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Certain Applications via Rational Type Contraction Fixed Point Theorems in A_b-Metric Spaces

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Abstract. We showed the existence of common fixed point theorems for four mappings involving rational type contractive conditions in A_b-metric spaces by extending and generalising previous work. Furthermore, we provide an instance demonstrating the applicability of the obtained results, as well as applications to integral equations and homotopy.

1. Introduction

The traditional notion of Banach contraction [1] has been crucial in obtaining a distinct resolution to approximation theory and fixed point theory outcomes. It is without a doubt a crucial and well-liked method for resolving modern nonlinear analysis problems in various mathematical domains. Since then, numerous generalizations of the Banach contraction principle in metric fixed point theory have been achieved through improvements to the underlying contraction condition. Then, rigorous research work was obtained and is soon to be used to support previously published findings (see. [2]- [7]) by weakening its hypotheses on a wide range of spaces, including pseudo-metric spaces, rectangular metric spaces, fuzzy metric spaces, quasi-semi-metric spaces, quasi-metric spaces, probabilistic metric spaces, F-metric spaces, cone metric spaces D-metric spaces, and G-metric spaces.

For certain rational type contractive conditions, Dass and Gupta [2] extended the Banach contraction principle in a metric space in 1975. In 1989, I.A. Bakhtin [8] has introduced b-metric space. The introduction of b-metric space led to the development of numerous metric space generalizations. In 2015, M.Abbas et al. [9] introduced and studied the topological characteristics of the n-tuple metric space. A_b-metric spaces are a generalized version of n-tuple metric spaces, as first

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proposed by M. Ughade et al. [10]. Then, in partially ordered A_b-metric spaces, N.Mlaiki et al. [11] and K.Ravibabu et al. ([12], [13]) obtained unique coupled common fixed point theorems. Afterwards, P. Naresh et al. [14] and N. Mangapathi et al. [15] used compatible and weakly compatible mappings to develop the first linked common fixed point theorems.

By expanding and generalizing several findings from the literature, the current work aims to provide common fixed point theorems for four mappings involving contractive conditions of Rational type in A_b-metric spaces. Furthermore, we could offer pertinent examples, integral equations, and homotopy applications.

2. Preliminaries

Definition 2.1. *([10]) Let $\Im$ be a non-empty set, and $\vartheta \geq 1$ be a real number. An A_b-metric on $\Im$ is defined as a mapping $A_b : \Im^n \to [0, \infty)$ that meets the following constraints for every $æ_z, ß \in \Im$ $z = 1, 2, 3, ..n$.*

(A_b1) $A_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) \geq 0,$

(A_b2) $A_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) = 0 \Leftrightarrow æ_1 = æ_2 = \cdots\cdots = æ_{n-1} = æ_n,$

$$(A_b3)\ \ A_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) \leq \vartheta \begin{pmatrix} A_b\left(æ_1, æ_1, ..., (æ_1)_{n-1}, ß\right) \\ +A_b\left(æ_2, æ_2, ..., (æ_2)_{n-1}, ß\right) \\ +\cdots\cdots + A_b\left(æ_{n-1}, æ_{n-1}, ..., (æ_{n-1})_{n-1}, ß\right) \\ +A_b\left(æ_n, æ_n, ..., (æ_n)_{n-1}, ß\right) \end{pmatrix}$$

Then the pair $(\Im, A_b)$ is called an A_b-metric space.

Remark 2.2. *([10]) It is worth noting that the class of A_b-metric spaces is significantly bigger than that of A-metric spaces. Each A-metric space is a A_b-metric space, where $\vartheta = 1$. However, the opposite is not always true. In addition, A_b-metric space is a n-dimensional S_b-metric space. Therefore, the S_b-metric are special examples of a A_b-metric with $n = 3$.*

The following example demonstrates that a A_b-metric on $\Im$ does not necessarily imply a A-metric on $\Im$.

Example 2.3. *([10]) Let $\Im = [0, +\infty)$, define $A_b : \Im^n \to [0, +\infty)$ as $A_b\left(æ_1, æ_2, ..., æ_{n-1}, æ_n\right) = \sum_{z=1}^{n} \sum_{z<\iota} |æ_z - æ_\iota|^2\ \forall\ æ_z \in \Im, z = 1, 2, \cdots, n$. Then $(\Im, A_b)$ is a A_b-metric space with $\vartheta = 2 > 1$.*

Definition 2.4. *([10]) A metric space $(\Im, A_b)$ is said to be symmetric if $A_b\left(æ, æ, \cdots, (æ)_{n-1}, œ\right) = A_b\left(œ, œ, \cdots, (œ)_{n-1}, æ\right)\ \forall\ æ, œ \in \Im$.*

Definition 2.5. *([10]) Let $(\Im, A_b)$ denote a A_b-metric space. Then, for $æ \in \Im$, $\delta > 0$, we defined the open ball $B_{A_b}(æ, \delta)$ and closed ball $B_{A_b}[æ, \delta]$ with centre æ and radius δ as follows:*

$$B_{A_b}(æ, \delta) = \{œ \in \Im : A_b(œ, œ, \cdots, (œ)_{n-1}, æ) < \delta\},$$

and

$$B_{A_b}[æ, \delta] = \{œ \in \Im : A_b(œ, œ, \cdots, (œ)_{n-1}, æ) \leq \delta\}.$$

Lemma 2.6. *([10]) In a A_b-metric space, we have*

(1) $A_b(æ, æ, \cdots, (æ)_{n-1}, œ) \leq \vartheta A_b(œ, œ, \cdots, (œ)_{n-1}, æ)$;
(2) $A_b(æ, æ, \cdots, (æ)_{n-1}, ß) \leq \vartheta(n-1)A_b(æ, æ, \cdots, (æ)_{n-1}, œ) + \vartheta^2 A_b(œ, œ, \cdots, (œ)_{n-1}, ß)$.

Definition 2.7. *([10]) Let $(\Im, A_b)$ be a metric space over A_b. A $\{æ_z\}$ sequence in $\Im$ is defined as follows:*

(1) *If $n_0 \in \mathbb{N}$ exists such that $A_b(æ_z, æ_z, \cdots\cdots (æ_z)_{n-1}, æ_\iota) < \epsilon$ for each $\iota, z \geq n_0$, then $\{æ_z\}$ is A_b-Cauchy sequence.*
(2) *We denote $\lim_{z\to\infty} æ_z = æ$. A_b-convergent to a point $æ \in \Im$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $A_b(æ_z, æ_z, \cdots\cdots (æ_z)_{n-1}, æ) < \epsilon$ for all $z \geq n_0$.*
(3) *An metric space A_b $(\Im, A_b)$ if all A_b-Cauchy sequences in $\Im$ are A_b-convergent, then A_b is said to be complete.*

Lemma 2.8. *([10]) Assuming that $\{æ_k\}$ is a A_b-convergent to æ and $\{œ_k\}$ is a A_b-convergent to œ, and $(\Im, A_b)$ is a A_b-metric space with $\vartheta \geq 1$, we have*

