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Chiti-type Reverse Hölder Inequality and Saint-Venant Theorem for Wedge Domains on Spheres

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Abstract. In this paper, we prove a new weighted reverse Hölder inequality for the first eigenfunction of the Dirichlet eigenvalue problem in a domain completely contained in a wedge in the sphere $\mathbb{S}^2$. This inequality is known as the Payne-Rayner inequality or Chiti-type inequality. We also prove an extension of Saint-Venant inequality for the relative torsional rigidity of such domains.

1. Introduction

In 1960, Payne and Weinberger [1] proved the curious inequality

$$\lambda \geq \left(\frac{4\alpha(\alpha+1)}{\pi} \int_D r^{2\alpha+1} \sin^2 \alpha\theta \, dr \, d\theta \right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \tag{1.1}$$

where λ is the fundamental eigenvalue of the Dirichlet eigenvalue boundary problem in a two-dimensional bounded domain D completely contained in a wedge of angle $\frac{\pi}{\alpha}$, $\alpha > 1$. Here (r, θ) are polar coordinates taken at the apex of the wedge, and $j_{\alpha,1}$ the first zero of the Bessel function $J_\alpha(x)$. Equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$. This inequality refines the Rayleigh-Faber-Krahn inequality for specific domains, such as certain triangles, and can be interpreted as a version of Faber-Krahn inequality in solids of rotation in higher dimensional Weinstein fractional spaces [2,3]. Note that α need not be an integer.

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Using weighted symmetrization and Payne-Weinberger inequality (1.1), A. Hasnaoui and L. Hermi [4] proved the following inequality

Theorem 1.1. *Let D be a bounded domain in the wedge $\mathcal{W}$. Let p, q be real numbers such that $q \geq p > 0$, then the eigenfunction u associated with the first Dirichlet eigenvalue λ in D satisfies the inequality*

$$\left(\int_D u^q r^{(2-q)\alpha+1} \sin^{2-q} \alpha\theta \, drd\theta\right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha) \left(\int_D u^p \, r^{(2-p)\alpha+1} \sin^{2-p} \alpha\theta \, drd\theta\right)^{\frac{1}{p}} \tag{1.2}$$

with

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r)dr\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r)dr\right)^{\frac{1}{p}}}.$$

The result is isoperimetric in the sense that equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$.

This is a wedge-like version of earlier results of Payne-Rayner [5] and Chiti [6], which have the interpretation of being the classical results in Weinstein fractional space [7] for the particular case of α being an integer.

One of the most recent generalization of Payne-Weinberger inequality is that of J.Ratzkin and A.Treibergs [8] for domains contained in a wedge in the sphere $\mathbb{S}^2$. In the second section, we proceed as in [4] to prove a new weighted reverse Hölder inequality for the first eigenfunction of the Dirichlet problem on such domains by using Ratzkin-Treibergs inequality.

To complete the earlier work of Philippin [9], A. Hasnaoui and L. Hermi [4] introduced the relative torsional rigidity, stated its various formulations and extended some isoperimetric inequalities for this quantity (see also [10]). In the third section, we introduce the relative torsional rigidity of wedge-like domains in the sphere $\mathbb{S}^2$, prove a weighted version of Saint-venant inequality and give the comparison theorem for the warping functions in such domains.

2. Chiti-type inequality for wedge-like domains in the sphere $\mathbb{S}^2$

In order to present our results, it is necessary to introduce certain notation and definitions. Let (ρ, θ) represent the polar coordinates on the sphere $\mathbb{S}^2$, then the round metric is given by

$$ds^2 = d\rho^2 + \sin^2 \rho d\theta^2. \tag{2.1}$$

Let $\mathcal{W}$ be a wedge in the sphere $\mathbb{S}^2$ defined by

$$\mathcal{W} = \{(\rho, \theta) \ : \quad 0 < \rho < \pi, \ \ 0 < \theta < \frac{\pi}{\alpha}\}, \tag{2.2}$$

where $\alpha > 1$. For $0 < r < \pi$, define the sector

$$\mathcal{S}_r = \{(\rho, \theta) \ : \quad 0 < \rho < r, \ \ 0 < \theta < \frac{\pi}{\alpha}\}, \tag{2.3}$$

and let $D \subset \mathbb{S}^2$ be a domain that lies in the wedge $\mathcal{W}$. Note that

$$h(\rho, \theta) = \tan^\alpha(\frac{\rho}{2}) \sin \alpha\theta, \tag{2.4}$$

is a positive harmonic function in $\mathcal{W}$, and its boundary values are zero. The Dirichlet eigenvalue problem in a domain $D \subset \mathcal{W}$ is given by

$$\mathcal{P}_1 : \begin{cases} \Delta u + \lambda u &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{cases}$$

To state our result, we need to introduce the substitution

$$u(\rho,\theta) = v(\rho,\theta)\, h(\rho,\theta) \quad \text{for } (\rho,\theta) \in D, \tag{2.5}$$

with $v \in C^2(D)$ and vanishing on $\partial D \cap \mathcal{W}$.

Theorem 2.1. *Let $D \subset \mathbb{S}^2$ be a domain that lies in the wedge $\mathcal{W}$. Let p, q be real numbers such that $q \geq p > 0$, then v satisfies the inequality*

$$\left(\int_D v^q h^2 da\right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha) \left(\int_D v^p\, h^2 da\right)^{\frac{1}{p}} \tag{2.6}$$

with

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \frac{\left(\int_0^{\rho_1} {}_2F_1^q\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right) \tan^{2\alpha}(\frac{\rho}{2}) \sin\rho\, d\rho\right)^{\frac{1}{q}}}{\left(\int_0^{\rho_1} {}_2F_1^p\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right) \tan^{2\alpha}(\frac{\rho}{2}) \sin\rho\, d\rho\right)^{\frac{1}{p}}}$$

and ρ_1 is the first positive root of the hypergeometric function

$$\rho \mapsto {}_2F_1\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right)$$

The result is isoperimetric in the sense that equality holds if and only if D is a sector of angle $\frac{\pi}{\alpha}$.

