepsilon \in L^2(0,T;H^1_0(\Omega)), \qquad u^\varepsilon_t \in L^2(0,T;H^{-1}(\Omega)).$$

The inviscid problem corresponding to (2.1) is easily obtained by setting formally $\varepsilon = 0$ in (2.1), we obtain the following transport equation

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + \dfrac{\partial u^0}{\partial x} = f, \quad \text{in } (0,T)\times\Omega, \\ u^0_{|_{t=0}} = u_0. \end{cases} \tag{2.2}$$

We need to impose a boundary condition to (2.2) for the well-posedness of this problem. As suggested by the theory of the transport equations, we prescribe the boundary condition ad the characteristics enter the domain. Hence, we propose the following boundary condition for u^0:

$$u^0(x=0) = 0. \tag{2.3}$$

The convergence in $L^2(\Omega)$ of u^ε, the solution of (2.1), to u^0, the solution of (2.2), as ε approaches zero, can be somewhat easy to derive. However, the convergence in $H^1(\Omega)$ is not true due to the boundary layer phenomena. To prove the convergence in $H^1(\Omega)$, we usually need to introduce a function called a *corrector* which aim to recover the disparity between the boundary values of the regularizing and limit solutions namely u^ε and u^0 respectively. For more details about corrector, see e.g. [13], [14], [1], [9], [10], [18] and [12].

2.1. **Construction of the correctors.** It is clear that the limit solution u^0 does not generally satisfy the boundary conditions at $x = 1$, and $y = 0, 1$. To resolve these discrepancies, we will introduce the ordinary boundary layer (OBLs) at $x = 1$, and the parabolic boundary layers (PBLs), at $y = 0, 1$. We propose the following formal approximation:

$$u^\varepsilon = u^0 + \Psi^\varepsilon + R^\varepsilon, \tag{2.4}$$

with

$$\Psi^\varepsilon = \theta^\varepsilon + \varphi_0^\varepsilon + \varphi_1^\varepsilon,$$

where θ^ε (respectively φ_0^ε, and φ_1^ε) is the corrector of the boundary layer at $x = 1$ (respectively $y = 0$, and $y = 1$). We start with the parabolic boundary layer generated near $y = 0$. A heuristic argument suggests that φ_0^ε be the solution of the following system:

$$\begin{cases} \dfrac{\partial \varphi_0^\varepsilon}{\partial t} - \varepsilon \Delta \varphi_0^\varepsilon + \varphi_{0x}^\varepsilon = 0, & \text{in } (0,T) \times \Omega, \\ \varphi_0^\varepsilon(t,x,y) = -u^0, & \text{at } y = 0, \\ \varphi_0^\varepsilon(t,x,y) = 0, & \text{at } y = 1, \\ \varphi_0^\varepsilon(t,x,y) = 0, & \text{at } x = 0, \\ \varphi^\varepsilon_{0|_{t=0}} = 0. \end{cases} \tag{2.5}$$

Using the local variable $\overline{y} = y/\sqrt{\varepsilon}$, we define an approximation $\widetilde{\varphi}_0^\varepsilon$ of φ_0^ε which satisfies:

$$\begin{cases} \dfrac{\partial \widetilde{\varphi}_0^\varepsilon}{\partial t} - \dfrac{\partial^2 \widetilde{\varphi}_0^\varepsilon}{\partial \overline{y}^2} + \dfrac{\widetilde{\varphi}_0^\varepsilon}{\partial x} = 0, & \text{in } (0,T) \times (0,1) \times \mathbb{R}_+^*, \\ \widetilde{\varphi}_0^\varepsilon(t,x,\overline{y}) = -u^0, & \text{at } \overline{y} = 0, \\ \widetilde{\varphi}_0^\varepsilon(t,x,\overline{y}) \to 0, & \text{as } \overline{y} \to +\infty, \\ \widetilde{\varphi}_0^\varepsilon(t,x,\overline{y}) = 0, & \text{at } x = 0, \\ \widetilde{\varphi}^\varepsilon_{0|_{t=0}} = 0. \end{cases} \tag{2.6}$$

To handle the error analysis later on, we need further estimates on the spatial derivatives of $\widetilde{\varphi}_0^\varepsilon$. for that purpose, it is useful to obtain the expression of the $\widetilde{\varphi}_0^\varepsilon$ which is provided by the lemma below.

Lemma 2.1. *Let φ be a solution of the following system:*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} - \dfrac{\partial^2 \varphi}{\partial y^2} + \dfrac{\partial \varphi}{\partial x} = 0, & \text{in } (0,+\infty) \times (0,1) \times \mathbb{R}_+^*, \\ \varphi(t;x,0) = g(t,x), & \text{at } y = 0, \\ \varphi(t;x,y) \to 0, & \text{as } y \to +\infty, \\ \varphi(t;x,y) = 0, & \text{at } x = 0, \\ \varphi_{|_{t=0}} = 0. \end{cases} \tag{2.7}$$

Then, the solution of (2.7) is unique and it expression is given by:

$$\varphi(t;x,y) = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2x}}^{\infty} e^{-s^2/2} \widetilde{g}(t - \frac{y^2}{2s^2}, x - \frac{y^2}{2s^2}, 0)\, ds. \tag{2.8}$$

where $\widetilde{g}$ denotes the extension of the function g as follows:

$$\widetilde{g} = \begin{cases} g & \text{if } t \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.9}$$