(i)

$$\begin{aligned} \frac{1}{\vartheta^2}A_b(æ, æ, \cdots, (æ)_{n-1}, œ) &\leq \lim_{z\to\infty}\inf\ A_b(æ_z, æ_z, \cdots, (æ_z)_{n-1}, œ_z) \\ &\leq \lim_{z\to\infty}\sup A_b(æ_z, æ_z, \cdots, (æ_z)_{n-1}, œ_z) \\ &\leq \vartheta^2 A_b(æ, æ, \cdots, (æ)_{n-1}, œ) \end{aligned}$$

in particular, if $œ_z = œ$ is constant, then

(ii)

$$\begin{aligned} \frac{1}{\vartheta^2}A_b(æ, æ, \cdots, (æ)_{n-1}, œ) &\leq \lim_{z\to\infty}\inf\ A_b(æ_z, æ_z, \cdots, (æ_z)_{n-1}, œ) \\ &\leq \lim_{z\to\infty}\sup A_b(æ_z, æ_z, \cdots, (æ_z)_{n-1}, œ) \\ &\leq \vartheta^2 A_b(æ, æ, \cdots, (æ)_{n-1}, œ) \end{aligned}$$

Definition 2.9. *[3] If $\Gamma : [0, \infty) \to [0, \infty)$ satisfies the following conditions, it is referred to as a comparison function.*

(a) *Γ is monotonically increasing;*
(b) *for all $s \in [0, \infty)$, the sequence $\{\Gamma^n(s)\}_{n=1}^{\infty}$ converges to zero, where Γ^n denotes the n^{th} iterate of Γ. If Γ satisfie (a) and*
(c) *the sequence $\sum_{i=1}^{n} \Gamma^i(s)$ converges for all $s \in [0, \infty)$, then is referred to as the (c)-comparison function.*

Every comparison function is (c) comparison.

The comparison function prototype example is $\Gamma(s) = \delta s$, where $s \in [0, \infty)$ and $\delta \in [0, 1)$.

Lemma 2.10. *[3] Assume that $\Gamma : [0, \infty) \to [0, \infty)$ be a comparison function, then $\Gamma(s) < s$ for all $s > 0$, and $\Gamma(s) = 0 \iff s = 0$.*

The following factors must be taken into account in order to get our results.

3. Main Results

In this segment, we examine the presence and singularity of a common fixed point for four self-mappings and demonstrate it in A_b-metric spaces that incorporate rational type contractive contraction.

Let $F(\mathfrak{T},\mathfrak{f}) = \{æ \in \mathfrak{I} : \mathfrak{T}æ = \mathfrak{f}æ = æ\}$ be the set of common fixed points and $C(\mathfrak{T},\mathfrak{f}) = \{æ \in \mathfrak{I} : \mathfrak{T}æ = \mathfrak{f}æ = æ\}$ be the set of coincidence points.Then, the following definitions must be included in the sequel.

Definition 3.1. *Let $(\mathfrak{I}, A_b)$ be a A_b-metric space. If $\mathfrak{T}\mathfrak{f}æ = \mathfrak{f}\mathfrak{T}æ\ \forall\ æ \in \mathfrak{I}$, then two self-maps $\mathfrak{T}$ and $\mathfrak{f}$ on a nonempty set $\mathfrak{I}$ are said to commute each other.*

Definition 3.2. *In $(\mathfrak{I}, A_b)$, two self-maps $\mathfrak{T}$ and $\mathfrak{f}$ on a nonempty set $\mathfrak{I}$ are considered ω-compatible if they commute at their coincidence points.*
i.e If $æ \in \mathfrak{I}$, $\mathfrak{T}\mathfrak{f}æ = \mathfrak{f}\mathfrak{T}æ$ whenever $\mathfrak{T}æ = \mathfrak{f}æ$.

Theorem 3.3. *Consider $(\mathfrak{I}, A_b)$ as a complete A_b-metric space with four mappings $\mathfrak{T}, \mathfrak{S}, \mathfrak{f}, \mathfrak{g} : \mathfrak{I} \to \mathfrak{I}$ satisfy the following conditions:*

$$\begin{aligned} &A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \mathfrak{S}œ\right) \\ &\quad \leq \alpha\Gamma\left(A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}œ\right)\right) \\ &\quad\quad + \beta\Gamma\left(\max\left\{\begin{array}{l} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}œ\right), \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}\right\}\right) \\ &+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}œ\right)\left[1 + \sqrt{\begin{array}{l} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}œ\right) \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}}\right]^2}{\left(1 + A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}œ\right)\right)^2}\right) \end{aligned} \tag{3.1}$$

for all $æ, œ \in \mathfrak{I}$, $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ and Γ is a comparison function.
a) $\mathfrak{T}(\mathfrak{I}) \subseteq \mathfrak{f}(\mathfrak{I})$ and $\mathfrak{S}(\mathfrak{I}) \subseteq \mathfrak{g}(\mathfrak{I})$;
b) Either $(\mathfrak{T}, \mathfrak{g})$ or $(\mathfrak{S}, \mathfrak{f})$ are ω-compatible;
c) one of $\mathfrak{f}(\mathfrak{I})$, $\mathfrak{g}(\mathfrak{I})$ is closed subset of $\mathfrak{I}$;
Then, $\mathfrak{T}$, $\mathfrak{S}$, $\mathfrak{f}$, and $\mathfrak{g}$ have a unique common fixed point in $\mathfrak{I}$.

Proof. Let $\imath_0 \in \mathfrak{I}$ and from (a) we construct the sequences $\{\imath_{2p}\}$ and $\{\varkappa_{2p}\}$ in $\mathfrak{I}$ as

$$\mathfrak{T}\imath_{2p} = \mathfrak{f}\imath_{2p+1} = \varkappa_{2p} \quad \mathfrak{S}\imath_{2p+1} = \mathfrak{g}\imath_{2p+2} = \varkappa_{2p+1} \text{ for } p = 0, 1, 2, \cdots .$$

Then from (3.1), we can get

$$A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \varkappa_{2p+1}\right) = A_b\left(\mathfrak{T}\imath_{2p}, \mathfrak{T}\imath_{2p}, \cdots, (\mathfrak{T}\imath_{2p})_{n-1}, \mathfrak{S}\imath_{2p+1}\right)$$

$$
\begin{aligned}
\leq\ & \alpha\Gamma\left(A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{f}\imath_{2p+1}\right)\right)\\
&\qquad +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{f}\imath_{2p+1}\right),\\ A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{T}\imath_{2p}\right)\end{array}\right\}\right)\\
&+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{f}\imath_{2p+1}\right)\left[1+\sqrt{\begin{array}{c}A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{f}\imath_{2p+1}\right)\\ A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{T}\imath_{2p}\right)\end{array}}\right]^2}{\left(1+A_b\left(\mathfrak{g}\imath_{2p},\mathfrak{g}\imath_{2p},\cdots,(\mathfrak{g}\imath_{2p})_{n-1},\mathfrak{f}\imath_{2p+1}\right)\right)^2}\right)\\
\leq\ & \alpha\Gamma\left(A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\right)\\
&\qquad +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right),\\ A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\end{array}\right\}\right)\\
&+\gamma\Gamma\left(\frac{A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,\varkappa_{2p}\right)\left[1+\sqrt{\begin{array}{c}A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,\varkappa_{2p}\right)\\ A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,\varkappa_{2p}\right)\end{array}}\right]^2}{\left(1+A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,\varkappa_{2p}\right)\right)^2}\right)\\
\leq\ & \alpha\Gamma\left(A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\right)+\beta\Gamma\left(A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\right)\\
&+\gamma\Gamma\left(A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\right).
\end{aligned}
$$

Since $\Gamma(s)\leq s$ for all $s\geq 0$, then we obtain

$$A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right)\leq(\alpha+\beta+\gamma)A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right).$$

