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Almost Quasi (τ_1, τ_2)-Continuity for Multifunctions

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Abstract. This paper is concerned with the concept of almost quasi (τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of almost quasi (τ_1, τ_2)-continuous multifunctions are investigated.

1. Introduction

Stronger and weaker forms of open sets play an important role in topological spaces. By utilizing these sets several authors introduced and investigated various types of generalizations of continuity. In [5], the present authors studied some properties of (Λ, sp)-open sets. Viriyapong and Boonpok [30] investigated some characterizations of (Λ, sp)-continuous functions by using (Λ, sp)-open sets and (Λ, sp)-closed sets. Furthermore, some characterizations of strongly $\theta(\Lambda, p)$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, pairwise almost M-continuous functions and almost $(\mu, \mu')^{(m,n)}$-continuous functions were studied in [28], [4], [9], [13] and [14], respectively. Marcus [19] introduced and investigated the notion of quasi continuous functions. Popa [26] introduced and studied the notion of almost quasi continuous functions. Neubrunnovaá [20] showed that quasi continuity is equivalent to semi-continuity due to Levine [17]. Popa and Stan [27] introduced and investigated the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [18] which are independent of each other. It is shown in [23] that weak quasi continuity is equivalent to weak semi-continuity due to Arya and Bhamini [1] and Kar and Bhattacharyya [15]. Bânzara and Crivăţ [2] introduced and studied the concept of quasi continuous multifunctions. Popa and Noiri [24] introduced

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the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Noiri and Popa [22] introduced and studied the notion of weakly quasi continuous multifunctions. Several characterizations of weakly quasi continuous multifunctions have been obtained in [24]. Popa and Noiri [25] introduced and studied the concepts of upper and lower θ-quasi continuous multifunctions. Moreover, some characterizations of upper and lower θ-quasi continuous multifunctions were presented in [21]. In [10], the author introduced and studied the concepts of almost quasi $\star$-continuous multifunctions and weakly quasi $\star$-continuous multifunctions. Additionally, several characterizations of almost $\star$-continuous multifunctions and almost $\beta(\star)$-continuous multifunctions were investigated in [11] and [7], respectively. Laprom et al. [16] introduced and investigated the concept of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [31] introduced and studied the concept of almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions. In particular, some characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [8] and [6], respectively. In this paper, we introduce the concept of almost quasi (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of almost quasi (τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [12] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [12] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [12] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [12] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.*
(2) *If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.*
(3) *$\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.*
(5) *$\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*r-open* [31] (resp. (τ_1, τ_2)*p-open* [8], $(\tau_1, \tau_2)\beta$*-open* [8], $\alpha(\tau_1, \tau_2)$*-open* [29]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$). A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*s-open* [8] if $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$.

The complement of a $(\tau_1, \tau_2)s$-open set is called $(\tau_1, \tau_2)s$-closed. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the (τ_1, τ_2)*s-closure* [8] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the (τ_1, τ_2)*s-interior* [8] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 2.2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) (τ_1, τ_2)*-sCl*$(A) = \tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A)) \cup A$ [6];
(2) (τ_1, τ_2)*-sInt*$(A) = \tau_1\tau_2$*-Cl*$(\tau_1\tau_2$*-Int*$(A)) \cap A$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [3] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Let $\mathscr{P}(X)$ be the collection of all nonempty subsets of X. For any $\tau_1\tau_2$-open set V of a bitopological space (X, τ_1, τ_2), we denote $V^+ = \{B \in \mathscr{P}(X) \mid B \subseteq V\}$ and

$$V^- = \{B \in \mathscr{P}(X) \mid B \cap V \neq \emptyset\}.$$

3. Almost quasi (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notion of almost quasi (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of almost quasi (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$, $F(G) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in G$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost quasi (τ_1, τ_2)-continuous if F is almost quasi (τ_1, τ_2)-continuous at each point of X.*

Theorem 3.1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is almost quasi (τ_1, τ_2)-continuous at $x \in X$;*
(2) *for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V_1)$ and (σ_1, σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$;*
(3) *$x \in (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2)))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$;*
(4) *$x \in \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V_1)) \cap F^-((\sigma_1, \sigma_2)$-sCl$(V_2))))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.*

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{U}(x)$ the family of all $\tau_1\tau_2$-open sets of X containing x. Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\tau_1\tau_2$-open set

G_H such that $G_H \subseteq H$, $F(G_H) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and (σ_1,σ_2)-sCl$(V_2) \cap F(y) \neq \emptyset$ for each $y \in G_H$. Let $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then W is $\tau_1\tau_2$-open in X, $x \in \tau_1\tau_2$-Cl(W), $F(W) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and

$$(\sigma_1,\sigma_2)\text{-sCl}(V_2) \cap F(w) \neq \emptyset$$

for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq \tau_1\tau_2$-Cl$(W)$. Therefore, we obtain U is a $(\tau_1,\tau_2)s$-open set of X containing x such that $F(U) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and (σ_1,σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$.