Proof. Let φ be the solution of (2.7). Using (2.9), we define an extension $\widetilde{\varphi}$ of φ and then, we derive the Fourier transform of $\widetilde{\varphi}^{\varepsilon}$:

$$\widehat{\widetilde{\varphi}}(\tau; x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\tau t} \widetilde{\varphi}(t; x, y)\, dt.$$

Hence, we obtain the following system:

$$\begin{cases} i\tau\widehat{\widetilde{\varphi}} - \dfrac{\partial^2 \widehat{\widetilde{\varphi}}}{\partial y^2} + \dfrac{\partial \widehat{\widetilde{\varphi}}}{\partial x} = 0, & \text{in } \mathbb{R} \times (0,1) \times \mathbb{R}_+^*, \\ \widehat{\widetilde{\varphi}}(\tau; x, y) = \widehat{\widetilde{g}}, & \text{at } y = 0, \\ \widehat{\widetilde{\varphi}}(\tau; x, y) \to 0, & \text{as } y \to +\infty, \\ \widehat{\widetilde{\varphi}}(\tau; x, y) = 0, & \text{at } x = 0, \\ \widehat{\widetilde{\varphi}}_{|_{\tau=0}} = 0. \end{cases} \tag{2.10}$$

Note that

$$i\tau\widehat{\widetilde{\varphi}} + \frac{\partial \widehat{\widetilde{\varphi}}}{\partial x} = e^{-i\tau x} \frac{\partial}{\partial x}\Big(e^{i\tau x} \widehat{\widetilde{\varphi}} \Big).$$

Then, we set $v^{\varepsilon} = e^{i\tau x}\widehat{\widetilde{\varphi}}$, which satisfies:

$$\begin{cases} \dfrac{\partial v^{\varepsilon}}{\partial x} - \dfrac{\partial^2 v^{\varepsilon}}{\partial y^2} = 0, & \text{in } \mathbb{R} \times (0,1) \times \mathbb{R}_+^*, \\ v^{\varepsilon}(\tau; x, y) = e^{i\tau x}\widehat{\widetilde{g}} & \text{at } y = 0, \\ v^{\varepsilon}(\tau; x, y) \to 0, & \text{as } y \to +\infty, \\ v^{\varepsilon}(\tau; x, y) = 0, & \text{at } x = 0, \\ v^{\varepsilon}_{|_{t=0}} = 0. \end{cases} \tag{2.11}$$

Now, we observe that v^{ε} satisfies the heat equation supplemented with nonhomogeneous boundary conditions of which the expression is given by:

$$v^{\varepsilon}(\tau; x, y) = 2 \int_0^x \frac{\partial K}{\partial y}(x - \xi, y) e^{i\tau\xi} \widehat{\widetilde{g}}(\tau; \xi, 0)\, d\xi,$$

where K is the fundamental solution of the heat equation

$$K(x, y) = \frac{1}{4\pi x} e^{-y^2/4x}.$$

Hence, we deduce that

$$\begin{aligned} \widehat{\widetilde{\varphi}}(\tau; x, y) &= e^{-i\tau x} v^{\varepsilon}(\tau; x, y) \\ &= 2 \int_0^x \frac{\partial K}{\partial y}(x - \xi, y) e^{i\tau(\xi - x)} \widehat{\widetilde{g}}(\tau; \xi, 0)\, d\xi, \end{aligned} \tag{2.12}$$

then

$$\begin{aligned}
\widetilde{\varphi}(t;x,y) &= 2\int_{-\infty}^{+\infty}\int_0^x \frac{\partial K}{\partial y}(x-\xi,y)e^{i\tau t}e^{-i\tau(x-\xi)}\widehat{\widetilde{g}}(\tau;\xi,0)\,d\tau\,d\xi \\
&= 2\int_0^x \frac{\partial K}{\partial y}(x-\xi,y)\int_{-\infty}^{+\infty} e^{i\tau(t-(x-\xi))}\widehat{\widetilde{g}}(\tau;\xi,0)\,d\tau\,d\xi \\
&= 2\int_0^x \frac{\partial K}{\partial y}(x-\xi,y)\widetilde{g}(t-(x-\xi);\xi,0)\,d\xi \\
&= 2\int_0^x (\frac{-y}{2(x-\xi)}\frac{e^{-y^2/4(x-\xi)}}{\sqrt{4\pi(x-\xi)}}\widetilde{g}(t-(x-\xi);\xi,0)\,d\xi \\
&= \text{setting } s=\frac{y}{\sqrt{2(x-\xi)}} \\
&= \sqrt{\frac{2}{\pi}}\int_{y/\sqrt{2x}}^{\infty} e^{-s^2/2}\widetilde{g}(t-\frac{y^2}{2s^2};x-\frac{y^2}{2s^2},0)\,ds.
\end{aligned} \tag{2.13}$$

This ends the proof of Lemma 2.1. •

From the previous result, we deduce the following lemma.