Put $\kappa=\alpha+\beta+\gamma$, then we have $0\leq\kappa<1$. So that

$$
\begin{aligned}
A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) &\leq \kappa A_b\left(\varkappa_{2p-1},\varkappa_{2p-1},\cdots,(\varkappa_{2p-1})_{n-1},\varkappa_{2p}\right)\\
&\leq \kappa^2 A_b\left(\varkappa_{2p-2},\varkappa_{2p-2},\cdots,(\varkappa_{2p-2})_{n-1},\varkappa_{2p-1}\right)\\
&\ \vdots\\
&\leq \kappa^{2p}A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right)\to 0 \text{ as } p\to\infty.
\end{aligned}
$$

Thus

$$\lim_{p\to\infty}A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right)=0.$$

Likewise, we can demonstrate that

$$\lim_{p\to\infty}A_b\left(\varkappa_{2p+1},\varkappa_{2p+1},\cdots,(\varkappa_{2p+1})_{n-1},\varkappa_{2p}\right)=0.$$

Now we can show $\{\varkappa_{2n}\}$ is a Cauchy sequence in $\Im$. Now, using (A_b3), we have for $q > p$

$$A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2q}\right) \leq \vartheta\begin{pmatrix} A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\ +A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\ +\cdots+A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\ +A_b\left(\varkappa_{2q},\varkappa_{2q},........,(\varkappa_{2q})_{n-1},\varkappa_{2p+1}\right)\end{pmatrix}$$

$$\begin{aligned}
\leq\ & \vartheta(n-1)A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\
& +\vartheta A_b\left(\varkappa_{2q},\varkappa_{2q},........,(\varkappa_{2q})_{n-1},\varkappa_{2p+1}\right) \\
\leq\ & \vartheta(n-1)A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\
& +\vartheta^2 A_b\left(\varkappa_{2p+1},\varkappa_{2p+1},........,(\varkappa_{2p+1})_{n-1},\varkappa_{2q}\right) \\
\leq\ & \vartheta(n-1)A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\
& +\vartheta^3(n-1)A_b\left(\varkappa_{2p+1},\varkappa_{2p+1},........,(\varkappa_{2p+1})_{n-1},\varkappa_{2p+2}\right) \\
& +\vartheta^4 A_b\left(\varkappa_{2p+2},\varkappa_{2p+2},........,(\varkappa_{2p+2})_{n-1},\varkappa_{2q}\right) \\
\leq\ & \vartheta(n-1)A_b\left(\varkappa_{2p},\varkappa_{2p},........,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right) \\
& +\vartheta^3(n-1)A_b\left(\varkappa_{2p+1},\varkappa_{2p+1},........,(\varkappa_{2p+1})_{n-1},\varkappa_{2p+2}\right) \\
& +\vartheta^5(n-1)A_b\left(\varkappa_{2p+2},\varkappa_{2p+2},........,(\varkappa_{2p+2})_{n-1},\varkappa_{2p+3}\right) \\
& +\vartheta^7(n-1)A_b\left(\varkappa_{2p+3},\varkappa_{2p+3},........,(\varkappa_{2p+3})_{n-1},\varkappa_{2p+4}\right) \\
& +\ldots+\vartheta^{2q-2p-3}(n-1)A_b\left(\varkappa_{2q-2},\varkappa_{2q-2},........,(\varkappa_{2q-2})_{n-1},\varkappa_{2q-1}\right) \\
& +\vartheta^{2q-2p-2}A_b\left(\varkappa_{2q-1},\varkappa_{2q-1},........,(\varkappa_{2q-1})_{n-1},\varkappa_{2q}\right) \\
\leq\ & (n-1)\left(\vartheta\kappa^{2p}+\vartheta^3\kappa^{2p+1}+\vartheta^5\kappa^{2p+2}+\ldots+\vartheta^{2q-2p-3}\kappa^{2q-2}\right)A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right) \\
& +\vartheta^{2q-2p-3}\kappa^{2q-1}A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right) \\
\leq\ & (n-1)\vartheta\kappa^{2p}\left(1+b^2\kappa+b^4\kappa^2+\ldots+b^{2q-2p-4}\kappa^{2q-2p-2}\right)A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right) \\
& +\vartheta^{2q-2p-2}\kappa^{2q-1}A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right) \\
\leq\ & (n-1)\vartheta\kappa^{2p}\left(1+\vartheta^2\kappa+\vartheta^4\kappa^2+\vartheta^6\kappa^3+\ldots\right)A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right) \\
\leq\ & \frac{(n-1)\vartheta\kappa^{2p}}{1-\vartheta^2\kappa}A_b\left(\varkappa_0,\varkappa_0,\cdots,(\varkappa_0)_{n-1},\varkappa_1\right)\to 0 \text{ as } p,q\to\infty.
\end{aligned}$$

Therefore The Cauchy sequence $\left\{\varkappa_{2p}\right\}$ is in $\Im$. Assuming that $\mathfrak{g}(\Im)$ is a complete subspace of $(\Im, A_b)$, Δ is the convergence point of the sequence $\left\{\varkappa_{2p}\right\}$ in $\mathfrak{g}(\Im)$. As a result, $\lim_{p\to\infty}\varkappa_{2p}=\Delta=\mathfrak{g}æ$ exists for every $æ\in\mathfrak{g}(\Im)$.

Now we show that $\mathfrak{T}æ=\Delta$. By using (3.1), we have

$$\begin{aligned}
& A_b\left(\mathfrak{T}æ,\mathfrak{T}æ\cdots(\mathfrak{T}æ)_{n-1},\varkappa_{2p+1}\right)=A_b\left(\mathfrak{T}æ,\mathfrak{T}æ\cdots(\mathfrak{T}æ)_{n-1},\mathfrak{S}\imath_{2p+1}\right) \\
\leq\ & \alpha\Gamma\left(A_b\left(\mathfrak{g}æ,\mathfrak{g}æ,\cdots,(\mathfrak{g}æ)_{n-1},\mathfrak{f}\imath_{2p+1}\right)\right)
\end{aligned}$$

$$+\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}\iota_{2p+1}\right), \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}\right\}\right)$$

$$+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}\iota_{2p+1}\right)\left[1+\sqrt{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}\iota_{2p+1}\right) \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}}\right]^2}{\left(1+A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{f}\iota_{2p+1}\right)\right)^2}\right)$$

$$\leq \alpha\Gamma\left(A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \varkappa_{2p}\right)\right)$$

$$+\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \varkappa_{2p}\right), \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}\right\}\right)$$

$$+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \varkappa_{2p}\right)\left[1+\sqrt{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \varkappa_{2p}\right) \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}}\right]^2}{\left(1+A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \varkappa_{2p}\right)\right)^2}\right).$$

If we allow $p \to \infty$ in the inequality above, we obtain

$$A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \Delta\right) \leq \alpha\Gamma\left(A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \Delta\right)\right)$$

$$+\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \Delta\right), \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}\right\}\right)$$

$$+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \Delta\right)\left[1+\sqrt{\begin{array}{c} A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \Delta\right) \\ A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}}\right]^2}{\left(1+A_b\left(\mathfrak{g}æ, \mathfrak{g}æ, \cdots, (\mathfrak{g}æ)_{n-1}, \Delta\right)\right)^2}\right).$$

Since $\mathfrak{T}æ = \mathfrak{g}æ$ and $\Gamma(s) \leq s$ for all $s \geq 0$, then

$$A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \Delta\right) \leq (\alpha+\beta+\gamma) A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \Delta\right).$$

