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A Generalization of m-Bi-Ideals in b-Semirings and Its Characterizing Extension

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Abstract. In this study, we introduce new types of m-quasi-ideals and m-bi-ideals in b-semirings and provide some characterizations of these ideals. We use an algebraic method to express the fundamental properties of m-bi-ideals in b-semirings. We also discuss the m-ideals in terms of their algebraic structures. Moreover, we examine the m-bi-ideals and their generators and provide some characterizations regarding bi-ideals. We further discuss the m-bi-ideal generated by a non-empty subset $\mathcal{S}$, which is denoted by $< \mathcal{S} >_m = \mathcal{S} \cup \sum_{\text{finite}} \mathcal{B}\mathcal{S}^m\mathcal{B}$, where $\mathcal{B}$ is the set of all bi-ideals.

1. Introduction

Vandiver [15] introduced the concept of a semiring in 1934. Regular rings have been extensively studied for their own sake and connection to operator algebras. In 2009, Ronnason [2] proposed the idea of b-semirings. In an article submitted for publication, Mohanraj et al. [8] established the concepts of weak-1 ideals and weak-2 ideals in b-semirings. This study characterizes different regular b-semirings using multiple weak ideals. Semigroups, which emerged as a generalization of group theory in the early 20th century, are basic structures widely recognized in various areas of science and mathematics, as noted by Munir and Habib [9]. Due to their inherent connection to finite automata, they have numerous applications in theoretical computer science. Examples include time-invariant processes, abstract evolution equations, and graph theory.

Semigroups are algebraic structures with an essential ideal similar to other ones. Steinfeld [12,13] was one of the pioneers of the concept of semigroups and rings as quasi-ideals. Iséki [5] extended this idea to semirings with no zero and explored significant semiring descriptions based on quasi-ideals. Mathematicians have found it useful and fascinating to generalize the ideals found in

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algebraic structures. This generalization of values led to one-sided ideals and pseudo-ideals, see [4].

Lajos developed the concept of bi-ideals as a more general version of quasi-ideals in associative rings. Later, Lajos and Szasz [7] and other mathematicians applied these ideas to study various semigroups. Kar and Maity [6] introduced generalized bi-ideals for ternary semigroups. In [11], the study of semirings and ordered semirings through the hypothesis of an ordered b-semiring is described. The paper attempts an in-depth analysis of Type-1 and Type-2 bi-ideals over ordered b-semirings. Many mathematicians have used various ideals to prove significant results and characterizations of algebraic structures, see [10]. They proved that the intersection of almost hyperideals need not be an almost hyperideal, but the union of almost hyperideals is an almost hyperideal. This is distinct from the classical concept of ideal theory.

In this paper, we delve into the significant classical results in bi-ideals, m-bi-ideals, and their relationship with the elements and subsets of a b-semiring. We examine the conversion of bi-ideal and quasi-ideal concepts into m-bi-ideal. The paper is divided into five sections. The first section provides an overview of the topic, while the second section explores b-semirings and their relevant definitions and results. In the third section, we cover m-bi-ideal and m-quasi-ideal generated by a single element and subset with numerical examples. Finally, we conclude our study in the fourth section. The primary objective of this paper is to establish the relationship between bi-ideals and m-bi-ideals in b-semirings and demonstrate the relationship between m-quasi ideals and m-bi-ideals in b-semirings. Next, we will characterize the generator of bi-ideal, weak-1 left ideal, weak-1 right ideal, weak-2 left ideal, and weak-2 right ideal.

2. Preliminaries

In this section, we will introduce the concept of m-bi-ideals in b-semirings. We will provide an overview of the key theories and concepts explained in [2,3] relevant to this topic. Here $\mathcal{S}$ denotes a b-semiring unless otherwise mentioned. Also, $*_1$ and $*_2$ denote the MinMax-product and the MaxMin-product, respectively.

Definition 2.1. *[1] Let $\mathcal{S}$ be a non-empty set and $*_1$ and $*_2$ be binary operations on S. Then $(\mathcal{S}, *_1, *_2)$ is called a b-semiring if $(\mathcal{S}, *_1)$ and $(\mathcal{S}, *_2)$ are semigroups and for all $a, b, c \in \mathcal{S}$, $a *_1 (b *_2 c) = (a *_1 b) *_2 (a *_1 c)$, $(b *_2 c) *_1 a = (b *_1 a) *_2 (c *_1 a)$, $a *_2 (b *_1 c) = (a *_2 b) *_1 (a *_2 c)$, and $(b *_1 c) *_2 a = (b *_2 a) *_1 (c *_2 a)$.*