Lemma 2.2. *Let $\widetilde{\varphi}_0^\varepsilon$ be the solution of (2.6). Then $\widetilde{\varphi}_0^\varepsilon$ admits the following integral representation:*

$$\widetilde{\varphi}_0^\varepsilon(t,x,\frac{y}{\sqrt{\varepsilon}}) = -\sqrt{\frac{2}{\pi}}\int_{y/\sqrt{2\varepsilon x}}^{\infty}\exp(-s^2/2)\widetilde{u}^0(t-\frac{y^2}{2\varepsilon s^2},x-\frac{y^2}{2\varepsilon s^2},0)\,ds. \tag{2.14}$$

Furthermore, we have:

$$|\widetilde{\varphi}_0^\varepsilon(t,x,\frac{y}{\sqrt{\varepsilon}})| \le k\exp(-\frac{y}{\sqrt{\varepsilon x}}) \quad \forall\,(t,x,y)\in(0,T)\times\Omega. \tag{2.15}$$

Proof. The explicit expression of $\widetilde{\varphi}_0^\varepsilon$, as in (2.14), is simply deduced from Lemma 2.1. Now, since u^0 is smooth as necessary, we have

$$\begin{aligned}
|\widetilde{\varphi}_0^\varepsilon(t,x,\frac{y}{\sqrt{\varepsilon}})| &\le \kappa\int_{y/\sqrt{2\varepsilon x}}^{\infty} e^{-s^2/2}\,ds \\
&\le \kappa\int_{y/\sqrt{2\varepsilon x}}^{\infty} e^{-cs}\,ds, \text{ for all } c>0 \\
&\le \kappa e^{-\frac{y}{\sqrt{\varepsilon x}}}.
\end{aligned}$$

•

Remark 2.1. Similarly at $y=1$, we introduce an approximation $\widetilde{\varphi}_1^\varepsilon$ of φ_1^ε, having the same structure as $\widetilde{\varphi}_0^\varepsilon$, with the role of $\overline{y}=y/\sqrt{\varepsilon}$ and $\widetilde{y}=(1-y)/\sqrt{\varepsilon}$ being exchanged.

Now, we want to solve the discrepancies at the boundary $x=1$. For this, we propose a corrector θ^ε which satisfies:

$$-\varepsilon\Delta\theta^\varepsilon + \theta_x^\varepsilon = 0,$$

$$\theta^\varepsilon(t, x = 1, y) = -u^0 - \widetilde{\varphi}_0^\varepsilon - \widetilde{\varphi}_1^\varepsilon, \qquad \theta^\varepsilon(t, x = 0, y) = 0.$$

Using the local variable $\overline{x} = (1 - x)/\varepsilon$, we define an approximation $\widetilde{\theta}^\varepsilon$ of θ^ε which verifies

$$\begin{cases} -\dfrac{\partial^2 \widetilde{\theta}\varepsilon}{\partial \overline{x}^2} + \dfrac{\partial \widetilde{\theta}^\varepsilon}{\partial \overline{x}} = 0, & \text{in } (0,T) \times \mathbb{R}_+^* \times (0,1), \\ \widetilde{\theta}^\varepsilon(t, \overline{x}, y) = -u^0(t, 1, y) - \widetilde{\varphi}_0^\varepsilon(t, 1, \overline{y}) - \widetilde{\varphi}_1^\varepsilon(t, 1, \widetilde{y}), & \text{at } \overline{x} = 0 \\ \widetilde{\theta}^\varepsilon(t, \overline{x}, y) \to 0, & \text{as } \overline{x} \to \infty, \\ \widetilde{\theta}^\varepsilon_{|_{t=0}} = 0. \end{cases} \tag{2.16}$$

The solution of (2.16) is given by:

$$\widetilde{\theta}^\varepsilon(t, \overline{x}, y) = -(u^0(t, 1, y) + \widetilde{\varphi}_0^\varepsilon(t, 1, \overline{y}) + \widetilde{\varphi}_1^\varepsilon(t, 1, \widetilde{y}))e^{-\overline{x}}.$$

At this stage, the function u^ε is tentatively approximated by $u^0 + \widetilde{\varphi}_0^\varepsilon + \widetilde{\varphi}_1^\varepsilon + \widetilde{\theta}^\varepsilon$. In the next section, we will provide norm estimates on the derivatives of all these correctors which will be used below.

2.2. **Asymptotic behavior of the correctors.** We start by studying the asymptotic behavior of the parabolic boundary layer at $y = 0$. We assume that the following condition holds:

$$u^0(t, 0, 0) = 0. \tag{2.17}$$

Then, we have the following lemma

Lemma 2.3. *Assume that the condition (2.17) holds. Then, there exists a positive constant κ independent of ε such that the following inequalities hold:*

$$\left| \frac{\partial}{\partial t} \widetilde{\varphi}_0^\varepsilon(t, x, \frac{y}{\sqrt{\varepsilon}}) \right| \leq \kappa \exp(-\frac{y}{\sqrt{\varepsilon}}), \tag{2.18}$$

$$\left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \widetilde{\varphi}_0^\varepsilon(t, x, \frac{y}{\sqrt{\varepsilon}}) \right| \leq \kappa\, \varepsilon^{-\frac{m}{2}} \exp(-\frac{y}{2\sqrt{\varepsilon}}), \quad \forall (x, y) \in \Omega, \tag{2.19}$$

for all $i, m \in \mathbb{N}$, with $0 \leq i + m \leq 1$.

As consequence, we have the following lemma.