Since $0 \leq \kappa = \alpha+\beta+\gamma < 1$, the above inequality is possible if
$A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \Delta\right) = 0$. implies that $\mathfrak{T}æ = \Delta$. Thus, $\mathfrak{T}æ = \Delta = \mathfrak{g}æ$. By the ω-compatibility of $\mathfrak{T}$ and $\mathfrak{g}$, we have $\mathfrak{T}\mathfrak{g}æ = \mathfrak{g}\mathfrak{T}æ$. Then $\mathfrak{T}\Delta = \mathfrak{T}\mathfrak{g}æ = \mathfrak{g}\mathfrak{T}æ = \mathfrak{g}\Delta$. Which implies that Δ is a coincidence point of $\mathfrak{T}$ and $\mathfrak{g}$. Now we prove that $\mathfrak{T}\Delta = \Delta$, then from (3.1), we have

$$A_b\left(\mathfrak{T}\Delta, \mathfrak{T}\Delta \cdots (\mathfrak{T}\Delta)_{n-1}, \varkappa_{2p+1}\right) = A_b\left(\mathfrak{T}\Delta, \mathfrak{T}\Delta \cdots (\mathfrak{T}\Delta)_{n-1}, \mathfrak{S}\iota_{2p+1}\right)$$

$$\leq \alpha\Gamma\left(A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \varkappa_{2p}\right)\right)$$

$$+\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \varkappa_{2p}\right), \\ A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \mathfrak{T}\Delta\right) \end{array}\right\}\right)$$

$$+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,(\mathfrak{g}\Delta)_{n-1},\varkappa_{2p}\right)\left[1+\sqrt{\frac{A_b\left(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,(\mathfrak{g}\Delta)_{n-1},\varkappa_{2p}\right)}{A_b\left(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,(\mathfrak{g}\Delta)_{n-1},\mathfrak{T}\Delta\right)}}\right]^2}{\left(1+A_b\left(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,(\mathfrak{g}\Delta)_{n-1},\varkappa_{2p}\right)\right)^2}\right).$$

By allowing $p\to\infty$ in the previous inequality and applying the property $\Gamma(s)\le s$ for all $s\ge 0$, we obtain

$$A_b\left(\mathfrak{T}\Delta,\mathfrak{T}\Delta\cdots(\mathfrak{T}\Delta)_{n-1},\Delta\right)\le(\alpha+\beta+\gamma)A_b\left(\mathfrak{T}\Delta,\mathfrak{T}\Delta\cdots(\mathfrak{T}\Delta)_{n-1},\Delta\right)$$

Since $0\le\kappa=\alpha+\beta+\gamma<1$,so which is possible if $A_b\left(\mathfrak{T}\Delta,\mathfrak{T}\Delta\cdots(\mathfrak{T}\Delta)_{n-1},\Delta\right)=0$ implies $\mathfrak{T}\Delta=\Delta$. Hence $\mathfrak{T}\Delta=\Delta=\mathfrak{g}\Delta$. Therefore, Δ is a common fixed point of $\mathfrak{T}$ and $\mathfrak{g}$. Since $\mathfrak{T}(\mathfrak{I})\subseteq\mathfrak{f}(\mathfrak{I})$ so there exist $\Upsilon\in\mathfrak{I}$ such that $\mathfrak{T}\Delta=\Delta=\mathfrak{f}\Upsilon$. Then from (3.1), we have

$$\begin{aligned}
&A_b\left(\varkappa_{2p+1},\varkappa_{2p+1}\cdots(\varkappa_{2p+1})_{n-1},\mathfrak{S}\Upsilon\right)=A_b\left(\mathfrak{T}\imath_{2p+1},\mathfrak{T}\imath_{2p+1}\cdots(\mathfrak{T}\imath_{2p+1})_{n-1},\mathfrak{S}\Upsilon\right)\\
\le\ &\alpha\Gamma\left(A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,(\mathfrak{g}\imath_{2p+1})_{n-1},\mathfrak{f}\Upsilon\right)\right)\\
&+\beta\Gamma\left(\max\left\{\begin{array}{c}A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,(\mathfrak{g}\imath_{2p+1})_{n-1},\mathfrak{f}\Upsilon\right),\\ A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,(\mathfrak{g}\imath_{2p+1})_{n-1},\mathfrak{T}\imath_{2p+1}\right)\end{array}\right\}\right)\\
&+\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,\mathfrak{f}\Upsilon\right)\left[1+\sqrt{\frac{A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,\mathfrak{f}\Upsilon\right)}{A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,\mathfrak{T}\imath_{2p+1}\right)}}\right]^2}{\left(1+A_b\left(\mathfrak{g}\imath_{2p+1},\mathfrak{g}\imath_{2p+1},\cdots,(\mathfrak{g}\imath_{2p+1})_{n-1},\mathfrak{f}\Upsilon\right)\right)^2}\right)\\
\le\ &\alpha\Gamma\left(A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,\mathfrak{f}\Upsilon\right)\right)\\
&+\beta\Gamma\left(\max\left\{\begin{array}{c}A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\mathfrak{f}\Upsilon\right),\\ A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right)\end{array}\right\}\right)\\
&+\gamma\Gamma\left(\frac{A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\mathfrak{f}\Upsilon\right)\left[1+\sqrt{\frac{A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\mathfrak{f}\Upsilon\right)}{A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\varkappa_{2p+1}\right)}}\right]^2}{\left(1+A_b\left(\varkappa_{2p},\varkappa_{2p},\cdots,(\varkappa_{2p})_{n-1},\mathfrak{f}\Upsilon\right)\right)^2}\right).
\end{aligned}$$

Taking $p\to\infty$ in the preceding inequality, we find that

$$0\le A_b\left(\Delta,\Delta\cdots,(\Delta)_{n-1},\mathfrak{S}\Upsilon\right)\le(\alpha+\beta+\gamma)\Gamma\left(A_b\left(\Delta,\Delta\cdots,(\Delta)_{n-1},\mathfrak{f}\Upsilon\right)\right)$$

Since $\kappa=\alpha+\beta+\gamma<1$ and using the property $\Gamma(s)=0$, iff $s=0$, which implies $\mathfrak{S}\Upsilon=\Delta$. Since $\{\mathfrak{S},\mathfrak{f}\}$ weakly compatible pair, we have $\mathfrak{S}\mathfrak{f}\Upsilon=\mathfrak{f}\mathfrak{S}\Upsilon$. Then $\mathfrak{S}\Delta=\mathfrak{S}\mathfrak{f}\Upsilon=\mathfrak{f}\mathfrak{S}\Upsilon=\mathfrak{f}\Delta$. Which implies that Δ is a coincidence point of $\mathfrak{S}$ and $\mathfrak{f}$. Now we prove that $\mathfrak{S}\Delta = \Delta$, then from (3.1), we have

$$\begin{aligned}
& A_b\left(\varkappa_{2p+1}, \varkappa_{2p+1} \cdots (\varkappa_{2p+1})_{n-1}, \mathfrak{S}a\right) = A_b\left(\mathfrak{T}\imath_{2p+1}, \mathfrak{T}\imath_{2p+1} \cdots (\mathfrak{T}\imath_{2p+1})_{n-1}, \mathfrak{S}a\right) \\
\leq\ & \alpha\Gamma\left(A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, (\mathfrak{g}\imath_{2p+1})_{n-1}, \mathfrak{f}a\right)\right) \\
& +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, (\mathfrak{g}\imath_{2p+1})_{n-1}, \mathfrak{f}a\right), \\ A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, (\mathfrak{g}\imath_{2p+1})_{n-1}, \mathfrak{T}\imath_{2p+1}\right) \end{array}\right\}\right) \\
& +\gamma\Gamma\left(\frac{A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, \mathfrak{f}a\right)\left[1 + \sqrt{\begin{array}{c} A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, \mathfrak{f}a\right) \\ A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, \mathfrak{T}\imath_{2p+1}\right) \end{array}}\right]^2}{\left(1 + A_b\left(\mathfrak{g}\imath_{2p+1}, \mathfrak{g}\imath_{2p+1}, \cdots, (\mathfrak{g}\imath_{2p+1})_{n-1}, \mathfrak{f}a\right)\right)^2}\right) \\
\leq\ & \alpha\Gamma\left(A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, \mathfrak{f}a\right)\right) \\
& +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \mathfrak{f}a\right), \\ A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \varkappa_{2p+1}\right) \end{array}\right\}\right) \\
& +\gamma\Gamma\left(\frac{A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \mathfrak{f}a\right)\left[1 + \sqrt{\begin{array}{c} A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \mathfrak{f}a\right) \\ A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \varkappa_{2p+1}\right) \end{array}}\right]^2}{\left(1 + A_b\left(\varkappa_{2p}, \varkappa_{2p}, \cdots, (\varkappa_{2p})_{n-1}, \mathfrak{f}\Upsilon\right)\right)^2}\right).
\end{aligned}$$