(2) ⇒ (3): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y and $F(x) \in V_1^+ \cap V_2^-$. Then, there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and (σ_1,σ_2)-sCl$(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))$ and hence

$$x \in U \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))).$$

(3) ⇒ (4): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y and $F(x) \in V_1^+ \cap V_2^-$. By (3),

$$x \in (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))).$$

Now, put $U = (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))$. Then, we have U is $(\tau_1,\tau_2)s$-open in X and by Lemma 2.2,

$$x \in U \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(U)) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))).$$

(4) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X containing x and V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$.
Then, we have $x \in \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))))$ and hence
$U \cap \tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))) \neq \emptyset$. Put

$$W = U \cap \tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))).$$

Then, we have W is a nonempty $\tau_1\tau_2$-open set of X such that $W \subseteq U$, $F(W) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and (σ_1,σ_2)-sCl$(V_2) \cap F(w) \neq \emptyset$ for every $w \in W$. This shows that F is almost quasi (τ_1,τ_2)-continuous at x. □

Theorem 3.2. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *F is almost quasi (τ_1,τ_2)-continuous;*
(2) *for each $x \in X$ and every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1,\sigma_2)$-sCl$(V_1)$ and*

$$(\sigma_1,\sigma_2)\text{-sCl}(V_2) \cap F(z) \neq \emptyset$$

for every $z \in U$;
(3) *$F^+(V_1) \cap F^-(V_2)$ is $(\tau_1,\tau_2)s$-open in X for every $(\sigma_1,\sigma_2)r$-open sets V_1, V_2 of Y;*
(4) *$F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y;*

(5)

$$(\tau_1,\tau_2)sCl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_2)))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(B_2))$$

for every subsets B_1, B_2 *of* Y*;*

(6) $F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(F^+((\sigma_1,\sigma_2)\text{-}sCl(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-}sCl(V_2))))$ *for every* $\sigma_1\sigma_2$*-open sets* V_1, V_2 *of* Y.

Proof. (1) ⇒ (2): The proof follows from Theorem 3.1.

(2) ⇒ (3): Let V_1, V_2 be any $(\sigma_1,\sigma_2)r$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have $F(x) \in V_1^+ \cap V_2^-$ and there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in (\tau_1,\tau_2)\text{-sInt}(F^+(V_1) \cap F^-(V_2))$. Therefore, $F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+(V_1) \cap F^-(V_2))$. This shows that $F^+(V_1) \cap F^-(V_2)$ is $(\tau_1,\tau_2)s$-open in X.

(3) ⇒ (4): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have

$$F(x) \subseteq V_1 \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V_1)$$

and $\emptyset \neq V_2 \cap F(x) \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V_2) \cap F(x)$. Therefore, $x \in F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1))$ and $x \in F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))$. By Lemma 2.2, $(\sigma_1,\sigma_2)\text{-sCl}(V_1)$ and $(\sigma_1,\sigma_2)\text{-sCl}(V_2)$ are $(\sigma_1,\sigma_2)r$-open sets. By (3),

$$F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))$$

is $(\tau_1,\tau_2)s$-open in X and $x \in (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2)))$. Thus,

$$F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(V_2))).$$

(4) ⇒ (5): Let B_1, B_2 be any subsets of Y. Then, we have $Y - \sigma_1\sigma_2\text{-Cl}(B_1)$ and $Y - \sigma_1\sigma_2\text{-Cl}(B_2)$ are $\sigma_1\sigma_2$-open sets of Y. By (4) and Lemma 2.2,

$$\begin{aligned}
&X - (F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)))\\
&= (X - F^-(\sigma_1\sigma_2\text{-Cl}(B_1))) \cap (X - F^+(\sigma_1\sigma_2\text{-Cl}(B_2)))\\
&= F^+(Y - \sigma_1\sigma_2\text{-Cl}(B_1)) \cap F^-(Y - \sigma_1\sigma_2\text{-Cl}(B_2))\\
&\subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B_1))) \cap F^-((\sigma_1,\sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B_2))))\\
&= (\tau_1,\tau_2)\text{-sInt}((X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_1\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1))))) \cap (X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2))))))\\
&= X - (\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))))
\end{aligned}$$

and hence

$$(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1)))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)).$$

(5) ⇒ (6): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y. Then $Y - V_1$ and $Y - V_2$ are $\sigma_1\sigma_2$-closed sets of Y. By (5) and Lemma 2.2, we have

$$\begin{aligned}
&\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_1))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_2)))))\\
&\subseteq F^-(Y - V_1) \cup F^+(Y - V_2)\\
&= (X - F^+(V_1)) \cup (X - F^-(V_2))\\
&= X - (F^+(V_1) \cap F^-(V_2)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_1))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}((X - F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1))) \cup (X - F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))))\\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(X - (F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))))\\
&= X - \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))).
\end{aligned}$$

Thus,

$$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))).$$

(6) ⇒ (1): Let $x \in X$ and V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By (6), we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2))))$$

and by Lemma 2.2,

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2))).$$

Put $U = (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V_1)) \cap F^-((\sigma_1, \sigma_2)\text{-sCl}(V_2)))$. Then, U is a $(\tau_1, \tau_2)s$-open set of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V_1)$ and $(\sigma_1, \sigma_2)\text{-Cl}(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. This shows that F is almost quasi (τ_1, τ_2)-continuous. □

Theorem 3.3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is almost quasi (τ_1, τ_2)-continuous;*
(2) *$(\tau_1, \tau_2)\text{-}sCl(F^-(V_1) \cap F^+(V_2)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$ for every $(\sigma_1, \sigma_2)\beta$-open sets V_1, V_2 of Y;*
(3) *$(\tau_1, \tau_2)\text{-}sCl(F^-(V_1) \cap F^+(V_2)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$ for every $(\sigma_1, \sigma_2)s$-open sets V_1, V_2 of Y;*
(4) *$F^+(V_1) \cap F^-(V_2) \subseteq (\tau_1, \tau_2)\text{-}sInt(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cap F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$ for every $(\sigma_1, \sigma_2)p$-open sets V_1, V_2 of Y.*

Proof. The proof follows from Theorem 3.2. □

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