Definition 2.2. *Let A and B be subsets of $(S, *_1, *_2)$. Then the $*_1$-product and $*_2$-product of A and B, denoted by $A *_1 B$ and $A *_2 B$, respectively; are defined as follows:*

$$A *_1 B = \Big\{ a *_1 b \mid a \in A \text{ and } b \in B \Big\} \text{ and } A *_2 B = \Big\{ a *_2 b \mid a \in A \text{ and } b \in B \Big\}.$$

Definition 2.3. *A sub b-semiring Q of $\mathcal{S}$ is called an m_1-quasi ideal (resp.,m_2-quasi ideal) of $\mathcal{S}$ if $Q *_2 \mathcal{S}^m \cap \mathcal{S}^m *_2 Q \subseteq Q$ (resp., $Q *_1 \mathcal{S}^m \cap \mathcal{S}^m *_1 Q \subseteq Q$).*

Definition 2.4. *A subset $\mathcal{B}$ of $\mathcal{S}$ is called an m_1-bi-ideal (resp., m_2-bi-ideal) of $\mathcal{S}$ if $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}$ and $\mathcal{B} *_2 \mathcal{S}^m *_2 \mathcal{B} \subseteq \mathcal{B}$ (resp., $\mathcal{B} *_1 \mathcal{S}^m *_1 \mathcal{B} \subseteq \mathcal{B}$), where m is a positive integer.*

Definition 2.5. *A subset $\mathcal{B}$ of $\mathcal{S}$ is called an m-bi-ideal of $\mathcal{S}$ if it satisfies both m_1-bi-ideal and m_2-bi-ideal of $\mathcal{S}$.*

For a subset A of S and $i = 1,2,3,...,n$, $\sum A = \{(a_1 *_1 a_2 *_1 ... *_1 a_n) \mid a_i \in A\}$ and $\prod A = \{(a_1 *_2 a_2 *_2 ... *_2 a_n) \mid a_i \in A\}$.

3. m_1-Bi-Ideals of b-Semirings

This section introduces m_1-bi-ideals of b-semirings and their generalizations. Examples are provided to illustrate the results.

Note 3.1. *The binary operations $\wedge$, $\vee$, and $*_1$ are defined as follows: $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \end{pmatrix} *_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & 0 & 0 & 0 \\ b_4 & b_5 & b_6 & 0 & 0 & 0 \\ b_7 & b_8 & b_9 & b_{10} & 0 & 0 \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 & 0 \\ c_2 & c_3 & 0 & 0 & 0 & 0 \\ c_4 & c_5 & c_6 & 0 & 0 & 0 \\ c_7 & c_8 & c_9 & c_{10} & 0 & 0 \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & 0 \end{pmatrix}, \textit{where}$$

$c_1 = a_1 \wedge b_1 \wedge b_2 \wedge b_4 \wedge b_7 \wedge b_{11}$; $c_2 = a_2 \wedge (a_3 \vee b_1) \wedge b_2 \wedge b_4 \wedge b_7 \wedge b_{11}$; $c_3 = a_2 \wedge a_3 \wedge b_3 \wedge b_5 \wedge b_8 \wedge b_{12}$; $c_4 = a_4 \wedge (a_5 \vee b_1) \wedge (a_6 \vee b_2) \wedge b_4 \wedge b_7 \wedge b_{11}$; $c_5 = a_4 \wedge a_5 \wedge (a_6 \vee b_3) \wedge b_5 \wedge b_8 \wedge b_{12}$; $c_6 = a_4 \wedge a_5 \wedge a_6 \wedge b_6 \wedge b_9 \wedge b_{13}$; $c_7 = a_7 \wedge (a_8 \vee b_1) \wedge (a_9 \vee b_2) \wedge (a_{10} \vee b_4) \wedge b_7 \wedge b_{11}$; $c_8 = a_7 \wedge a_8 \wedge (a_9 \vee b_3) \wedge (a_{10} \vee b_5) \wedge b_8 \wedge b_{12}$; $c_9 = a_7 \wedge a_8 \wedge a_9 \wedge (a_{10} \vee b_6) \wedge b_9 \wedge b_{13}$; $c_{10} = a_7 \wedge a_8 \wedge a_9 \wedge a_{10} \wedge b_{10} \wedge b_{14}$; $c_{11} = a_{11} \wedge (a_{12} \vee b_1) \wedge (a_{13} \vee b_2) \wedge (a_{14} \vee b_4) \wedge (a_{15} \vee b_7) \wedge b_{11}$; $c_{12} = a_{11} \wedge a_{12} \wedge (a_{13} \vee b_3) \wedge (a_{14} \vee b_5) \wedge (a_{15} \vee b_8) \wedge b_{12}$; $c_{13} = a_{11} \wedge a_{12} \wedge a_{13} \wedge (a_{14} \vee b_6) \wedge (a_{15} \vee b_9) \wedge b_{13}$; $c_{14} = a_{11} \wedge a_{12} \wedge a_{13} \wedge a_{14} \wedge (a_{15} \vee b_{10}) \wedge b_{14}$; $c_{15} = a_{11} \wedge a_{12} \wedge a_{13} \wedge a_{14} \wedge a_{15} \wedge b_{15}$.