We are now prepared to demonstrate the key Theorem 2.1. Prior to proving this result, we will perform a series of reductions.

Recall the function v defined by (2.5). For $0 \leq t \leq \overline{v} = \sup v$, let $D_t = v^{-1}((t,\overline{v}]) = \Big\{(\rho,\theta) \in D |\ v(\rho,\theta) > t\Big\}$. Define the function

$$\xi(t) = \int_{D_t} h^2 da. \tag{2.7}$$

Using the co-area formula, we get

$$\xi(t) = \int_{D_t} h^2 da = \int_t^{\overline{v}} \int_{\partial D_\tau} \frac{h^2}{|\nabla v|} ds\, d\tau \tag{2.8}$$

As D has finite measure, the above shows that the function

$$t \mapsto \int_{\partial D_t} \frac{h^2}{|\nabla v|} ds. \tag{2.9}$$

is integrable, which implies that ξ is absolutely continuous. Then, ξ is differentiable almost everywhere, and

$$\frac{d\xi}{dt} = -\int_{\partial D_t} \frac{h^2}{|\nabla v|} ds \ < \ 0 \tag{2.10}$$

for almost all $t \in [0, \overline{v}]$. The function ξ is, therefore, a nonincreasing function and has an inverse denoted as $t(\xi)$. Note that

$$h^2 = \left(\sqrt{|\nabla v|} h \right) \left(\frac{h}{\sqrt{|\nabla v|}} \right).$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\left(\int_{\partial D_t} h^2 ds \right)^2 \leq \left(\int_{\partial D_t} \frac{h^2}{|\nabla v|} ds \right) \left(\int_{\partial D_t} h^2 |\nabla v| ds \right). \tag{2.11}$$

Therefore,

$$-t'(\xi) = -\frac{1}{\xi'(t)} \leq \frac{\int_{\partial D_t} h^2 |\nabla v| ds}{\left(\int_{\partial D_t} h^2 ds \right)^2}. \tag{2.12}$$

We will now use a geometric inequality presented by J.Ratzkin and A.Treibergs in the following lemma.

Lemma 2.1. *(Ratzkin-Treibergs [8].) Let $D \subset \mathcal{W}$ be a domain with compact closure. Then*

$$\int_{\partial D} h^2 ds \geq \frac{\pi}{2\alpha} \Upsilon_\alpha \left(\frac{2\alpha}{\pi} \int_D h^2 da \right). \tag{2.13}$$

Here $\Upsilon_\alpha = f' \circ f^{-1}$, where $f(r) = \int_0^r \tan^{2\alpha}(\frac{\rho}{2}) \sin \rho \, d\rho$. Equality holds if and only if D is a sector almost everywhere.

Applying this lemma, we obtain

$$-t'(\xi) \leq \left(\frac{2\alpha}{\pi} \right)^2 \frac{\int_{\partial D_t} h^2 |\nabla v| ds}{\Upsilon_\alpha^2 \left(\frac{2\alpha}{\pi} \int_{D_t} h^2 da \right)}. \tag{2.14}$$

Using the divergence theorem and the fact that $\Delta h = 0$, we obtain

$$\begin{aligned} \int_{\partial D_t} h^2 |\nabla v| ds &= -\int_{D_t} \operatorname{div} \left(h^2 \nabla v \right) da \\ &= -\int_{D_t} h \, \left(h \triangle v + 2 \langle \nabla v, \nabla h \rangle \right) da \\ &= \lambda \int_{D_t} v \, h^2 da. \end{aligned} \tag{2.15}$$

Remark 2.1. $\forall p \geq 0$, *we have*

$$\int_{D_t} v^p \, h^2 da = \int_t^{\overline{v}} \tau^p \int_{\partial D_\tau} \frac{h^2}{|\nabla v|} \, ds \, d\tau = -\int_t^{\overline{v}} \tau^p \xi'(\tau) d\tau. \tag{2.16}$$

The change of variable $\eta = \xi(\tau)$ gives

$$\int_{D_t} v^p h^2 da = \int_0^{\xi(t)} \left(t(\eta) \right)^p d\eta. \tag{2.17}$$

Applying this remark for $p = 1$ in inequality (2.14), we obtain

$$-t'(\xi) \leq \left(\frac{2\alpha}{\pi}\right)^2 \lambda \frac{\int_0^{\xi} t(\eta)d\eta}{\Upsilon_\alpha^2(\frac{2\alpha}{\pi}\xi)}, \tag{2.18}$$

for almost all $\xi \in [0, \xi_0]$, with $\xi_0 = \xi(0) = \int_D h^2 da$.