Lemma 2.4. *Assume that the condition (2.17) holds. For $0 \leq \sigma < 1$, we set*

$$\Omega^\sigma = (0, 1) \times (\sigma, 1).$$

Then, there exists a positive constant κ independent of ε such that:

$$\left\| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \widetilde{\varphi}_0^\varepsilon \right\|_{L^2(\Omega^\sigma)} \leq \kappa\, \varepsilon^{-m/2+1/4} \exp(-\frac{\sigma}{2\sqrt{\varepsilon}}).$$

In particular, we have

$$\left\| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \widetilde{\varphi}_0^\varepsilon \right\|_{L^2(\Omega)} \leq \kappa\, \varepsilon^{-m/2+1/4},$$

and

$$\left\| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \widetilde{\varphi}_0^\varepsilon \right\|_{L^\infty(0,T; L^2(\Omega))} \leq \kappa\, \varepsilon^{-m/2+1/4}.$$

Proof. Since we have

$$|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_0^{\varepsilon}(t,x,\frac{y}{\sqrt{\varepsilon}})| \leq \kappa\,\varepsilon^{-m/2}\exp(\frac{-y}{2\sqrt{\varepsilon}}),$$

then

$$\begin{aligned}\int_0^1\int_\sigma^1 |\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_0^{\varepsilon}(t,x,\frac{y}{\sqrt{\varepsilon}})|^2\,dydx &\leq \kappa\,\varepsilon^{-m}\int_0^1\int_\sigma^1 \exp(\frac{-y}{\sqrt{\varepsilon}})\,dydx \\ &\leq \kappa\,\varepsilon^{-m}\Big(-\sqrt{\varepsilon}\exp(\frac{-1}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon}\exp(\frac{-\sigma}{\sqrt{\varepsilon}})\Big) \\ &\leq \kappa\,\varepsilon^{-m+1/2}\exp(-\frac{\sigma}{\sqrt{\varepsilon}}),\end{aligned}$$

which implies

$$\Big\|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_0^{\varepsilon}(t)\Big\|_{L^2(\Omega^\sigma)} \leq \kappa\,\varepsilon^{-m/2+1/4}\exp(-\frac{\sigma}{2\sqrt{\varepsilon}}). \tag{2.20}$$

Letting $\sigma \to 0^+$ in (2.20), we obtain:

$$\Big\|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_0^{\varepsilon}(t)\Big\|_{L^2(\Omega)} \leq \kappa\,\varepsilon^{-m/2+1/4},$$

and thus

$$\Big\|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_0^{\varepsilon}\Big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\,\varepsilon^{-m/2+1/4}.$$

•

Remark 2.2. Similarly for $\widetilde{\varphi}_1^{\varepsilon}$, we need to assume the following condition:

$$u^0(t,0,1) = 0. \tag{2.21}$$

Under the hypothesis (2.21), the estimates obtained in Lemma 2.3 and Lemma 2.4 are valid for $\widetilde{\varphi}_1^{\varepsilon}$, that is

$$|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_1^{\varepsilon}(t,x,\frac{1-y}{\sqrt{\varepsilon}})| \leq \kappa\,\varepsilon^{-m/2}\exp(-\frac{1-y}{2\sqrt{\varepsilon}}), \tag{2.22}$$

and

$$\Big\|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_1^{\varepsilon}(t)\Big\|_{L^2(\Omega)} \leq \kappa\,\varepsilon^{-m/2+1/4}. \tag{2.23}$$

Remark 2.3. As for $\Omega^\sigma = (0,1)\times(\sigma,1)$, we introduce $\Omega^{\sigma'} = (0,1)\times(0,1-\sigma')$. Therefore, we have

$$\Big\|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\varphi}_1^{\varepsilon}(t)\Big\|_{L^2(\Omega^{\sigma'})} \leq \kappa\,\varepsilon^{-m/2+1/4}\exp(-\frac{\sigma'}{\sqrt{2\varepsilon}}).$$

Now, setting $\sigma' = \sigma = k\varepsilon^\alpha$ with $0 < \alpha < \frac{1}{2}$, we deduce that the parabolic boundary layer correctors $\widetilde{\varphi}_0^{\varepsilon}$ and $\widetilde{\varphi}_1^{\varepsilon}$ are exponentially small on $(0,1)\times(\sigma,1-\sigma)$. This implies that we only need to take care of the parabolic boundary layers near the boundaries $y = 0$ and $y = 1$.

Now, we derive the norm estimates for the ordinary boundary layer corrector θ^ε defined by (2.16). We have the following lemma.

Lemma 2.5. *Assume that the conditions (2.17) and (2.21) hold. Then, there exist a positive constant κ independent of ε, such that the following inequalities hold:*

$$\left|\frac{\partial}{\partial t}\widetilde{\theta}^{\varepsilon}(t,\frac{1-x}{\varepsilon},y)\right| \leq \kappa\, \exp(-\frac{1-x}{\varepsilon})(1+\exp(-\frac{y}{2\sqrt{\varepsilon}})+\exp(-\frac{1-y}{2\sqrt{\varepsilon}})),$$

$$\left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}\widetilde{\theta}^{\varepsilon}(t,\frac{1-x}{\varepsilon},y)\right| \leq \kappa\, \varepsilon^{-i}\exp(-\frac{1-x}{\varepsilon})(1+\varepsilon^{-m/2}\exp(-\frac{y}{2\sqrt{\varepsilon}})+\varepsilon^{-m/2}\exp(-\frac{1-y}{2\sqrt{\varepsilon}})).$$

for $i,m \in \mathbb{N}$ and $0 \leq i+m \leq 1$.