Letting $p \to \infty$ in the above inequality and using the property $\Gamma(s) \leq s$, for all $s \geq 0$, we get

$$0 \leq A_b\left(\Delta, \Delta \cdots (\Delta)_{n-1}, \mathfrak{S}\Delta\right) \leq (\alpha + \beta + \gamma) A_b\left(\Delta, \Delta \cdots (\Delta)_{n-1}, \mathfrak{S}\Delta\right).$$

Since $0 \leq \kappa = \alpha + \beta + \gamma < 1$, which is possible if $A_b\left(\Delta, \Delta, \cdots, (\Delta)_{n-1}, \mathfrak{S}\Delta\right) = 0$ implies that $\mathfrak{S}\Delta = \Delta$. Hence $\mathfrak{S}\Delta = \Delta = \mathfrak{f}\Delta$. Thus, Δ is a common fixed point between $\mathfrak{S}$ and $\mathfrak{f}$. As a result, Δ is the common fixed point of $\mathfrak{T}, \mathfrak{S}, \mathfrak{f}$, and $\mathfrak{g}$ in $\mathfrak{I}$.

The following will demonstrate the uniqueness of the common fixed point in $\mathfrak{I}$. Consider another fixed point $a^\star$ of $\mathfrak{T}, \mathfrak{S}, \mathfrak{f}$, and $\mathfrak{g}$ in $\mathfrak{I}$. Based on (3.1), we have $\mathfrak{g}$ in $\mathfrak{I}$.

In the following we will show the uniqueness of common fixed point in $\mathfrak{I}$. For this purpose, assume that there is another fixed point $\Delta^\star$ of $\mathfrak{T}, \mathfrak{S}, \mathfrak{f}$ and $\mathfrak{g}$ in $\mathfrak{I}$. Then from (3.1), we have

$$\begin{aligned}
& A_b\left(\Delta, \Delta \cdots (\Delta)_{n-1}, \Delta^\star\right) = A_b\left(\mathfrak{T}\Delta, \mathfrak{T}\Delta \cdots (\mathfrak{T}\Delta)_{n-1}, \mathfrak{S}\Delta^\star\right) \\
\leq\ & \alpha\Gamma\left(A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \mathfrak{f}\Delta^\star\right)\right) \\
& +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \mathfrak{f}\Delta^\star\right), \\ A_b\left(\mathfrak{g}\Delta, \mathfrak{g}\Delta, \cdots, (\mathfrak{g}\Delta)_{n-1}, \mathfrak{T}\Delta\right) \end{array}\right\}\right)
\end{aligned}$$

$$+\gamma\Gamma\left(\frac{A_b(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,\mathfrak{f}\Delta^\star)\left[1+\sqrt{\frac{A_b(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,\mathfrak{f}\Delta^\star)}{A_b(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,\mathfrak{T}\Delta)}}\right]^2}{(1+A_b(\mathfrak{g}\Delta,\mathfrak{g}\Delta,\cdots,(\mathfrak{g}\Delta)_{n-1},\mathfrak{f}\Delta^\star))^2}\right)$$

$$\leq (\alpha+\beta+\gamma)\Gamma\left(A_b\left(\Delta,\Delta\cdots(\Delta)_{n-1},\Delta^\star\right)\right).$$

Since $0 \leq \kappa = \alpha+\beta+\gamma < 1$ and using the property $\Gamma(s) \leq s$, for all $s \geq 0$, so which is possible if $A_b(\Delta,\Delta,\cdots,(\Delta)_{n-1},\Delta^\star) = 0$ implies $\Delta = \Delta^\star$. Thus Δ is UCFP of $\mathfrak{T},\mathfrak{S},\mathfrak{f}$ and $\mathfrak{g}$ in $\mathfrak{I}$. □

Corollary 3.4. *Let $(\mathfrak{I},A_b)$ be a complete A_b-metric space with two mappings $\mathfrak{T},\mathfrak{f}:\mathfrak{I}\to\mathfrak{I}$ that meet the following conditions:*

$$A_b(\mathfrak{T}æ,\mathfrak{T}æ\cdots(\mathfrak{T}æ)_{n-1},\mathfrak{T}œ) \leq \alpha\Gamma\left(A_b(\mathfrak{f}x,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{f}œ)\right)$$
$$+\beta\Gamma\left(\max\left\{\begin{array}{l}A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{f}œ),\\ A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{T}æ)\end{array}\right\}\right)$$
$$+\gamma\Gamma\left(\frac{A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{f}œ)\left[1+\sqrt{\frac{A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{f}œ)}{A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{T}æ)}}\right]^2}{(1+A_b(\mathfrak{f}æ,\mathfrak{f}æ,\cdots,(\mathfrak{f}æ)_{n-1},\mathfrak{f}œ))^2}\right)$$

for all $æ,œ \in \mathfrak{I}$, $\alpha,\beta,\gamma \geq 0$ with $\alpha+\beta+\gamma<1$ and Γ is a comparison function.
a) $\mathfrak{T}(\mathfrak{I}) \subseteq \mathfrak{f}(\mathfrak{I})$;
b) The pair $(\mathfrak{T},\mathfrak{f})$ has ω-compatibility, and $\mathfrak{f}(\mathfrak{I})$ is a closed subset of $\mathfrak{I}$.
Then, $\mathfrak{T}$ and $\mathfrak{f}$ have a UCFP in $\mathfrak{I}$.

Corollary 3.5. *Let $(\mathfrak{I},A_b)$ be a complete A_b-metric space. The self-mapping $\mathfrak{T}:\mathfrak{I}\to\mathfrak{I}$ is such that*

$$A_b(\mathfrak{T}æ,\mathfrak{T}æ\cdots(\mathfrak{T}æ)_{n-1},\mathfrak{T}œ) \leq \alpha\Gamma\left(A_b(æ,æ,\cdots,(æ)_{n-1},œ)\right)$$

for all $æ,œ \in \mathfrak{I}$, $\alpha \in (0,1)$ and Γ is a comparison function.Then there is a UFP of $\mathfrak{T}$ in $\mathfrak{I}$.