Note 3.2. *The binary operation $*_2$ is defined as follows:*

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & a_6 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} *_2 \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 & b_8 & b_9 \\ 0 & 0 & 0 & b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 & 0 & b_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & c_5 & c_6 & c_7 \\ 0 & 0 & 0 & 0 & c_8 & c_9 \\ 0 & 0 & 0 & 0 & 0 & c_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \textit{where}$$

$c_1 = b_6 \wedge a_1$; $c_2 = (b_7 \wedge a_1) \vee (b_{10} \wedge a_2)$; $c_3 = (b_8 \wedge a_1) \vee (b_{11} \wedge a_2) \vee (b_{13} \wedge a_3)$; $c_4 = (b_9 \wedge a_1) \vee (b_{12} \wedge a_2) \vee (b_{14} \wedge a_3) \vee (b_{15} \wedge a_4)$; $c_5 = b_{10} \wedge a_6$; $c_6 = (b_{11} \wedge a_6) \vee (b_{13} \wedge a_7)$; $c_7 = (b_{12} \wedge a_6) \vee (b_{14} \wedge a_7) \vee (b_{15} \wedge a_8)$; $c_8 = b_{13} \wedge a_{10}$; $c_9 = (b_{14} \wedge a_{10}) \vee (b_{15} \wedge a_{11})$; $c_{10} = b_{15} \wedge a_{13}$.

Theorem 3.1. *Every bi-ideal of $\mathcal{S}$ is an m_1-bi-ideal.*

Proof. Let $\mathcal{B}$ be a bi-ideal of $\mathcal{S}$. Then $\mathcal{B} *_2 \mathcal{S} *_2 \mathcal{B} \subseteq \mathcal{B}$. Now, it is also true that $\mathcal{B} *_2 \mathcal{S}^1 *_2 \mathcal{B} \subseteq \mathcal{B}$. Similarly, we can see that $\mathcal{B} *_2 \mathcal{S}^2 *_2 \mathcal{B} \subseteq \mathcal{B} *_2 \mathcal{S}^1 *_2 \mathcal{B} \subseteq \mathcal{B}$. In general, $\mathcal{B} *_2 \mathcal{S}^m *_2 \mathcal{B} \subseteq \mathcal{B} *_2 \mathcal{S}^{m-1} *_2 \mathcal{B} \subseteq \mathcal{B}$. Hence, $\mathcal{B}$ is an m_1-bi-ideal of $\mathcal{S}$. □

Remark 3.1. *The reverse implication of the Theorem 3.1 does not satisfy; see Example 3.1.*

Example 3.1. *Consider the b-semiring* $(\mathcal{S}_1, *_1, *_2)$*, where* $*_1$ *and* $*_2$ *are defined in the above Note 3.1. Let*

$$\mathcal{S}_1 = \left\{ \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in \mathbb{Z}^* \right\},$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in \mathbb{Z}^* \right\}.$$

Then $\mathcal{B}$ *is a sub b-semiring of* $\mathcal{S}_1$*. Now,*

$$\mathcal{B} *_2 \mathcal{S}_1^2 *_2 \mathcal{B} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathcal{B}.$$

Thus, $\mathcal{B}$ *is an* 2*-bi-ideal of* $\mathcal{S}_1$ *but it may not necessarily be a bi-ideal of* $\mathcal{S}_1$ *by*

$$(\mathcal{B} *_2 \mathcal{S}_1 *_2 \mathcal{B}) = \left\{ \begin{pmatrix} 0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_1 \in \mathbb{Z}^* \right\} \nsubseteq \mathcal{B}.$$