Proof. Using the estimates mentioned in Lemma 2.3 and Remark 2.2, we find that

$$\begin{aligned}\left|\frac{\partial}{\partial t}\widetilde{\theta}^{\varepsilon}(t,\frac{1-x}{\varepsilon},y)\right| &\leq \left|\frac{\partial}{\partial t}(u^0(t,1,y)+\widetilde{\varphi}_0^{\varepsilon}(t,1,\frac{y}{\sqrt{\varepsilon}})+\widetilde{\varphi}_1^{\varepsilon}(t,1,\frac{1-y}{\sqrt{\varepsilon}}))\exp(-\frac{1-x}{\varepsilon})\right| \\ &\leq \kappa\, \exp(-\frac{1-x}{\varepsilon})\Big(1+\exp(-\frac{y}{2\sqrt{\varepsilon}})+\exp(-\frac{1-y}{2\sqrt{\varepsilon}})\Big).\end{aligned}$$

Furthermore,

$$\begin{aligned}\left|\frac{\partial^{i+m}\widetilde{\theta}^{\varepsilon}}{\partial x^i \partial y^m}(t,\frac{1-x}{\varepsilon},y)\right| &= \left|\frac{\partial^{i+m}}{\partial x^i \partial y^m}(u^0(t,1,y)+\widetilde{\varphi}_0^{\varepsilon}(t,1,\frac{y}{\sqrt{\varepsilon}})+\widetilde{\varphi}_1^{\varepsilon}(t,1,\frac{1-y}{\sqrt{\varepsilon}})\exp(-\frac{1-x}{\varepsilon}))\right| \\ &\leq \kappa\, \varepsilon^{-i}\exp(-\frac{1-x}{\varepsilon})(1+\varepsilon^{-m/2}\exp(-\frac{y}{2\sqrt{\varepsilon}})+\varepsilon^{-m/2}\exp(-\frac{1-y}{2\sqrt{\varepsilon}}).\end{aligned}$$

•

The following norm estimate is deduced immediately from Lemma 2.5.

Lemma 2.6. *For $0 \leq \sigma_1, \sigma_2 < 1$, we define*

$$\Omega^{\sigma_1,\sigma_2} = (0, 1-\sigma_1) \times (\sigma_2, 1-\sigma_2).$$

Assume that the conditions (2.17) and (2.21) hold. Then, there exists a positive constant κ independent of ε such that the following estimates hold:

$$\left\|\frac{\partial\widetilde{\theta}^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega^{\sigma_1,\sigma_2})} \leq \kappa\, \varepsilon^{1/2}\exp(-\frac{\sigma_1}{\varepsilon})(1+\varepsilon^{1/4}\exp(\frac{-\sigma_2}{4\sqrt{\varepsilon}}),$$

$$\left\|\frac{\partial^{i+m}\widetilde{\theta}^{\varepsilon}}{\partial x^i \partial y^m}\right\|_{L^2(\Omega^{\sigma_1,\sigma_2})} \leq \kappa\, \varepsilon^{-i+1/2}(1+\varepsilon^{-m/2+1/4}\exp(-\frac{\sigma_2}{\sqrt{\varepsilon}}))\exp(-\frac{\sigma_1}{\varepsilon}).$$

In particular, as $\sigma_1, \sigma_2 \to 0$, we obtain

$$\left\|\frac{\partial\widetilde{\theta}^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega)} \leq \kappa\, \varepsilon^{1/2}(1+\varepsilon^{1/4}),$$

$$\left\|\frac{\partial^{i+m}\widetilde{\theta}^{\varepsilon}}{\partial x^i \partial y^m}\right\|_{L^2(\Omega)} \leq \kappa\, \varepsilon^{-i+1/2}(1+\varepsilon^{-m/2+1/4}),$$

$$\left\|\frac{\partial^{i+m}\widetilde{\theta}^{\varepsilon}}{\partial x^i \partial y^m}\right\|_{L^{\infty}(0,T;L^2(\Omega))} \leq \kappa\, \varepsilon^{-i+1/2}(1+\varepsilon^{-m/2+1/4}),$$

for $i, m \in \mathbb{N}$, *with* $0 \le i + m \le 1$.

In the next section, we state and prove the convergence results for the solution of the system (2.1).

2.3. **Convergence result.** We conclude this study stating and proving the following theorem which provides the asymptotic approximation for which we will justify later on the validity of our choice of the boundary layer correctors.

Theorem 2.1. *For the Dirichlet boundary value problem (2.1), let* f *be any smooth function and assume that the conditions (2.17) and (2.21) hold. Then,*

$$\|u^\varepsilon - (u^0 + \widetilde{\theta}^\varepsilon + \widetilde{\varphi}_0^\varepsilon + \widetilde{\varphi}_1^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} \le \kappa\, \varepsilon^{3/4}, \tag{2.24}$$

$$\|u^\varepsilon - (u^0 + \widetilde{\theta}^\varepsilon + \widetilde{\varphi}_0^\varepsilon + \widetilde{\varphi}_1^\varepsilon)\|_{L^2(0,T;H^1(\Omega)} \le \kappa\, \varepsilon^{1/4}, \tag{2.25}$$

where κ *is a positive constant independent of* ε.