With λ still represent the first eigenvalue of $\mathcal{P}_1$, we consider the sector

$$S_\lambda = \Big\{(\rho, \theta) : \ \ 0 < \rho < \rho_1, \ 0 < \theta < \frac{\pi}{\alpha} \quad \Big\},$$

where ρ_1 is the first positive root of the hypergeometric function

$$\rho \mapsto \ {}_2F_1(\frac{1 - \sqrt{1+4\lambda}}{2}, \frac{1 + \sqrt{1+4\lambda}}{2}; \alpha + 1; \frac{1 - \cos\rho}{2})$$

The eigenvalue problem in S_λ is given by

$$\mathcal{P}_2 : \begin{cases} \Delta u + \mu u &= \ 0 \quad \text{in } S_\lambda \\ u &= \ 0 \quad \text{on } \partial S_\lambda. \end{cases}$$

The sector S_λ is defined in such a way that its first eigenvalue is equal to λ. The corresponding eigenfunction can be explicitly expressed as follows:

$$z(\rho, \theta) = h(\rho, \theta)\, R(\rho). \tag{2.19}$$

Here, R denotes the radial function defined by

$$R(\rho) = c \ {}_2F_1(\frac{1 - \sqrt{1+4\lambda}}{2}, \frac{1 + \sqrt{1+4\lambda}}{2}; \alpha + 1; \frac{1 - \cos\rho}{2}), \tag{2.20}$$

and c is a normalizing constant. For all $0 \leq s \leq \overline{R} = \sup R$, let

$$S_{\lambda,s} = \Big\{(\rho, \theta) | \ \ R(\rho) > s, \ 0 < \theta < \frac{\pi}{\alpha}\Big\} \quad \text{and} \quad \zeta(s) = \int_{S_{\lambda,s}} h^2 da.$$

We can now follow the same steps as in the proof of inequality (2.18) to establish that ζ is a decreasing function and possesses an inverse denoted by $s(\zeta)$. This inverse function satisfies the integro-differential inequality

$$-s'(\zeta) \leq \left(\frac{2\alpha}{\pi}\right)^2 \lambda \frac{\int_0^{\zeta} t(\eta)d\eta}{\Upsilon_\alpha^2(\frac{2\alpha}{\pi}\zeta)}, \tag{2.21}$$

for almost all $\zeta \in [0, \zeta_0]$, with $\zeta_0 = \zeta(0) = \int_{S_\lambda} h^2 da$.

Let $S_0 = \Big\{(\rho, \theta) \ | \ \ 0 < \rho < \rho_0, \ 0 < \theta < \frac{\pi}{\alpha}\Big\}$ such that

$$\int_{S_0} h^2 da = \int_D h^2 da = \xi_0. \tag{2.22}$$

From an explicit computation, we get that $\xi_0 = \frac{\pi}{2\alpha} f(\rho_0)$. Let us introduce the function $u^\star$ defined on S_0 as follows

$$u^\star(\rho, \theta) = v^\star(\rho)\, h(\rho, \theta), \tag{2.23}$$

where $v^\star$ is the radial and decreasing function defined by

$$v^\star(\rho) = t\Big(\frac{\pi}{2\alpha} f(\rho)\Big), \qquad \forall \rho \in [0, \rho_0]. \tag{2.24}$$

It is clear that the level set

$$S_{0,\tau} = \Big\{(\rho, \theta) \in S_0 \ \mid \ v^\star(\rho) > \tau, \ \ 0 < \theta < \frac{\pi}{\alpha}\Big\}$$

is a sector, and for all $p \geq 0$ we have

$$\begin{aligned}
\int_{S_{0,\tau}} v^{\star p} h^2 da &= \frac{\pi}{2\alpha} \int_{\left\{\rho>0, \ t\left(\frac{\pi}{2\alpha} f(\rho)\right)>\tau\right\}} \Big(t\Big(\frac{\pi}{2\alpha} f(\rho)\Big)\Big)^p f'(\rho)\ d\rho \\
&= \int_{\left\{\eta>0, \quad t(\eta)>\tau\right\}} t^p(\eta) d\eta \\
&= \int_0^{\xi(\tau)} t^p(\eta) d\eta \\
&= \int_{D_\tau} v^p h^2 da
\end{aligned} \tag{2.25}$$

for all $\tau \in [0, \overline{v}]$.

Lemma 2.2. *Choose c in* (2.20) *such that $R(0) = v^\star(0)$. Then*

$$z(\rho, \theta) \leq u^\star(\rho, \theta), \quad \forall\ (\rho, \theta) \in S_\lambda. \tag{2.26}$$

Proof. In order to prove this lemma, it is necessary to introduce the following remark.

Remark 2.2. *We have*

$$\xi_0 \geq \zeta_0. \tag{2.27}$$

Consider the lowest eigenvalue λ_0 of the eigenvalue problem for S_0. Assume that $\xi_0 < \zeta_0$ and note that S_0 and S_λ are concentric sectors with fixed angle $\frac{\pi}{\alpha}$. Hence, $S_0 \subset S_\lambda$ and from domains monotonicity of eigenvalues, we deduce that $\lambda_0 > \lambda$ which contradicts the Ratzkin-Treibergs Theorem in [8].

Using Remark 2.2, two cases occur:

If $\xi_0 = \zeta_0$. Since D and S_λ share the same Dirichlet first eigenvalue, and according to the Ratzkin-Treibergs Theorem, we conclude that $D = S_\lambda = S_0$. Now, considering $\Delta h = 0$ and applying the divergence theorem, we obtain

$$\int_{S_0} |\nabla u^\star|^2 da = \int_{S_0} |\nabla v^\star|^2 h^2 da, \tag{2.28}$$

and

$$\begin{aligned}
\int_{S_0} |\nabla v^\star|^2 h^2 da &= \int_{S_0} \Big|\nabla t\Big(\frac{\pi}{2\alpha} f(\rho)\Big)\Big|^2 h^2(\rho, \theta) \sin\rho d\rho d\theta \\
&= \Big(\frac{\pi}{2\alpha}\Big)^3 \int_0^{\rho_0} \left(f'(\rho)\right)^3 \Big(t'\Big(\frac{\pi}{2\alpha} f(\rho)\Big)\Big)^2 d\rho
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\pi}{2\alpha}\right)^2 \int_0^{\xi_0} \Upsilon_\alpha^2(\frac{2\alpha}{\pi}\xi)\,(t'(\xi))^2\,d\xi \\
&\leq \lambda \int_0^{\xi_0} (-t'(\xi)) \int_0^{\xi} t(\eta)d\eta d\xi \\
&= \lambda \int_0^{\xi_0} \Big(t(\xi)\Big)^2 d\xi \\
&= \lambda \int_{S_0} u^{\star 2} da.
\end{aligned} \tag{2.29}
$$