Theorem 3.2. *The product of any two* m_1*-bi-ideal and* ${m_1}'$*-bi-ideal of* $\mathcal{S}$ *with the identity element e is a* $\max\{m_1, {m_1}'\}$*-bi-ideal of* $\mathcal{S}$*.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be an m_1-bi-ideal and an ${m_1}'$-bi-ideal of $\mathcal{S}$, respectively. Now, $\mathcal{B}_1 *_2 \mathcal{S}^{m_1} *_2 \mathcal{B}_1 \subseteq \mathcal{B}_1$ and $\mathcal{B}_2 *_2 \mathcal{S}^{m_1} *_2 \mathcal{B}_2 \subseteq \mathcal{B}_2$. From Note 3.2, $(\mathcal{B}_1 *_2 \mathcal{B}_2)^2 = (\mathcal{B}_1 *_2 \mathcal{B}_2) *_2 (\mathcal{B}_1 *_2 \mathcal{B}_2) \subseteq (\mathcal{B}_1 *_2 \mathcal{S} *_2 \mathcal{B}_1) *_2 \mathcal{B}_2 \subseteq (\mathcal{B}_1 *_2 \mathcal{S} *_2 e *_2 ... *_2 e *_2 \mathcal{B}_1 *_2 \mathcal{B}_2) \subseteq (\mathcal{B}_1 *_2 \mathcal{S} *_2 \mathcal{S} *_2 ... *_2 \mathcal{S} *_2 \mathcal{B}_1) *_2 \mathcal{B}_2 \subseteq (\mathcal{B}_1 *_2 \mathcal{S}^m *_2 \mathcal{B}_1) *_2 \mathcal{B}_2 \subseteq \mathcal{B}_1 *_2 \mathcal{B}_2$. Also, $(\mathcal{B}_1 *_2 \mathcal{B}_2) *_2 \mathcal{S}^{\max\{m_1, {m_1}'\}} *_2 (\mathcal{B}_1 *_2 \mathcal{B}_2) \subseteq \mathcal{B}_1 *_2 \mathcal{S} *_2 \mathcal{S}^{\max\{m_1, {m_1}'\}} *_2 \mathcal{B}_1 *_2 \mathcal{B}_2 \subseteq \mathcal{B}_1 *_2 \mathcal{B}_2$. Therefore, $\mathcal{B}_1 *_2 \mathcal{B}_2$ is a $\max\{m_1, {m_1}'\}$-bi-ideal of $\mathcal{S}$. □

Theorem 3.3. *If* $\mathcal{B}$ *is a* $*_1$ *closure of* $\mathcal{S}$*,* $\mathcal{R}$ *is a subset of* $\mathcal{S}$*, and* $\mathcal{B}$ *is an* m_1*-bi-ideal of* $\mathcal{S}$*, then* $\mathcal{B} *_2 \mathcal{R}$ *and* $\mathcal{R} *_2 \mathcal{B}$ *are* m_1*-bi-ideals of* $\mathcal{S}$*.*

Proof. Now, $(\mathcal{B} *_2 \mathcal{R})^2 = (\mathcal{B} *_2 \mathcal{R}) *_2 (\mathcal{B} *_2 \mathcal{R}) = (\mathcal{B} *_2 \mathcal{R} *_2 \mathcal{B}) *_2 \mathcal{R} \subseteq (\mathcal{B} *_2 \mathcal{R} *_2 \mathcal{S}) *_2 \mathcal{B} \subseteq (\mathcal{B} *_2 (\mathcal{S}^m) *_2 \mathcal{B}) *_2 \mathcal{R} \subseteq \mathcal{B} *_2 \mathcal{R}$. Also, $(\mathcal{B} *_2 \mathcal{R}) *_2 \mathcal{S}^m *_2 (\mathcal{B} *_2 \mathcal{R}) *_2 \subseteq \mathcal{B} *_2 \mathcal{S} *_2 \mathcal{S}^m *_2 (\mathcal{B} *_2 \mathcal{R}) \subseteq \mathcal{B} *_2 \mathcal{S}^m *_2 \mathcal{B} *_2 \mathcal{R} \subseteq \mathcal{B} *_2 \mathcal{R}$. Therefore, $\mathcal{B} *_2 \mathcal{R}$ is an m_1-bi-ideal of $\mathcal{S}$. Similarly, we can demonstrate that $\mathcal{R} *_2 \mathcal{B}$ is an m_1-bi-ideal of $\mathcal{S}$. □

Theorem 3.4. *If $\mathcal{B}$ is an intersection of all bi-ideals with bipotencies $m_1, m_2...$, then $\mathcal{B}$ is also bi-ideal with bipotency* $\max\{m_1, m_2, ...\}$.