Example 3.6. *Let $\mathfrak{I} = [0,4]$, define $A_b : \mathfrak{I}^n \to [0,+\infty)$ as $A_b(æ_1,æ_2,........,æ_{n-1},æ_n) = |æ_n - \max\{æ_1,æ_2,æ_3,\cdots,æ_{n-1}\}|^2\ \forall\ æ_i \in \mathfrak{I}$, $i = 1,2,\cdots n$. Then $(\mathfrak{I},A_b)$ is an complete A_b-metric space with $\vartheta = 2$.*
Let $\mathfrak{T}:\mathfrak{I}\to\mathfrak{I}$ by $\mathfrak{T}(æ) = \begin{cases} 0 & \text{for } 0 \leq æ \leq 3 \\ \frac{6n-5}{6n} & \text{for } 3 < æ \leq 4\end{cases}$
and $\mathfrak{f}:\mathfrak{I}\to\mathfrak{I}$ be given by $\mathfrak{f}(æ) = \begin{cases} \frac{4æ}{3n} & \text{for } 0 \leq æ \leq 3 \\ \frac{32n-4}{32n} & \text{for } 3 < æ \leq 4\end{cases}$
if choose $\alpha = \frac{1}{3}$ $\beta = \frac{1}{3}$, $\gamma = \frac{1}{4}$, and $\Gamma : [0,\infty) \to [0,\infty)$ be as $\Gamma(s) = \frac{3s}{4}$ $\forall s \in [0,\infty)$. Then obviously,$\mathfrak{T}0 = \mathfrak{f}0 = 0$ implies that 0 is a coincidence point of $\mathfrak{T}$ and $\mathfrak{f}$. Moreover, $\mathfrak{f}(\mathfrak{I}) = [0,4] \cup (0,1]$ and $\mathfrak{T}(\mathfrak{I}) = \{0\} \cup (0,1]$. Hence, $\mathfrak{T}(\mathfrak{I}) \subseteq \mathfrak{f}(\mathfrak{I})$ and also $\mathfrak{T}0 = \mathfrak{T}\mathfrak{f}0 = \mathfrak{f}\mathfrak{T}0 = \mathfrak{f}0 = 0$,then $(\mathfrak{T},\mathfrak{f})$ is ω-compatible.

Case (i): Suppose $æ, œ \in [0,3]$ *then*
$A_b(\mathfrak{T}æ, \mathfrak{T}æ, \cdots, (\mathfrak{T}æ)_{n-1}, \mathfrak{T}œ) = A_b(0, 0, \cdots (0)_{n-1}, 0) = 0.$

$$\begin{aligned} A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right) &= A_b\left(\frac{4æ}{3n}, \frac{4æ}{3n}, \cdots, (\frac{4æ}{3n})_{n-1}, \frac{4œ}{3n}\right) \\ &= |\frac{4œ}{3n} - \frac{4æ}{3n}|^2 = \frac{16}{9n^2}|œ - æ|^2 \end{aligned}$$

and

$$\begin{aligned} A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{T}æ\right) &= A_b\left(\frac{4æ}{3n}, \frac{4æ}{3n}, \cdots, (\frac{4æ}{3n})_{n-1}, 0\right) \\ &= \frac{16}{9n^2}|æ|^2 \end{aligned}$$

Then by contractive condition of Corollary 3.4 we have

$$\begin{aligned} A_b\left(\mathfrak{T}æ, \mathfrak{T}æ \cdots (\mathfrak{T}æ)_{n-1}, \mathfrak{T}œ\right) \le \alpha\Gamma\left(A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right)\right) \\ +\beta\Gamma\left(\max\left\{\begin{array}{c} A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right), \\ A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}\right\}\right) \\ +\gamma\Gamma\left(\frac{A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right)\left[1 + \sqrt{\begin{array}{c} A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right) \\ A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{T}æ\right) \end{array}}\right]^2}{\left(1 + A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right)\right)^2}\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 &\le \frac{16}{36n^2}|œ - æ|^2 + \frac{16}{36n^2}\max\left\{\ |œ - æ|^2, |æ|^2\ \right\} \\ &+ \frac{3}{16}\left(\frac{\frac{16}{9n^2}|œ - æ|^2\left[1 + \sqrt{\begin{array}{c} \frac{16}{9n^2}|œ - æ|^2 \\ \frac{16}{9n^2}|æ|^2 \end{array}}\right]^2}{\left(1 + \frac{16}{9n^2}|œ - æ|^2\right)^2}\right) \\ &\le \frac{16}{45n^2}|œ - æ|^2 + \frac{16}{45n^2}\max\left\{\ |œ - æ|^2, |æ|^2\ \right\} + \frac{1}{3n^2}|œ - æ|^2. \end{aligned}$$

This is always true for all $æ, œ \in [0,3]$.
Case (ii): Suppose $æ, œ \in (3,4]$, *then the hypothesis of corollary 3.4 trivially holds. Case (iii): if* $æ \in [0,3]$ *and* $œ \in (3,4]$ *then*
$A_b(\mathfrak{T}æ, \mathfrak{T}æ, \cdots, (\mathfrak{T}æ)_{n-1}, \mathfrak{T}œ) = A_b(0, 0, \cdots (0)_{n-1}, \frac{6n-5}{6n}) = |\frac{6n-5}{6n}|^2.$

$$A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right) = |\frac{32n-4}{32n} - \frac{4æ}{3n}|^2$$

and

$$A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{T}æ\right) = \frac{16}{9n^2}|æ|^2$$

Then clearly,

$$\begin{aligned} |\frac{6n-5}{6n}|^2 &\leq \frac{1}{4}|\frac{32n-4}{32n}-\frac{4æ}{3n}|^2+\frac{1}{4}\max\left\{ |\tfrac{32n-4}{32n}-\tfrac{4æ}{3n}|^2, \tfrac{16}{9n^2}|æ|^2 \right\} \\ &\quad +\frac{3}{16}|\frac{32n-4}{32n}-\frac{4æ}{3n}|^2 \\ &\leq \frac{7}{16}|\frac{32n-4}{32n}-\frac{4æ}{3n}|^2+\frac{1}{4}\max\left\{ |\tfrac{32n-4}{32n}-\tfrac{4æ}{3n}|^2, \tfrac{16}{9n^2}|æ|^2 \right\} \end{aligned}$$

which is true for all $æ \in [0,3]$ *and* $œ \in (3,4]$

Case (iv): if $æ \in (3,4]$ *and* $œ \in [0,3]$ *then*

$A_b(\mathfrak{T}æ, \mathfrak{T}æ, \cdots, (\mathfrak{T}æ)_{n-1}, \mathfrak{T}œ) = A_b(\frac{6n-5}{6n}, \frac{6n-5}{6n}, \cdots (\frac{6n-5}{6n})_{n-1}, 0) = |\frac{6n-5}{6n}|^2.$

$$A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{f}œ\right) = |\frac{4æ}{3n}-\frac{32n-4}{32n}|^2$$

and

$$A_b\left(\mathfrak{f}æ, \mathfrak{f}æ, \cdots, (\mathfrak{f}æ)_{n-1}, \mathfrak{T}æ\right) = |\frac{68}{96n}|^2$$

Then clearly,

$|\frac{6n-5}{6n}|^2 \leq \frac{7}{16}|\frac{4æ}{3n}-\frac{32n-4}{32n}|^2+\frac{1}{4}\max\left\{ |\frac{32n-4}{32n}-\frac{4æ}{3n}|^2, |\frac{68}{96n}|^2 \right\}$

true for all $æ \in (3,4]$ *and* $œ \in [0,3]$. *Put* $n = 2$, $\vartheta = 2$ *since* $\alpha+\beta+\gamma < 1$,*from cases (i)-(iv) all the conditions of Corollary 3.4 are satisfied and 0 is the UCFP of* $\mathfrak{T}$ *and* $\mathfrak{f}$.

4. Application to Integral Equations

In this part, we explore the existence of a unique solution to an initial value problem and apply it to Corollary 3.5.