In the above steps, we used (2.18), $t(\xi_0) = 0$, and integration by parts. Thus

$$
\frac{\int_{S_0} |\nabla u^\star|^2 da}{\int_{S_0} u^{\star 2} da} \leq \lambda. \tag{2.30}
$$

Since λ is also the minimum of the Rayleigh quotient on S_0, it implies that this minimum is attained by $u^\star$. Therefore, $u^\star$ is indeed the eigenfunction associated with λ on S_0. From equation (2.25), we have $u^\star = u$ and $z = z^\star$. Furthermore, the fact that $u^\star$ and z are the first Dirichlet eigenfunctions in S_0 implies the existence of a positive constant c' such that $R(r) = c' v^\star(r)$ for all $r \in (0, \rho_0)$. Finally, the assumption of the lemma yields $c' = 1$ and $z = u = u^\star$.

If $\xi_0 > \zeta_0$**.** We observe that $t(\zeta_0) > 0$ while $s(\zeta_0) = 0$. Additionally, considering the fact that

$$
s(0) = R(0) = v^\star(0) = \sup v = t(0), \tag{2.31}
$$

we can find a constant $\kappa \geq 1$ such that

$$
\kappa\, t(\zeta) \geq s(\zeta) \quad \forall \zeta \in [0, \zeta_0]. \tag{2.32}
$$

Let us define the constant c'' as follows

$$
c'' = \inf\{\kappa \geq 1\ ; \quad \kappa\, t(\zeta) \geq s(\zeta), \quad \forall \zeta \in [0, \zeta_0]\} \tag{2.33}
$$

Then, using the definition of c'', we can find $\zeta_1 \in [0, \zeta_0)$ such that $c''\, t(\zeta_1) = s(\zeta_1)$.
We introducee now the function φ_1 defined by

$$
\varphi_1(\zeta) = \begin{cases} c''\, t(\zeta); & \text{if } \zeta \in [0, \zeta_1] \\ s(\zeta); & \text{if } \zeta \in [\zeta_1, \zeta_0]. \end{cases}
$$

The properties of t and s imply that φ_1 is monotonically decreasing and $\varphi_1(\zeta_0) = 0$. Further, from inequalities (2.18) and (2.21), we observe the following

$$
-\varphi_1'(\zeta) \leq \left(\frac{2\alpha}{\pi}\right)^2 \lambda \frac{\int_0^{\zeta} \varphi_1(\eta) d\eta}{\Upsilon_\alpha^2(\frac{2\alpha}{\pi}\zeta)},, \tag{2.34}
$$

for almost all $\zeta \in [0, \zeta_0]$. Now, let Φ_1 defined in S_λ by

$$
\Phi_1(\rho, \theta) = \varphi_1\Big(\frac{\pi}{2\alpha} f(\rho)\Big)\, h(\rho, \theta), \tag{2.35}
$$

then, Φ_1 is an admissible function for the Rayleigh quotient on S_λ. Following the same steps as in the proof of inequality (2.29), we obtain

$$\frac{\int_{S_\lambda} |\nabla \Phi_1|^2 da}{\int_{S_\lambda} \Phi_1^2 da} \leq \lambda. \tag{2.36}$$

Furthermore, due to the definition of S_λ, we can deduce that Φ_1 is an eigenfunction associated with λ on S_λ. Therefore, Φ_1 is a multiple of z, and by considering the definition of φ_1, we conclude that $c'', t(\zeta) = s(\zeta)$ for $0 \leq \zeta \leq \zeta_1$. Since $t(0) = s(0)$, it follows that $c'' = 1$, and thus $t(\zeta) \geq s(\zeta)$ for all $0 \leq \zeta \leq \zeta_0$. This completes the proof of the lemma. The following result is an extension of Chiti Comparison Lemma [6], for wedge domains on spheres. □

Theorem 2.2. *For $p > 0$, let c be chosen in* (2.20) *such that*

$$\int_D v^p h^2 da = \int_{S_\lambda} R^p h^2 da, \tag{2.37}$$

Then, there exists $\rho_2 \in (0, \rho_1)$ such that

$$u^\star(\rho, \theta) \leq z(\rho, \theta), \quad \forall (\rho, \theta) \in (0, \rho_2] \times (0, \tfrac{\pi}{\alpha})\ ; \tag{2.38}$$

$$u^\star(\rho, \theta) \geq z(\rho, \theta), \quad \forall (\rho, \theta) \in [\rho_2, \rho_1] \times (0, \tfrac{\pi}{\alpha}). \tag{2.39}$$

Remark 2.3. *The normalization condition* (2.37) *gives*

$$\int_0^{\xi_0} t^p(\xi) d\xi = \int_0^{\zeta_0} s^p(\zeta) d\zeta. \tag{2.40}$$

Since the functions t and s are nonnegative, and considering that $\zeta_0 \leq \xi_0$ as stated in Remark 2.2, it is evident that

$$\int_0^{\zeta_0} t^p(\zeta) d\zeta \leq \int_0^{\zeta_0} s^p(\zeta) d\zeta. \tag{2.41}$$

Proof. Let us consider the claim that $s(0) \geq t(0)$.