Proof. Let $\{\mathcal{B}_\zeta \mid \zeta \in \wedge\}$ be a family of m_ζ-bi-ideals of $\mathcal{S}$. Then $\mathcal{B} = \bigcap \mathcal{B}_\zeta$. Thus, $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}$. Since $\mathcal{B}_\zeta *_2 \mathcal{S}^{m_\zeta} *_2 \mathcal{B}_\zeta \subseteq \mathcal{B}_\zeta \subseteq \mathcal{B}$ for all $\zeta \in \wedge$, we have $\mathcal{B} *_2 \mathcal{S}^{\max\{m_\zeta|\zeta\in\wedge\}} *_2 \mathcal{B} \subseteq \mathcal{B}_\zeta \mathcal{S}^{m_\zeta} *_2 \mathcal{B}_\zeta \subseteq \mathcal{B}_\zeta$ for all $\zeta \in \wedge$. This implies that $\mathcal{B} *_2 \mathcal{S}^{\max\{m_\zeta|\zeta\in\wedge\}} *_2 \mathcal{B} \subseteq \mathcal{B}$. Therefore, $\mathcal{B}$ is an m_1-bi-ideal of $\mathcal{S}$ with bipotency $\max\{m_1, m_2, ...\}$. □

Theorem 3.5. *Every m_1-quasi ideal of $\mathcal{S}$ is an m_1-bi-ideal.*

Proof. Let $\mathcal{Q}$ be an m_1-quasi ideal of $\mathcal{S}$. Clearly, $\mathcal{Q}$ is a sub b-semiring of $\mathcal{S}$. Now, $\mathcal{Q} *_2 \mathcal{S}^m *_2 \mathcal{Q} \subseteq \mathcal{Q} *_2 \mathcal{S}^m *_2 \mathcal{S} = \mathcal{Q} *_2 \mathcal{S}^{m+1} \subseteq \mathcal{Q} *_2 \mathcal{S}^m$. Similarly, $\mathcal{Q} *_2 \mathcal{S}^m *_2 \mathcal{Q} \subseteq \mathcal{S}^m *_2 \mathcal{Q}$. We get $\mathcal{Q} *_2 \mathcal{S}^m *_2 \mathcal{Q} \subseteq (\mathcal{Q} *_2 \mathcal{S}^m) \cap (\mathcal{S}^m *_2 \mathcal{Q}) \subseteq \mathcal{Q}$. Hence, $\mathcal{Q}$ is an m_1-bi-ideal of $\mathcal{S}$. □

Remark 3.2. *The reverse implication of the Theorem 3.5 does not hold, see Example 3.2.*

Example 3.2. *Let $\mathcal{S}_1$ be a b-semiring and $\mathcal{B}$ be a sub b-semiring as in Example 3.1. Then*

$$\mathcal{B} *_2 \mathcal{S}_1^2 *_2 \mathcal{B} = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\} \subseteq \mathcal{B}.$$

Thus, $\mathcal{B}$ is a 2_1-bi-ideal of $\mathcal{S}_1$ but it may not be a m_1-quasi ideal by

$$(\mathcal{B} *_2 \mathcal{S}_1^2) \cap (\mathcal{S}_1^2 *_2 \mathcal{B}) = \left\{\begin{pmatrix} 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_1 \in \mathbb{Z}^* \right\} \nsubseteq \mathcal{B}.$$

Theorem 3.6. *If $\mathcal{Q}$ is a $*_2$ product of any (m_1, m_2)-quasi ideal and (n_1, n_2)-quasi ideal of $\mathcal{S}$, then $\mathcal{S}$ has a $\max\{m_1, m_2, n_1, n_2\}$-bi-ideal of $\mathcal{S}$ with the identity element.*

Proof. By Theorem 3.5, we have $(\mathcal{Q}_1 *_2 \mathcal{Q}_2)^2 = (\mathcal{Q}_1 *_2 \mathcal{Q}_2) *_2 (\mathcal{Q}_1 *_2 \mathcal{Q}_2) \subseteq \mathcal{Q}_1 *_2 (\mathcal{Q}_2 *_2 \mathcal{S} *_2 \mathcal{Q}_2) \subseteq \mathcal{Q}_1 *_2 \mathcal{Q}_2$. Therefore, $\mathcal{Q}_1 *_2 \mathcal{Q}_2$ is closed under $*_2$. Now, $(\mathcal{Q}_1 *_2 \mathcal{Q}_2) *_2 \mathcal{S}^{\max\{m_1,m_2,n_1,n_2\}} *_2 (\mathcal{Q}_1 *_2 \mathcal{Q}_2) \subseteq (\mathcal{Q}_1 *_2 \mathcal{Q}_2) *_2 \mathcal{S}^{\max\{m_1,m_2,n_1,n_2\}} *_2 (\mathcal{S} *_2 \mathcal{Q}_2) \subseteq \mathcal{Q}_1 *_2 (\mathcal{Q}_2 *_2 \mathcal{S}^{\max\{m_1,m_2,n_1,n_2\}+1} *_2 \mathcal{Q}_2) \subseteq \mathcal{Q}_1 *_2 \mathcal{Q}_2$. Therefore, $\mathcal{Q}_1 *_2 \mathcal{Q}_2$ is a $\max\{m_1, m_2, n_1, n_2\}$ bi-ideal of $\mathcal{S}$. □