Proof. We set

$$R^\varepsilon = u^\varepsilon - (u^0 + \widetilde{\theta}^\varepsilon + \widetilde{\varphi}_0^\varepsilon + \widetilde{\varphi}_1^\varepsilon),$$

which verifies

$$\frac{\partial R^\varepsilon}{\partial t} - \varepsilon \Delta R^\varepsilon + R_x^\varepsilon = \varepsilon \Delta u^0 + \varepsilon\Big(\frac{\partial^2 \widetilde{\varphi}_0^\varepsilon}{\partial x^2} + \frac{\partial^2 \widetilde{\varphi}_1^\varepsilon}{\partial x^2} + \frac{\partial^2 \widetilde{\theta}^\varepsilon}{\partial y^2}\Big) + \frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}.$$

The boundary conditions satisfied R^ε are given as follows

$$R^\varepsilon_{|_{x=0}} = 0, \qquad R^\varepsilon_{|_{x=1}} = 0,$$

$$R^\varepsilon_{|_{y=0}} = -(\widetilde{\varphi}_1^\varepsilon + \widetilde{\theta}^\varepsilon), \qquad R^\varepsilon_{|_{y=1}} = -(\widetilde{\varphi}_0^\varepsilon + \widetilde{\theta}^\varepsilon).$$

Since R^ε has nonhomogeneous boundary values, we introduce a supplementary corrector

$$v^\varepsilon(t; x, y) = (1-y) R^\varepsilon(t, x, 0) + y R^\varepsilon(t, x, 1)$$

and we define

$$\begin{aligned} \psi^\varepsilon(t; x, y) &= R^\varepsilon - (1-y) R^\varepsilon(t, x, 0) - y R^\varepsilon(t, x, 1) \\ &= R^\varepsilon(t, x, y) - v^\varepsilon(t, x, y). \end{aligned} \tag{2.26}$$

Thus, $\psi^\varepsilon = 0$ on $\partial\Omega$, and satisfies the following system:

$$\begin{cases} \dfrac{\partial \psi^\varepsilon}{\partial t} - \varepsilon \Delta \psi^\varepsilon + \psi_x^\varepsilon = R_0^\varepsilon + R_1^\varepsilon + R_2^\varepsilon \quad \text{in } (0,T) \times \Omega, \\ \psi^\varepsilon = 0, \qquad \text{on } \partial\Omega\,, \\ \psi^\varepsilon_{|_{t=0}} = 0, \end{cases} \tag{2.27}$$

where

$$\begin{aligned} L^{\varepsilon} &= \frac{\partial}{\partial t} - \varepsilon\Delta + \frac{\partial}{\partial x}, \\ R_0^{\varepsilon} &= \varepsilon\Delta u^0, \\ R_1^{\varepsilon} &= \varepsilon\Big(\frac{\partial^2\widetilde{\varphi}_0^{\varepsilon}}{\partial x^2} + \frac{\partial^2\widetilde{\varphi}_1^{\varepsilon}}{\partial x^2} + \frac{\partial^2\widetilde{\theta}^{\varepsilon}}{\partial y^2} + \varepsilon^{-1}L^{\varepsilon}v^{\varepsilon}\Big), \\ R_2^{\varepsilon} &= \frac{\partial\widetilde{\theta}^{\varepsilon}}{\partial t}. \end{aligned}$$

Here, we can easily verify that v^{ε} is exponentially small. Indeed, we find that for $i, m \in \mathbb{N}$ with $0 \le i + m \le 1$:

$$\begin{aligned} \left|\frac{\partial^{i+m}}{\partial x^i \partial y^m} v^{\varepsilon}(t,x,y)\right| &\le \Big\{ \Big|\frac{\partial^i}{\partial x^i}\widetilde{\varphi}_1^{\varepsilon}(t,x,y=0)\Big| + \Big|\frac{\partial^i}{\partial x^i}\widetilde{\varphi}_0^{\varepsilon}(t,x,y=1)\Big| \\ &+ \Big(|\widetilde{\varphi}_1^{\varepsilon}(t,0,y=0)| + |\widetilde{\varphi}_0^{\varepsilon}(t,0,y=1)|\Big)\Big|\frac{\partial^i}{\partial x^i} e^{-\frac{1-x}{\varepsilon}}\Big\} \\ &\le \kappa(c)\ \exp\Big(-\frac{c}{2\sqrt{\varepsilon}}\Big),\ \ \forall c > 0, \forall (t;x,y) \in (0,T)\times\Omega \end{aligned}$$

which yields

$$\Big\|\frac{\partial^{i+m}v^{\varepsilon}}{\partial x^i \partial y^i}\Big\|_{L^{\infty}(0,T;L^2(\Omega)} \le \kappa\, e^{-1/2\sqrt{\varepsilon}}. \tag{2.28}$$

Moreover, we have

$$\begin{aligned} \left|\frac{\partial}{\partial t}v^{\varepsilon}(t,x,y)\right| &\le \kappa\left|\frac{\partial\widetilde{\varphi}_0^{\varepsilon}}{\partial t}(t,x,y=1)\right| + \left|\frac{\partial\widetilde{\varphi}_1^{\varepsilon}}{\partial t}(t,x,y=0)\right| \\ &+ \left|\frac{\partial}{\partial t}\{\widetilde{\varphi}_0^{\varepsilon}(t,0,y=1) + \widetilde{\varphi}_1^{\varepsilon}(t,0,y=0)\}e^{-(1-x)/\varepsilon}\right| \\ &\le \kappa\, e^{-1/2\sqrt{\varepsilon}}. \end{aligned}$$

Hence, we deduce that

$$\Big\|\frac{\partial v^{\varepsilon}}{\partial t}\Big\|_{L^{\infty}(0,T;L^2(\Omega))} \le \kappa\, e^{-1/2\sqrt{\varepsilon}}. \tag{2.29}$$

We conclude from (2.28) and (2.29) that $L^{\varepsilon}v^{\varepsilon}$ is exponentially small; and this can be absorbed in other L^2-term or H^1-term.