Suppose, for the sake of contradiction, that $s(0) < t(0)$.

In such a case, there exists a constant $\kappa > 1$ such that $\kappa s(0) = t(0)$. According to Lemma 2.2, we have

$$\kappa\, s(\zeta) \leq t(\zeta) \quad \forall \zeta \in [0, \zeta_0]. \tag{2.42}$$

Therefore

$$\kappa^p \int_0^{\zeta_0} s^p(\zeta) d\zeta \leq \int_0^{\zeta_0} t^p(\zeta) d\zeta.$$

Combining this inequality with (2.41) leads to $\kappa^p \leq 1$, which is a contradiction.

Suppose now that $s(0) = t(0)$.

Using equality (2.40) and Lemma 2.2, we get

$$\int_0^{\xi_0} t^p(\zeta) d\zeta = \int_0^{\zeta_0} s^p(\zeta) d\zeta \leq \int_0^{\zeta_0} t^p(\zeta) d\zeta. \tag{2.43}$$

This implies that $\int_{\zeta_0}^{\xi_0} t^p(\zeta), d\zeta = 0$, and since $t > 0$ in $(0, \xi_0)$, it follows that $\xi_0 = \zeta_0$. Hence, we have $z = u^\star$, and the statements of the theorem become evident.

Now, let's consider the case where $s(0) > t(0)$.

In this case, it is evident from the proof of Lemma 2.2 that $\zeta_0 < \xi_0$. Therefore, $s(\zeta_0) = 0$ and $t(\zeta_0) > 0$. Using the continuity of t and s, we can conclude that there exists a neighborhood of 0 in which $s(\zeta) > t(\zeta)$. Furthermore, there exists $\zeta_1 \in (0, \zeta_0)$ such that $s(\zeta_1) = t(\zeta_1)$. We choose ζ_1 to be the largest such number satisfying the condition that $t(\zeta) \leq s(\zeta)$ for all $\zeta \in [0, \zeta_1]$. By the definition of ζ_1, there exists an interval immediately to the right of ζ_1 where $t(\zeta) > s(\zeta)$.

We will now prove that $t(\zeta) > s(\zeta)$ for all $\zeta \in (\zeta_1, \zeta_0]$. Suppose not, and let $\zeta_2 \in (\zeta_1, \zeta_0)$ be a point such that $t(\zeta_2) = s(\zeta_2)$ and $t(\zeta) > s(\zeta)$ for all $\zeta \in (\zeta_1, \zeta_2)$. In this case, we can define the function

$$\varphi_2(\zeta) = \begin{cases} s(\zeta), & \text{for } \zeta \in [0, \zeta_1] \cup [\zeta_2, \zeta_0], \\ t(\zeta), & \text{for } \zeta \in [\zeta_1, \zeta_2]. \end{cases}$$

It follows from (2.18) and (2.21) that φ_2 satisfies

$$-\varphi_2'(\zeta) \leq \left(\frac{2\alpha}{\pi}\right)^2 \lambda \frac{\int_0^\zeta \varphi_2(\eta) d\eta}{\Upsilon_\alpha^2(\frac{2\alpha}{\pi}\zeta)}, \tag{2.44}$$

From φ_2 define the function in S_λ by

$$\Phi_2(\rho, \theta) = \varphi_2\Big(\frac{\pi}{2\alpha} f(\rho)\Big) h(\rho, \theta). \tag{2.45}$$

Then Φ_2 is an admissible function for the Rayleigh quotient on S_λ. From this and proceed exactly as in the proof of the inequality (2.36), we have

$$\frac{\int_{S_\lambda} |\nabla \Phi_2|^2 da}{\int_{S_\lambda} \Phi_2^2 da} \leq \lambda. \tag{2.46}$$

It will follow that the Rayleigh quotient of Φ_2 is equal to λ and hence that Φ_2 is an eigenfunction for λ, Consequently, $t = s$ and so $t(\zeta) = s(\zeta)$ in $[\zeta_1, \zeta_2]$ contradicting the maximality of ζ_1. The proof of the theorem is now complete. □

Proof of Theorem 2.1. For $p > 0$, we choose c in (2.20) such that (2.37) is satisfied. This implies

$$\int_0^{\xi_0} t^p(\xi) d\xi = \int_0^{\zeta_0} s^p(\xi) d\xi \tag{2.47}$$

as mentioned in Remark 2.3.

Now, if we extend the function s by zero in $[\zeta_0, \xi_0]$, we get

$$\int_0^\xi t^p(\eta) d\eta \leq \int_0^\xi s^p(\eta) d\eta, \qquad \forall \xi \in [0, \xi_0]. \tag{2.48}$$

To establish (2.48), we observe that Theorem 2.2 yields the following

If $\xi \in [0, \zeta_1]$, then

$$t(\eta) \leq s(\eta) \quad \forall \eta \in [0, \xi],$$

and therefore, we have

$$\int_0^{\xi} t^p(\eta)d\eta \leq \int_0^{\xi} s^p(\eta)d\eta.$$

If $\xi \in [\zeta_1, \xi_0]$, then

$$\begin{aligned} \int_0^{\xi} t^p(\eta)d\eta &= \int_0^{\xi_0} t^p(\eta)d\eta - \int_{\xi}^{\xi_0} t^p(\eta)d\eta \\ &\leq \int_0^{\xi_0} s^p(\eta)d\eta - \int_{\xi}^{\xi_0} s^p(\eta)d\eta \\ &= \int_0^{\xi} s^p(\eta)d\eta. \end{aligned}$$