Theorem 3.7. *Every m_1-left ideal of $\mathcal{S}$ is an m_1-bi-ideal.*

Proof. Assume that $\mathcal{G}$ is an m_1-left ideal of $\mathcal{S}$. Then $\mathcal{G} *_2 \mathcal{S}^m *_2 \mathcal{G} \subseteq \mathcal{G} *_2 \mathcal{G} \subseteq \mathcal{G}$. This implies that $\mathcal{G}$ is an m_1-bi-ideal of $\mathcal{S}$. □

Theorem 3.8. *Every m_1-right ideal of $\mathcal{S}$ is an m_1-bi-ideal.*

Proof. The proof is similar to Theorem 3.7. □

Theorem 3.9. *If $\mathcal{G}$ is a q_1-left ideal and $\mathcal{H}$ is an r_1-right ideal of $\mathcal{S}$, then $\mathcal{G} \cap \mathcal{H}$ is a k_1-bi-ideal of $\mathcal{S}$ with $k = \max\{q, r\}$.*

Proof. Let $\mathcal{G}$ be a q_1-left ideal and $\mathcal{H}$ is an r_1-right ideal of $\mathcal{S}$. Then $\mathcal{G}$ and $\mathcal{H}$ are q_1-bi and r_1-bi-ideals of $\mathcal{S}$. By Theorem 3.4, the intersection is a $\max\{q, r\}_1$-bi-ideal of $\mathcal{S}$. Also, $\mathcal{G} \cap \mathcal{H} *_2 \mathcal{S}^{\max\{q,r\}} *_2 \mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} *_2 \mathcal{S}^{\max\{q,r\}} *_2 \mathcal{G} \subseteq \mathcal{S}^{\max\{q,r\}+1} *_2 \mathcal{G} \subseteq \mathcal{S}^q *_2 \mathcal{G} \subseteq \mathcal{G}$. Similarly, we can prove that $\mathcal{G} \cap \mathcal{H} *_2 \mathcal{S}^{\max\{q,r\}} *_2 \mathcal{G} \cap \mathcal{H} \subseteq \mathcal{H}$. Consequently, $\mathcal{G} \cap \mathcal{H} *_2 \mathcal{S}^{\max\{q,r\}} *_2 \mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} \cap \mathcal{H}$. Therefore, $\mathcal{G} \cap \mathcal{H}$ is a k_1-bi-ideal of $\mathcal{S}$ with $k = \max\{q, r\}$. □

Theorem 3.10. *Let $a \in \mathcal{S}$, then the m_1-bi-ideal of $\mathcal{S}$ generated by a is $< a >_{m_1 b} = \{na\} \cup \{n' a^2\} \cup a *_2 \mathcal{S}^m *_2 a$.*

Proof. Let $x, y \in < a >_{m_1 b}$, $x = n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)$, and $y = n_2 a \cup m_2 a^2 \cup (a *_2 s^m *_2 a)$. Then

$x *_1 y = \{n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)\} *_1 \{n_2 a \cup m_2 a^2 \cup (a *_2 s^m *_2 a)\} = \{n_1 a *_1 (n_2 a \cup m_2 a^2) \cup (a *_2 s^m *_2 a) \cup (m_1 a^2 *_1 (n_2 a \cup m_2 a^2 \cup (a *_2 s^m *_2 a))) \cup ((a *_2 s^m *_2 a) *_1 (n_2 a \cup m_2 a^2 \cup (a *_2 s^m *_2 a)))\} = \{(n_1 a *_1 n_2 a) \cup (n_1 a *_1 m_2 a^2) \cup (n_1 a *_1 a *_2 s^m *_2 a) \cup (m_1 a^2 *_1 n_2 a) \cup (m_1 a^2 *_1 m_2 a^2) \cup (m_1 a^2 *_1 (a *_2 s^m *_2 a)) \cup ((a *_2 s^m *_2 a) *_1 (a *_2 s^m *_2 a))\} \in < a >_{m_1 b}$,

$x *_2 y = \{n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)\} *_2 \{n_2 a \cup m_2 a^2 \cup (a *_2 s^m *_2 a)\} = \{(n_1 a *_2 n_2 a) \cup (n_1 a *_2 m_2 a^2) \cup (n_1 a *_2 a *_2 s^m *_2 a) \cup (m_1 a^2 *_2 n_2 a) \cup (m_1 a^2 *_2 m_2 a^2) \cup (m_1 a^2 *_2 (a *_2 s^m *_2 a)) \cup ((a *_2 s^m *_2 a) *_2 (a *_2 s^m *_2 a))\} \in < a >_{m_1 b}$, and

$x *_2 s' *_2 x = (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a) *_2 s' *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)) = ((n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a) *_2 s') *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)) = (n_1 a *_2 s') \cup (m_1 a^2 *_2 s') \cup (a *_2 s *_2 a) *_2 s' *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)) = \{(n_1 a *_2 s') *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a) \cup (m_1 a^2 *_2 s') *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a) \cup (a *_2 s^m *_2 a *_2 s') *_2 (n_1 a \cup m_1 a^2 \cup (a *_2 s^m *_2 a)\} \in < a >_{m_1 b}$.