Theorem 4.1. *Consider the initial value problem*

$$æ'(t) = \mathfrak{T}(t, æ(t)),\ t \in I = [0,1],\ æ(0) = æ_0, \tag{4.1}$$

$\mathfrak{T} : I \times \mathbb{R} \to \mathbb{R}$, *with* $æ_0 \in \mathbb{R}$. *The initial value problem (4.1) has a unique solution in* $C(I, \mathbb{R})$.

Proof. In the case of the initial value problem (4.1), the integral equation is

$$æ(t) = æ_0 + 2(n-1)^{\frac{1}{2}} \int_0^t \mathfrak{T}(s, æ(s))ds.$$

Let $\mathfrak{I} = C(I, \mathbb{R})$ and $A_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) = \sum_{i=1}^{n}\sum_{i<j}|æ_i - æ_j|^2$ for all $æ_i \in \mathfrak{I}, i = 1, 2, \cdots, n$,and $\Gamma : [0,\infty) \to [0,\infty)$ be as $\Gamma(s) = \frac{5s}{9}$ $\forall s \in [0,\infty)$ define $\mathfrak{R} : \mathfrak{I} \to \mathfrak{I}$ by

$$\mathfrak{R}(æ)(t) = \frac{æ_0}{2(n-1)^{\frac{1}{2}}} + \int_0^t \mathfrak{T}(s, æ(s))ds.$$

Clearly, for all $æ, œ \in \mathfrak{I}$, we have

$$\begin{aligned} & A_b\left(\Re(æ)(t), \Re(æ)(t), \cdots, \Re(œ)(t)\right) = (n-1)|\Re(æ)(t) - \Re(œ)(t)|^2 \\ = \; & (n-1)\left|\frac{æ_0}{2(n-1)^{\frac{1}{2}}} + \int_0^t \mathfrak{T}(s,(æ)(s))ds - \frac{œ_0}{2(n-1)^{\frac{1}{2}}} + \int_0^t \mathfrak{T}(s,(œ)(s))ds\right|^2 \\ = \; & \frac{1}{4}|æ(t) - œ(t)|^2 \leq \frac{5}{18}(n-1)|æ(t) - œ(t)|^2 \leq \frac{1}{2}\left(\frac{5}{9}A_b(æ, æ, \cdots, œ)\right) \\ \leq \; & \alpha\Gamma\left(A_b\left(æ, æ, \cdots, (æ)_{n-1}, œ\right)\right) \end{aligned}$$

Corollary 3.5 leads us to the conclusion that there is only one fixed point for $\Re$ in $\mathfrak{I}$.

□

5. Application to Homotopy

We examine the existence of a singular solution to homotopy theory in this section.

Theorem 5.1. *With $\vartheta \geq 1$, $\mathfrak{U}$ and $\overline{\mathfrak{U}}$ representing open and closed subsets of $\mathfrak{I}$ such that $\mathfrak{U} \subseteq \overline{\mathfrak{U}}$, let $(\mathfrak{I}, A_b)$ be the complete A_b-metric space. Assume that $\mathfrak{H} : \overline{\mathfrak{U}} \times [0,1] \to \mathfrak{I}$ is an operator with the following requirements:*
τ_0) For each $æ \in \partial\mathfrak{U}$ and $\kappa \in [0,1]$, we have $æ \neq \mathfrak{H}(æ,\kappa)$, (here $\partial\mathfrak{U}$ is boundary of $\mathfrak{U}$ in $\mathfrak{I}$);
τ_1) for all $æ, œ \in \overline{\mathfrak{U}}$ and $\kappa \in [0,1]$ such that

$$A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, (\mathfrak{H}(æ,\kappa))_{n-1}, \mathfrak{H}(œ,\kappa)\right) \leq \frac{1}{\vartheta^2}\Gamma\left(A_b\left(æ, æ, \cdots, œ\right)\right)$$

where $\Gamma : [0,\infty) \to [0,\infty)$ is a comparison function.
τ_2) $\exists\, M \geq 0 \ni A_b(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, (\mathfrak{H}(æ,\kappa))_{n-1}, \mathfrak{H}(æ,\zeta)) \leq M|\kappa - \zeta|$
for every $æ \in \overline{\mathfrak{U}}$and $\kappa, \zeta \in [0,1]$.
There is then a fixed point in $\mathfrak{H}(.,0)$. $\iff$ There is a fixed point in $\mathfrak{H}(.,1)$.

Proof. Let the set $\mathfrak{F} = \left\{\ \kappa \in [0,1] : \mathfrak{H}(æ,\kappa) = æ \text{ for some } æ \in \mathfrak{U}\ \right\}$.
Given a fixed point in $\mathfrak{U}$ for $\mathfrak{H}(.,0)$, we have $0 \in \mathfrak{F}$, for $\mathfrak{F}$ to be a non-empty set. We now demonstrate that, given the connectedness $\mathfrak{I} = [0,1]$, $\mathfrak{F}$ is both closed and open in $[0,1]$. Consequently, there is a fixed point for $\mathfrak{H}(.,1)$ in $\mathfrak{U}$. Initially, we demonstrate that $\mathfrak{F}$ closed in $[0,1]$. In order to observe this, let $\left\{\kappa_p\right\}_{p=1}^{\infty} \subseteq \mathfrak{I}$, where $p \to \infty$ and $\kappa_p \to \kappa \in [0,1]$. To prove that $\kappa \in \mathfrak{F}$, we must. Since $\kappa_p \in \mathfrak{F}$ for $p = 0,1,2,3,\cdots$, there exists sequences $\left\{æ_p\right\} \subseteq \mathfrak{U}$ with $æ_p = \mathfrak{H}(æ_p, \kappa_p)$.
Consider

$$\begin{aligned} & A_b(æ_p, æ_p, \cdots, (æ_p)_{n-1}, æ_{p+1}) \\ = \; & A_b(\mathfrak{H}(æ_p,\kappa_p), \mathfrak{H}(æ_p,\kappa_p), \cdots, (\mathfrak{H}(æ_p,\kappa_p))_{n-1}, \mathfrak{H}(æ_{p+1},\kappa_{p+1})) \\ \leq \; & \begin{array}{c} \vartheta(n-1)A_b\left(\ \mathfrak{H}(æ_p,\kappa_p), \mathfrak{H}(æ_p,\kappa_p), \cdots, \mathfrak{H}(æ_{p+1},\kappa_p)\ \right) \\ +\vartheta^2 A_b\left(\ \mathfrak{H}(æ_{p+1},\kappa_p), \mathfrak{H}(æ_{p+1},\kappa_p), \cdots, \mathfrak{H}(æ_{p+1},\kappa_{p+1})\ \right) \end{array} \end{aligned}$$

$$\leq \begin{array}{c} \vartheta(n-1)A_b\big(\mathfrak{H}(æ_p,\kappa_p),\mathfrak{H}(æ_p,\kappa_p),\cdots,\mathfrak{H}(æ_{p+1},\kappa_p) \big) \\ +\vartheta^2 M|\kappa_p-\kappa_{p+1}| \end{array}.$$

Given $p\to\infty$, we obtain

$$\begin{aligned}\lim_{p\to\infty}\frac{1}{\vartheta(n-1)}A_b(æ_p,æ_p,\cdots,æ_{p+1}) &\leq \lim_{p\to\infty}A_b\left(\mathfrak{H}(æ_p,\kappa_p),\mathfrak{H}(æ_p,\kappa_p),\cdots,\mathfrak{H}(æ_{p+1},\kappa_p)\right)\\ &\leq \lim_{p\to\infty}\frac{1}{\vartheta^2}\Gamma\left(A_b\left(æ_p,æ_p,\cdots,æ_{p+1}\right)\right)\end{aligned}$$