Now, we multiply (2.27) by ψ^{ε} and integrate over Ω, we obtain

$$\begin{aligned} \int_{\Omega}\frac{\partial\psi^{\varepsilon}}{\partial t}\psi^{\varepsilon}\, d\Omega + \varepsilon\int_{\Omega}|\nabla\psi^{\varepsilon}|^2\, d\Omega + \int_{\Omega}\psi_x^{\varepsilon}\psi^{\varepsilon}\, d\Omega &= \int_{\Omega}R_0^{\varepsilon}\psi^{\varepsilon}\, d\Omega + \int_{\Omega}R_1^{\varepsilon}\psi^{\varepsilon}\, d\Omega \\ &+ \int_{\Omega}R_2^{\varepsilon}\psi^{\varepsilon}\, d\Omega. \end{aligned} \tag{2.30}$$

Using $(2.27)_2$, we infer that

$$\int_{\Omega}\psi_x^{\varepsilon}\psi^{\varepsilon}\, d\Omega = 0. \tag{2.31}$$

By applying the Cauchy-Schwartz inequality for the first term in the right hand side in the energy equality (2.30), we find

$$\begin{aligned} \int_\Omega R_0^\varepsilon \psi^\varepsilon \, d\Omega &\leq \|R_0^\varepsilon\|_{L^2(\Omega)} \|\psi^\varepsilon\|_{L^2(\Omega)} \\ &\leq \kappa\, \varepsilon^2 + \frac{1}{2}\|\psi^\varepsilon\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.32}$$

Thanks to Lemma 2.4, Remark 2.2 and Lemma 2.6, we have

$$\begin{aligned} \int_\Omega R_1^\varepsilon \psi^\varepsilon \, d\Omega &= \varepsilon\,(\|\frac{\partial \widetilde{\varphi}_0^\varepsilon}{\partial x}\|^2_{L^2(\Omega)} + \|\frac{\partial \widetilde{\varphi}_1^\varepsilon}{\partial x}\|^2_{L^2(\Omega)} + \|\frac{\partial \widetilde{\theta}^\varepsilon}{\partial y}\|^2_{L^2(\Omega)} + \\ &\quad + \|\varepsilon^{-1} L^\varepsilon v^\varepsilon\|^2_{L^2(\Omega)}) + \frac{\varepsilon}{8}\|\nabla \psi^\varepsilon\|^2_{L^2(\Omega)} \\ &\leq \kappa\, \varepsilon^{3/2} + \frac{\varepsilon}{4}\|\nabla \psi^\varepsilon\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.33}$$

To estimate $\int_\Omega R_2^\varepsilon \psi^\varepsilon \, d\Omega$, we observe that have

$$\left|\frac{\partial}{\partial t}\widetilde{\theta}^\varepsilon(t;x,y)\right| \leq \kappa\, \exp(-\frac{1-x}{\varepsilon})(1+\exp(-\frac{y}{2\sqrt{\varepsilon}}) + \exp(-\frac{1-y}{2\sqrt{\varepsilon}})), \quad \forall\,(t;x,y) \in (0,T)\times\Omega.$$

Hence, we have

$$\begin{aligned} \int_\Omega |\frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}\psi^\varepsilon|\, d\Omega &= \int_\Omega |(1-x)\frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}\frac{\psi^\varepsilon}{(1-x)}|\, d\Omega \\ &\leq \left\|(1-x)\frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}\right\|_{L^2(\Omega)} \left\|\frac{\psi^\varepsilon}{(1-x)}\right\|_{L^2(\Omega)} \\ &\leq \text{(using Hardy's inequality)} \\ &\leq k\left\|(1-x)\frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}\right\|_{L^2(\Omega)} \left\|\nabla \psi^\varepsilon\right\|_{L^2(\Omega)}. \end{aligned}$$

Yet,

$$\begin{aligned} \|(1-x)\frac{\partial \widetilde{\theta}^\varepsilon}{\partial t}\|^2_{L^2(\Omega)} &\leq k\int_0^1\int_0^1 (1+\exp(-\frac{y}{\sqrt{\varepsilon}}) + \exp(-\frac{1-y}{\sqrt{\varepsilon}}))(1-x)^2 \exp(-\frac{2(1-x)}{\varepsilon})\, dydx \\ &\leq k\,(1+\sqrt{\varepsilon})\int_0^1 (1-x)^2 \exp(-\frac{2(1-x)}{\varepsilon})\, dx \\ &\leq k\,\varepsilon^3. \end{aligned}$$

Then, we have:

$$\begin{aligned} \int_\Omega |\frac{\partial}{\partial t}\widetilde{\theta}^\varepsilon \psi^\varepsilon|\, d\Omega &\leq k\,\varepsilon^{\frac{3}{2}}\|\nabla \psi^\varepsilon\|_{L^2(\Omega)} \\ &\leq k\,\varepsilon^2 + \frac{\varepsilon}{4}\|\nabla \psi^\varepsilon\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.34}$$

We conclude from (2.32), (2.33) and (2.34) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi^\varepsilon\|^2_{L^2(\Omega)}+\varepsilon\|\nabla\psi^\varepsilon(t)\|^2_{L^2(\Omega)} &\leq k\,\varepsilon^2+\frac{1}{2}\|\psi^\varepsilon\|^2_{L^2(\Omega)}+k\varepsilon^{3/2}+\frac{\varepsilon}{4}\|\nabla\psi^\varepsilon\|^2_{L^2(\Omega)}\\ &+ k\,\varepsilon^2+\frac{\varepsilon}{4}\|\nabla\psi^\varepsilon\|^2_{L^2(\Omega)}\\ &\leq \frac{1}{2}\|\psi^\varepsilon\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\|\nabla\psi^\varepsilon\|^2_{L^2(\Omega)}+k\,(\varepsilon^2+\varepsilon^{3/2}),\end{aligned}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|\psi^\varepsilon(t)\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\|\nabla\psi^\varepsilon(t)\|^2_{L^2(\Omega)}\leq\frac{1}{2}\|\psi^\varepsilon\|^2_{L^2(\Omega)}+k\,\varepsilon^{3/2}.$$

We now apply the Gronwall inequality and obtain:

$$\|\psi^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leq k\,\varepsilon^{3/4},$$

$$\|\psi^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\leq k\,\varepsilon^{1/4}.$$

This ends the proof of Theorem 2.1. •

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FFR-2024- 2731-01".