Therefore, $< a >_{m_1 b}$ is an m_1-bi-ideal of $\mathcal{S}$ generated by a. If B is an m_1-bi-ideal of $\mathcal{S}$ such that $a \in B$, then $< a >_{mb} \subseteq B$. Thus, $< a >_{mb}$ is the smallest m_1-bi-ideal of $\mathcal{S}$ generated by a. □

Theorem 3.11. *Let $\mathcal{P}$ be a non-empty subset of $\mathcal{S}$. Then m_1-bi-ideal generated by $\mathcal{P}$ is*

$$< \mathcal{P} >_{mb} = \sum \mathcal{P} \cup \Big[\sum \mathcal{P} *_2 \mathcal{P}\Big] \cup \Big[\sum \mathcal{P} *_2 \mathcal{S}^m *_2 \mathcal{P}\Big].$$

4. m_2-Bi-Ideals of b-Semirings

We introduce m_2-bi-ideals of b-semirings and their generalizations. Examples are provided to illustrate our results.

Theorem 4.1. *Every bi-ideal of $\mathcal{S}$ is an m_2-bi-ideal.*

Proof. The proof is similar to Theorem 3.1. □

Remark 4.1. *The reverse implication of the Theorem 4.1 does not satisfy; see Example 4.1.*

Example 4.1. *Consider the b-semiring $(\mathcal{S}_1, *_1, *_2)$, where $*_2$ and $*_1$ are defined in the above Note 3.1. Let*

$$\mathcal{S}_1 = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & 0 \\ s_8 & s_9 & s_{10} & s_{11} \\ s_{12} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\},$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_3 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}$$

Then $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}_1$. Now,

$$\mathcal{B} *_1 {\mathcal{S}_1}^2 *_1 \mathcal{B} = \left\{ \begin{pmatrix} n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \\ n_4 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\} \subseteq \mathcal{B}.$$

As a result, $\mathcal{B}$ is not a bi-ideal but 2_2-bi-ideal of $\mathcal{S}_1$ by

$$\mathcal{B} *_1 \mathcal{S}_1 *_1 \mathcal{B} = \left\{ \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & 0 & 0 & 0 \\ m_6 & m_7 & m_8 & m_9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'s} \in Z^* \right\} \nsubseteq \mathcal{B}.$$

Theorem 4.2. *The product of any two m_2-bi-ideal and ${m_2}'$-bi-ideal of $\mathcal{S}$ with the identity element e is a $\max\{m_2, {m_2}'\}$-bi-ideal of $\mathcal{S}$.*

Proof. The proof is similar to Theorem 3.2. □

Theorem 4.3. *If $\mathcal{B}$ is a $*_2$ closure of $\mathcal{S}$, $\mathcal{R}$ is a subset of $\mathcal{S}$, and $\mathcal{B}$ be m_2-bi-ideal of $\mathcal{S}$, then $\mathcal{B} *_1 \mathcal{R}$ and $\mathcal{R} *_1 \mathcal{B}$ are m_2-bi-ideals of $\mathcal{S}$.*

Proof. The proof is similar to Theorem 3.3. □

Theorem 4.4. *Every m_2-quasi ideal of $\mathcal{S}$ is an m_2-bi-ideal.*

Proof. The proof is similar to Theorem 3.5. □

Remark 4.2. *The reverse implication of Theorem 4.4 is not true, as shown in Example 4.2.*

Example 4.2. *Let $\mathcal{S}_1$ be a b-semiring and $\mathcal{B}$ be a sub b-semiring as shown in Example 3.1. Let*

$$\mathcal{S}_1 = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & 0 \\ s_8 & s_9 & s_{10} & s_{11} \\ s_{12} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\},$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_3 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}.$$

Then $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}_1$. Now,

$$\mathcal{B} *_1 {\mathcal{S}_1}^2 *_1 \mathcal{B} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathcal{B}.$$

As a result, $\mathcal{B}$ is a not a quasi-ideal but an m_2-bi-ideal of $\mathcal{S}_1$ by

$$\mathcal{B} *_1 \mathcal{S}_1 *_1 \mathcal{B} = \left\{ \begin{pmatrix} r_1 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 \\ r_3 & r_4 & r_5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \in Z^* \right\} \nsubseteq \mathcal{B}.$$