Using the property $\Gamma(s)\leq s$, for all $s\geq 0$, we get

$$\lim_{p\to\infty}(\frac{1}{\vartheta(n-1)}-\frac{1}{\vartheta^2})A_b(æ_p,æ_p,\cdots,æ_{p+1})\leq 0.$$

So that

$$\lim_{p\to\infty}A_b(æ_p,æ_p,\cdots,æ_{p+1})=0.$$

We now demonstrate that $\{æ_p\}$ in $(\Im,A_b)$ is a A_b-Cauchy sequence. Conversely, let us assume that $\{æ_p\}$ is not A_b-Cauchy. The natural numbers $\{p_k\}$ and $\{q_k\}$ have monotonic rising sequences such that $q_k>p_k$, and there exists $\epsilon>0$.

$$A_b\left(æ_{p_k},æ_{p_k}\cdots,(æ_{p_k})_{n-1},æ_{q_k}\right)\geq\epsilon \tag{5.1}$$

and

$$A_b\left(æ_{p_k},æ_{p_k}\cdots,(æ_{p_k})_{n-1},æ_{q_k-1}\right)<\epsilon \tag{5.2}$$

From (5.1) and (5.1), we have

$$\begin{aligned}\epsilon &\leq A_b\left(æ_{p_k},æ_{p_k}\cdots,(æ_{p_k})_{n-1},æ_{q_k}\right)\\ &\leq (n-1)\vartheta A_b\left(æ_{p_k},æ_{p_k}\cdots,(æ_{p_k})_{n-1},æ_{p_k+1}\right)+\vartheta^2 A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right)\end{aligned}$$

If we allow $k\to\infty$, we get

$$\frac{\epsilon}{\vartheta^2} \leq \lim_{p\to\infty}A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right) \tag{5.3}$$

But we have

$$\begin{aligned}&\lim_{p\to\infty}A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right)\\ \leq\ &\lim_{p\to\infty}A_b\left(\mathfrak{H}(æ_{p_k+1},\kappa_{p_k+1}),\mathfrak{H}(æ_{p_k+1},\kappa_{p_k+1}),\cdots,(\mathfrak{H}(æ_{p_k+1},\kappa_{p_k+1}))_{n-1},\mathfrak{H}(æ_{q_k},\kappa_{q_k})\right)\\ \leq\ &\lim_{p\to\infty}\frac{1}{\vartheta^2}\Gamma\left(A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right)\right)\end{aligned} \tag{5.4}$$

It follows that

$$(1-\frac{1}{\vartheta^2})\lim_{p\to\infty}A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right)\leq 0.$$

Thus

$$\lim_{p\to\infty}A_b\left(æ_{p_k+1},æ_{p_k+1},\cdots,(æ_{p_k+1})_{n-1},æ_{q_k}\right)=0$$

Therefore, $\epsilon \leq 0$, which is contradictory, follows from (5.3). Therefore Given the completeness of $(\mathfrak{I}, A_b)$ and the A_b-Cauchy sequence $\{æ_p\}$ in $(\mathfrak{I}, A_b)$, there exists $\varkappa \in \mathfrak{U}$ with

$$\lim_{p\to\infty} æ_{p+1} = \varkappa = \lim_{p\to\infty} æ_p.$$

From Lemma (2.8), we have

$$\begin{aligned} \frac{1}{\vartheta^2} A_b\left(\mathfrak{H}(\varkappa,\kappa), \mathfrak{H}(\varkappa,\kappa), \cdots, \varkappa\right) &\leq \lim_{p\to\infty} \inf A_b\Big(\mathfrak{H}(\varkappa,\kappa), \mathfrak{H}(\varkappa,\kappa), \cdots, \mathfrak{H}(æ_p,\kappa)\Big) \\ &\leq \lim_{p\to\infty} \inf \alpha\Gamma\Big(A_b\Big(\varkappa, \varkappa, \cdots, æ_p\Big)\Big) \\ &= 0. \end{aligned}$$

$\varkappa = \mathfrak{H}(\varkappa, \kappa)$ is the ensuing result. $\kappa \in \mathfrak{F}$ as a result. Thus, in $[0,1]$, $\mathfrak{F}$ is closed.
Let $\kappa_0 \in \mathfrak{F}$. Then, $æ_0 = \mathfrak{H}(æ_0, \kappa_0)$ exists for $æ_0 \in \mathfrak{U}$. Given that $\mathfrak{U}$ is open, $B_{A_b}(æ_0, r) \subseteq \mathfrak{U}$ for every $r > 0$. Select $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \kappa_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$. Next, for $æ \in \overline{B_{A_b}(æ_0, r)} = \Big\{æ \in \mathfrak{F} / A_b(æ, æ, \cdots, æ_0) \leq r + \vartheta^2 A_b(æ_0, æ_0, \cdots, æ_0)\Big\}$.
Now we have

$$\begin{aligned} A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, æ_0\right) &= A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, \mathfrak{H}(æ_0,\kappa_0)\right) \\ &\leq (n-1)\vartheta A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, \mathfrak{H}(æ,\kappa_0)\right) \\ &\quad + \vartheta^2 A_b\left(\mathfrak{H}(æ,\kappa_0), \mathfrak{H}(æ,\kappa_0), \cdots, \mathfrak{H}(æ_0,\kappa_0)\right) \\ &\leq \vartheta(n-1)M|\kappa - \kappa_0| + \vartheta^2 A_b\left(\mathfrak{H}(æ,\kappa_0), \mathfrak{H}(æ,\kappa_0), \cdots, \mathfrak{H}(æ_0,\kappa_0)\right) \\ &\leq \vartheta(n-1)\frac{1}{M^{p-1}} + \vartheta^2 A_b\left(\mathfrak{H}(æ,\kappa_0), \mathfrak{H}(æ,\kappa_0), \cdots, \mathfrak{H}(æ_0,\kappa_0)\right). \end{aligned}$$

Letting $p \to \infty$, we obtain

$$\begin{aligned} \frac{1}{\vartheta^2} A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, æ_0\right) &\leq A_b\left(\mathfrak{H}(æ,\kappa_0), \mathfrak{H}(æ,\kappa_0), \cdots, \mathfrak{H}(æ_0,\kappa_0)\right) \\ &\leq \frac{1}{\vartheta^2}\Gamma\left(A_b(æ, æ, \cdots, æ_0)\right). \end{aligned}$$

Utilizing Γ's property, we have

$$\begin{aligned} A_b\left(\mathfrak{H}(æ,\kappa), \mathfrak{H}(æ,\kappa), \cdots, æ_0\right) &\leq A_b(æ, æ, \cdots, æ_0) \\ &\leq r + \vartheta^2 A_b(æ_0, æ_0, \cdots, æ_0) \end{aligned}$$

Therefore, $\mathfrak{H}(.,\kappa) : \overline{B_{A_b}(æ_0, r)} \to \overline{B_{A_b}(æ_0, r)}$ holds for every fixed $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. Then, Theorem 5.1 is satisfied in all its conditions. As a result, we deduce that $\overline{\mathfrak{U}}$ has a fixed point for $\mathfrak{H}(.,\kappa)$. Therefore, since (τ_0) holds, this must be in $\mathfrak{U}$. For any $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$, $\kappa \in \mathfrak{F}$. Therefore, $(\kappa_0 + \epsilon, \kappa_0 - \epsilon) \subseteq \mathfrak{F}$. $\mathfrak{F}$ is obviously open in [0, 1]. We employ the identical method for the opposite inference.

□

Conclusion

In this paper, we use rational contractive type fixed point theorems in the set up of A_b-metric spaces to conclude certain applications to integral equations and homotopy theory.

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