Now, using the result of Hardy, Littlewood and Pólya proved in [11] (Theorem 10, p. 152), we obtain, for $q \geq p$, that

$$\int_0^{\xi_0} t^q(\eta)d\eta \leq \int_0^{\xi_0} s^q(\eta)d\eta = \int_0^{\zeta_0} s^q(\eta)d\eta. \tag{2.49}$$

Using this and equality (2.47), we obtain

$$\left(\int_D u^q\, h^{2-q}da\right)^{\frac{1}{q}} \leq K(p,q,\rho_1,\alpha)\left(\int_D u^p\, h^{2-p}da\right)^{\frac{1}{p}}, \tag{2.50}$$

with

$$\begin{aligned} K(p,q,\rho_1,\alpha) &= \frac{\left(\int_{S_\lambda} c^q\, R^q h^2 da\right)^{\frac{1}{q}}}{\left(\int_{S_\lambda} c^p R^p\, h^2 da\right)^{\frac{1}{p}}} \\ &= \frac{\left(\int_0^{\frac{\pi}{\alpha}}\int_0^{\rho_1} c^q\; {}_2F_1^q\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right)\tan^{2\alpha}\left(\frac{\rho}{2}\right) sin^2(\alpha\theta)\sin\rho\, d\rho d\theta\right)^{\frac{1}{q}}}{\left(\int_0^{\frac{\pi}{\alpha}}\int_0^{\rho_1} c^p\; {}_2F_1^p\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right)\tan^{2\alpha}\left(\frac{\rho}{2}\right) sin^2(\alpha\theta)\sin\rho\, d\rho d\theta\right)^{\frac{1}{p}}} \\ &= \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \frac{\left(\int_0^{\rho_1} {}_2F_1^q\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right)\tan^{2\alpha}\left(\frac{\rho}{2}\right)\sin\rho\, d\rho\right)^{\frac{1}{q}}}{\left(\int_0^{\rho_1} {}_2F_1^p\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos\rho}{2}\right)\tan^{2\alpha}\left(\frac{\rho}{2}\right)\sin\rho\, d\rho\right)^{\frac{1}{p}}} \end{aligned}$$

This completes the proof of Theorem 2.1.

3. The Saint-Venant theorem for wedge-like domains in the sphere $\mathbb{S}^2$.

In this section we are interested in the mathematical quantity given by

$$P_\alpha = \int_D ghda,$$

where $D \subset \mathbb{S}^2$ is domain that lies in the wedge $\mathcal{W}$ and g is a solution of the Dirichlet boundary value problem

$$\mathcal{P}_5 : \begin{cases} -\Delta g &= h & \text{in } D \\ g &= 0 & \text{on } \partial D. \end{cases}$$

Now, If we write $g = h \cdot w$, where w is a function in $C^2(D)$ satisfying the boundary condition $w = 0$ on $\partial D \cap \mathcal{W}$, a straightforward computation reveals that w is a solution to the following problem

$$\mathcal{P}_6 : \begin{cases} -\mathrm{div}(h^2 \nabla w) &= h^2 & \text{in } D \\ w &= 0 & \text{on } \partial D \cap \mathcal{W}. \end{cases}$$

With this substitution, it is clear that

$$P_\alpha = \int_D w \, d\mu,$$

where the measure $d\mu = h^2 da$ .For the planar case, P_α can be interpreted as torsional rigidity in dimension $(2\alpha + 2)$ for the particular cases $\alpha = 1$ and $\alpha = 2$, it's called the relative torsional rigidity of D (see [4,10]) . We now introduce the space $W_2(D, d\mu)$ which is the set of measurable functions φ which satisfy the following conditions:

(i) $\int_D |\nabla\varphi|^2 d\mu + \int_D |\varphi|^2 d\mu < +\infty$

(ii) There exists a sequence of functions $\varphi_n \in C^1(\overline{D})$ such that $\varphi_n(\rho, \theta) = 0$ on $\partial D \cap \mathcal{W}$ and

$$\lim_{n \to +\infty} \int_D |\nabla(\varphi - \varphi_n)|^2 d\mu + \int_D |\varphi - \varphi_n|^2 d\mu = 0. \tag{3.1}$$

The space $W_2(D, d\mu)$ was defined in the context of proving Talenti theorem in various cases, such as those explored in [4,12–14]). By that we can define P_α via the variational formulation

$$\frac{1}{P_\alpha} = \inf_{\varphi \in W_2(D, d\mu)} \frac{\int_D |\nabla\varphi|^2 \, d\mu}{\left(\int_D \varphi \, d\mu\right)^2}, \tag{3.2}$$

The proof of the given formulation (3.2) is similar to the (Remark 5.2. in [4]).

Now, we are ready to present and prove several new results for the relative torsional rigidity on spheres.

Theorem 3.1. *Let D be a domain, with a piecewise smooth boundary, completely contained in $\mathcal{W}$ then*

$$P_\alpha \, \lambda < A_\alpha, \tag{3.3}$$

where

$$A_\alpha = \int_D h^2 da.$$

Proof. The result follows immediately using the Cauchy-Schwartz inequality in the statement

$$\begin{aligned} P_\alpha = \frac{\left(\int_D g \, h \, da\right)^2}{\int_D |\nabla g|^2 da} &\leq \frac{\int_D g^2 da \int_D h^2 da}{\int_D |\nabla g|^2 da} \\ &\leq \lambda^{-1} A_\alpha. \end{aligned}$$

The last inequality was obtained applying the Rayleigh-Ritz principle for λ with g being a test function. □

Theorem 3.2. *Let $D \subset \mathcal{W}$ be a domain with a piecewise smooth boundary, then*

$$\frac{1}{P_\alpha} \leq \frac{2\alpha}{\pi}\lambda K^2(1,2,\lambda,\alpha). \tag{3.4}$$

Proof. As before, let u be the first eigenfunction of the Dirichlet problem. By the variational formulation one can see that

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2\, da}{\left(\int_D uh\, da\right)^2} \tag{3.5}$$

Using Theorem 2.1 we obtain

$$\begin{aligned} \frac{1}{P_\alpha} &\leq \frac{2\alpha}{\pi}K^2(1,2,\lambda,\alpha)\frac{\int_D |\nabla u|^2\, da}{\int_D u^2\, da} && (3.6)\\ &= \frac{2\alpha}{\pi}\lambda K^2(1,2,\lambda,\alpha). && (3.7)\end{aligned}$$

□

The next result is Saint-Venant inequality for wedge-like domains in the sphere.