Theorem 4.5. *If $\mathcal{Q}$ is a $*_1$-product of any (m_2, m_1)-quasi ideal and (n_1, n_2)-quasi ideal of $\mathcal{S}$, then $\mathcal{S}$ has a* $\max\{m_2, m_1, n_1, n_2\}$*-bi-ideal with the identity element.*

Proof. The proof is similar to Theorem 3.6. □

Theorem 4.6. *Every m_2-left ideal of $\mathcal{S}$ is an m_2-bi-ideal.*

Proof. The proof is similar to Theorem 3.7. □

Theorem 4.7. *Every m_2-right ideal of $\mathcal{S}$ is an m_2-bi-ideal.*

Proof. The proof is similar to Theorem 4.6. □

Theorem 4.8. *If $\mathcal{G}$ is a q_2-left ideal and $\mathcal{H}$ is an r_2-right ideal of $\mathcal{S}$, then $\mathcal{G} \cap \mathcal{H}$ is a k_2-bi-ideal of $\mathcal{S}$ with $k = \max\{q, r\}$.*

Proof. The proof is similar to Theorem 3.9. □

Theorem 4.9. *Let $a \in \mathcal{S}$, then m_2-bi-ideal of $\mathcal{S}$ generated by a is $< a >_{mb} = \{na\} \cup \{n' a^2\} \cup a *_1 \mathcal{S}^m *_1 a$.*

Proof. The proof is similar to Theorem 3.10. □

Theorem 4.10. *Let $\mathcal{P}$ be a non-empty subset of $\mathcal{S}$. Then m_2-bi-ideal generated by $\mathcal{P}$ is*

$$< \mathcal{P} >_{mb} = \sum \mathcal{P} \cup \Big[\sum \mathcal{P} *_1 \mathcal{P}\Big] \cup \Big[\sum \mathcal{P} *_1 \mathcal{S}^m *_1 \mathcal{P}\Big].$$

5. m-Bi-Ideals of b-Semirings

This section presents m-bi-ideals of b-semirings and their generalizations.

Theorem 5.1. *Every bi-ideal of $\mathcal{S}$ is an m-bi-ideal.*

Proof. The proof follows from Theorems 3.1 and 4.1. □

Theorem 5.2. *The product of any two m_1-bi-ideal and m_2-bi-ideal of $\mathcal{S}$ with the identity element e is a* $\max\{m_1, m_2\}$*-bi-ideal of $\mathcal{S}$.*

Proof. The proof follows from Theorems 3.2 and 4.2. □

Theorem 5.3. *Every m-quasi ideal of $\mathcal{S}$ is an m-bi-ideal.*

Proof. The proof follows from Theorems 3.5 and 4.4. □

Theorem 5.4. *If $\mathcal{G}$ is a q-left ideal and $\mathcal{H}$ is an r-right ideal of $\mathcal{S}$, then $\mathcal{G} \cap \mathcal{H}$ is a k-bi-ideal of $\mathcal{S}$ with* $k = \max\{q, r\}$.

Proof. The proof follows from Theorems 3.9 and 4.8. □

Theorem 5.5. *For $a \in \mathcal{S}$, m-bi-ideal of $\mathcal{S}$ generated by a is $< a >_{mb} = \{na\} \cup \{n' a^2\} \cup a *_2 \mathcal{S}^m *_2 a$.*

Proof. The proof follows from Theorems 3.10 and 4.9. □

Theorem 5.6. *Let $\mathcal{P}$ be a non-empty subset of $\mathcal{S}$. Then the m-bi-ideal generated by $\mathcal{P}$ is*

$$< \mathcal{P} >_{m} = \sum \mathcal{P} \cup \Big[\sum \mathcal{P} * \mathcal{P}\Big] \cup \Big[\sum \mathcal{P} * \mathcal{S}^m * \mathcal{P}\Big],$$

where $* \in \{*_1, *_2\}$.

Proof. The proof follows from Theorems 3.11 and 4.10. □

6. Conclusion

During our study, we established the concepts of m-quasi ideals and m-bi-ideals in b-semirings, generalisations of bi-ideals. We examined some important characteristics and used their m-bi-ideals to explain them. We also looked at the structures of m-b-semiring ideals formed when a subset of the b-semiring was provided.

We plan to use m-bi-ideals to characterise various forms of semirings, such as regular, irregular, and weakly regular semirings. We will also investigate additional classes of m-bi-ideals, such as prime, maximum, minimal, and main m-bi-ideals. Towards the end of our discussion, we explored the relationship between m-quasi ideals and m-bi-ideals. Our study will examine their research on hyper b-semirings using m-bi-ideals and m-bi-quasi ideals.

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