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] W. Eckhaus, Asymptotic Analysis of Singular Perturbations, North-Holland, (1979).

[2] J. Grasman, On the Birth of Boundary Layers, Mathematical Centre Tracts 36. Mathematisch Centrum, Amsterdam, (1971).

[3] E. Grenier, O. Guès, Boundary Layers for Viscous Perturbations of Noncharacteristic Quasilinear Hyperbolic Problems, J. Diff. Equ. 143 (1998), 110–146. https://doi.org/10.1006/jdeq.1997.3364.

[4] O. Guès, G. Métivier, M. Williams, K. Zumbrun, Boundary Layer and Long Time Stability for Multi-D Viscous Shocks, Discrete Contin. Dyn. Syst. A. 11 (2004), 131–160. https://doi.org/10.3934/dcds.2004.11.131.

[5] M. Hamouda, R. Temam, Boundary Layers for the Navier–stokes Equations. the Case of a Characteristic Boundary, Georgian Math. J. 15 (2008), 517–530. https://doi.org/10.1515/GMJ.2008.517.

[6] M. Hamouda, R. Temam, Some Singular Perturbation Problems Related to the Navier-Stokes Equations, in: Advances in Deterministic and Stochastic Analysis, World Scientific, 2007: pp. 197–227. https://doi.org/10.1142/9789812770493_0011.

[7] M. Hamouda, C.-Y. Jung, R. Temam, Boundary Layers for the 2D Linearized Primitive Equations, Commun. Pure Appl. Anal. 8 (2009), 335–359. https://doi.org/10.3934/cpaa.2009.8.335.

[8] M. Hamouda, C.Y. Jung, R. Temam, Asymptotic Analysis for the 3d Primitive Equations in a Channel, Discrete Contin. Dyn. Syst. - Ser. S. 6 (2013), 401–422.

[9] C.-Y. Jung, R. Temam, Convection–Diffusion Equations in a Circle: The Compatible Case, J. Math. Pures Appl. 96 (2011), 88–107. https://doi.org/10.1016/j.matpur.2011.03.006.

[10] C.-Y. Jung, R. Temam, Singular Perturbations and Boundary Layer Theory for Convection-Diffusion Equations in a Circle: The Generic Noncompatible Case, SIAM J. Math. Anal. 44 (2012), 4274–4296. https://doi.org/10.1137/110839515.

[11] T. Kato, Remarks on the Euler and Navier-Stokes equations in $\mathbf{R}^2$, in: Proceedings of Symposia in Pure Mathematics, vol. 45, pp. 1–7. American Mathematical Society, Providence (1986).

[12] R.B. Kellogg, M. Stynes, Corner Singularities and Boundary Layers in a Simple Convection–Diffusion Problem, J. Diff. Equ. 213 (2005), 81–120. https://doi.org/10.1016/j.jde.2005.02.011.

[13] P. Lagerstrom, Matched Asymptotics Expansion, Ideas and Techniques, Springer, New York, (1988).

[14] J.L. Lions, Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal, Springer, Berlin, Heidelberg, 1973. https://doi.org/10.1007/BFb0060528.

[15] N. Masmoudi, The Euler Limit of the Navier-Stokes Equations, and Rotating Fluids with Boundary, Arch. Ration. Mech. Anal. 142 (1998), 375–394. https://doi.org/10.1007/s002050050097.

[16] R.E. O'Malley, Singularly Perturbed Linear Two-Point Boundary Value Problems, SIAM Rev. 50 (2008), 459–482. https://doi.org/10.1137/060662058.

[17] O.A. Oleinik, V.N. Samokhin, Mathematical Models in Boundary Layer Theory, Vol. 15, Applied Mathematics and Mathematical Computation, Chapman & Hall/CRC, Boca Raton, (1999).

[18] G.M. Gie, M. Hamouda, A. Sboui, Asymptotic Analysis of the Stokes Equations in a Square at Small Viscosity, Appl. Anal. 95 (2016), 2683–2702. https://doi.org/10.1080/00036811.2015.1105963.

[19] R. Temam, X. Wang, Asymptotic Analysis of the Oseen Type Equations in a Channel at Small Viscosity, Indiana Univ. Math. J. 45 (1996), 863–914. https://www.jstor.org/stable/24899140.

[20] R. Temam, X. Wang, Asymptotic Analysis of the Linearized Navier–Stokes Equations in a General 2D Domain, Asympt. Anal. 14 (1997), 293–321. https://doi.org/10.3233/ASY-1997-14401.

[21] R. Temam, X.M. Wang, Asymptotic Analysis of the Linearized Navier-Stokes Equations in a Channel, Diff. Integral Equ. 8 (1995), 1591–1618. https://doi.org/10.57262/die/1368397749.