Theorem 3.3. *Let D be a domain completely contained in $\mathcal{W}$, with a piecewise smooth boundary, then*

$$P_\alpha \leq \frac{\pi}{2\alpha}\int_0^{\rho_0}\left(\int_\rho^{\rho_0}\frac{f(\eta)}{f'(\eta)}\, d\eta\right)f'(\rho)d\rho. \tag{3.8}$$

Equality is attained if and only if D is a sector of angle $\frac{\pi}{\alpha}$.

Proof. For $0 < t \leq m = \sup\{w(\rho,\theta),\ \ (\rho,\theta) \in D\}$, we introduce the ensemble

$$D_t = \{(\rho,\theta) \in D; \quad w(\rho,\theta) > t\}, \tag{3.9}$$

and we define the functions

$$A(t) = \int_{D_t} h^2 da, \qquad \Psi(t) = \int_{D_t} |\nabla w|^2 h^2 da. \tag{3.10}$$

The co-area formula gives

$$A(t) = \int_t^m \int_{\partial D_\tau} \frac{1}{|\nabla w|} ds d\tau, \qquad \Psi(t) = \int_t^m \int_{\partial D_\tau} |\nabla w| ds d\tau, \tag{3.11}$$

Differentiating with respect to t, we obtain

$$A'(t) = -\int_{\partial D_t} \frac{1}{|\nabla w|} ds, \qquad \Psi'(t) = -\int_{\partial D_t} |\nabla w| ds, \tag{3.12}$$

Let w^* the inverse function of A and $w^\star$ be the radial function define in $D^\star$ by $w^\star(\rho) = w^*(\frac{\pi}{2\alpha}f(\rho))$. From that we define the functions

$$A_\star(t) = \int_{S_{0,t}} h^2 da, \qquad \Psi_\star(t) = \int_{S_{0,t}} |\nabla w^\star|^2 h^2 da. \tag{3.13}$$

By the fact that the function $w^\star$ is radiale decreasing function we have that his level set $S_{0,t} = \{(\rho,\theta) \in S_0; \quad w^\star(\rho) > t \ \ , 0 < \theta < \frac{\pi}{\alpha}\}$ is a sector . Hence, we can state that

$$A_\star(t) = \frac{\pi}{2\alpha}\int_0^{\rho(t)} \tan^{2\alpha}(\frac{\tau}{2})\sin\tau\, d\tau \tag{3.14}$$

and

$$\Psi_\star(t) = \frac{\pi}{2\alpha}\int_0^{\rho(t)} (\frac{dw^\star}{d\tau})^2\tan^{2\alpha}(\frac{\tau}{2})\sin\tau\, d\tau, \tag{3.15}$$

where $\rho(t) = w^{\star,-1}(t)$. Differentiating with respect to t, we get

$$A'_\star(t) = -\frac{1}{|\nabla w^\star|(t)}\frac{\pi}{2\alpha}\tan^{2\alpha}(\frac{\rho(t)}{2})\sin(\rho(t)) = -\frac{1}{|\nabla w^\star|(t)}\int_{\partial S_{0,t}} h^2\, ds \tag{3.16}$$

and

$$\Psi'_\star(t) = -|\nabla w^\star|(t)\frac{\pi}{2\alpha}\tan^{2\alpha}(\frac{\rho(t)}{2})\sin(\rho(t)) = -|\nabla w^\star|(t)\int_{\partial S_{0,t}} h^2\, ds. \tag{3.17}$$

Multiplying (3.16) by (3.17), we get

$$\left(\int_{\partial S_{0,t}} h^2\, ds\right)^2 = A'_\star(t)\Psi'_\star(t). \tag{3.18}$$

On the other hind the Cauchy-Schwartz inequality gives

$$\left(\int_{\partial D_t} h^2\, ds\right)^2 \leq A'(t)\Psi'(t). \tag{3.19}$$

From the fact that $A_\star(t) = A(t)$, we have $A'(t) = A'_\star(t)$. From this, Lemma 2.1 , equality (3.18) and inequality (3.19) we have

$$\Psi'_\star(t) \geq \Psi'(t) \tag{3.20}$$

Integrating the last inequality from 0 to m and using the fact that $\Psi_\star(m) = 0 = \Psi(m)$ we obtain

$$\int_{S_0} |\nabla w^\star|^2 h^2\, da = \Psi_\star(0) \leq \Psi(0) = \int_D |\nabla w|^2 h^2\, da. \tag{3.21}$$

Now, from the fact that

$$\int_{S_0} w^\star h^2\, da = \int_D w h^2\, da, \tag{3.22}$$

and the variational formulation given in 3.2 we get

$$\frac{1}{P_\alpha(D)} = \frac{\int_D |\nabla w|^2 h^2\, da}{(\int_D wh^2\, da)^2} \geq \frac{\int_{S_0} |\nabla w^\star|^2 h^2\, da}{(\int_{S_0} w^\star h^2\, da)^2} \geq \frac{1}{P_\alpha(S_0)} \tag{3.23}$$

Using the definition of $P_\alpha(S_0)$ on the symmetrized domain S_0, we obtain that $P_\alpha(S_0) = \int_{S_0} \tilde{w}h^2 da$. Where $\tilde{w}$ is the solution of the following problem

$$\mathcal{P}_7 : \begin{cases} -\mathrm{div}(h^2\nabla\tilde{w}) &= h^2 \quad \text{in } S_0 \\ \tilde{w} &= 0 \quad \text{on } \partial S_0 \cap \mathcal{W}. \end{cases}$$

Now, by a little computation, it is not difficult to check that

$$\tilde{w}(\rho) = \int_{\rho}^{\rho_0} \frac{f(\eta)}{f'(\eta)}\, d\eta.$$

.

In the case of equality, when considering the proof of our theorem, integrating the inequality (3.20), and using the equalities $\Psi_\star(m) = \Psi(m)$ and $\Psi_\star(0) = \Psi(0)$, we can deduce that $\Psi'_\star(t) = \Psi'(t)$. By applying this result to equations (3.19) and (3.18), we obtain

$$\int_{\partial S_{0,t}} h^2\, ds \geq \int_{\partial D_t} h^2\, ds. \tag{3.24}$$

Finally, applying Lemma 2.1 , we obtain equality in (3.24) and that $D = S_0$ which completes the proof. □

In the last part, we give the comparison theorem for the warping functions in the case of wedge like domains in spheres. this theorem remounted to Talenti in the case of Euclidian space.

Theorem 3.4. *Let w be the solution to problem $\mathcal{P}_6$ and let $\tilde{w}$ be the solution to the problem $\mathcal{P}_7$. Then $w^\star$, the symmetrization of w satisfies*

$$w^\star \leq \tilde{w} \qquad a.e \;\; in \;\; S_0. \tag{3.25}$$

We obtain the equality if and only if D is a sector of angle $\frac{\pi}{\alpha}$.

Proof. Let φ be a test function in the weak formulation of our problem defined by

$$\varphi(\rho,\theta) = \begin{cases} w(\rho,\theta) - t, & \text{if } w(\rho,\theta) > t \\ 0, & \text{otherwise ,} \end{cases} \tag{3.26}$$

where $0 \leq t < m$. Since The solution w satisfies the equality

$$\int_D \langle \nabla w, \nabla \varphi \rangle h^2 da = \int_D \varphi h^2 da, \tag{3.27}$$

we have

$$\Psi(t) = \int_{D_t} |\nabla w|^2 h^2 da = \int_{w>t} (w-t)\, h^2 da.. \tag{3.28}$$

The function Ψ is decreasing of t then, for $\epsilon > 0$ we obtain

$$\frac{\Psi(t) - \Psi(t+\epsilon)}{\epsilon} = \int_{w>t+\epsilon} h^2 da + \int_{t<w\leq t+\epsilon} \left(\frac{w-t}{\epsilon}\right) h^2 da.$$

Letting ϵ go to zero, we obtain the following expression for the right derivative of $\Psi(t)$

$$-\Psi'_+(t) = \int_{w>t} h^2 da \quad \text{a.e.} \quad t > 0. \tag{3.29}$$

By the same computation, we can establish that the equality holds for the left derivative of $\Psi(t)$ as well. Thus

$$0 \leq -\Psi'(t) = A(t). \tag{3.30}$$

We use now the Cauchy-Schwarz inequality

$$\left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon} |\nabla w| h^2 da\right)^2 \leq \left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon} |\nabla w|^2 h^2 da\right)\left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon} h^2 da\right). \tag{3.31}$$

Therefore, letting $\epsilon \to 0$ and using (3.30), we get

$$\left(-\frac{d}{dt}\int_{w>t} |\nabla w| h^2 da\right)^2 \leq -A'(t)A(t). \tag{3.32}$$

From the co-area formula, we have

$$-\frac{d}{dt}\int_{w>t} |\nabla w| h^2 da = \int_{\partial D_t} h^2 ds, \quad \text{a.e.} \quad t > 0. \tag{3.33}$$

But by Lemma 2.1, we have

$$\int_{\partial D_t} h^2 ds \geq \int_{\partial S_{0,t}} h^2 ds = \frac{\pi}{2\alpha}\tan^{2\alpha}\left(\frac{\rho(t)}{2}\right)\sin(\rho(t)) = \frac{\pi}{2\alpha} f'(\rho(t)). \tag{3.34}$$

Combining this with (3.33) and (3.32), we obtain

$$-A'(t)A(t) \geq \left(\frac{\pi}{2\alpha} f'(\rho(t))\right)^2. \tag{3.35}$$

for almost every t in $(0, m)$. Using the equalities $\frac{\pi}{2\alpha} f(\rho(t)) = A(t)$ and $\rho(t) = w^{\star,-1}(t)$ we get

$$-w^{\star'}(\rho) \leq \frac{f(\rho)}{f'(\rho)}. \tag{3.36}$$

Now, for $\rho \in (0, \rho_0)$, integrating this inequality from ρ to ρ_0 we obtain

$$w^\star(\rho) \leq \int_\rho^{\rho_0} \frac{f(\eta)}{f'(\eta)} d\eta = \tilde{w}(\rho) \tag{3.37}$$

and the proof is complete. Now assume that we have equality integrating this we obtain

$$P_\alpha(D) = \int_D w h^2 da = \int_{S_0} w^\star h^2 da = \int_{S_0} \tilde{w} h^2 da = P_\alpha(S_0). \tag{3.38}$$

Finally applying Theorem 3.3 we conclude that $D = S_0$. □

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