Int. J. Anal. Appl. (2024), 22:119

International Journal of Analysis and Applications

Majorization in Analytic Functions Among Distinct Classes Defined by Modified Tremblay Fractional Operator

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Abstract. This paper presents and investigates three distinct kinds of analytic functions described by the Modified Tremblay Fractional Operator: $\mathfrak{I}_{\Upsilon}[A,B]$, $\mathfrak{Q}_{\Upsilon}[A,B]$, and $\mathfrak{P}_{\Upsilon}[A,B]$. We give a detailed knowledge of these unique categories features by exploring majorization difficulties within them. By means of a careful analysis of majorization phenomena, we present a range of novel findings that demonstrate the significance of parameter specialisation in these classes. This work greatly expands our understanding of analytic functions and improves the field of mathematical analysis as a whole. To sum up, this study offers a comprehensive investigation of new analytic function classes, clarifies certain aspects of majorization, and makes significant contributions that broaden our understanding of complex analysis and geometric function theory.

1. Introduction

In the domain related to analytic functions within the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. The notion of majorization is established for a pair of functions f and g, defined as per [12]:

$$f(z) \ll g(z) \quad (z \in \mathbb{E}) \tag{1.1}$$

If a function is analytic $\Psi(z)$ exists in the open unit disk $\mathbb{E}$, meeting the conditions

$$|\Psi(z)| \leq 1 \quad and \quad f(z) = \Psi(z)g(z) \quad (z \in \mathbb{E}) \tag{1.2}$$

For a pair of functions f and g, we declare that f is subordinated to g, denoted as $f(z) \prec g(z)$, if there exists an analytic function ω within the open unit disk $\mathbb{E}$. This function ω satisfies the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{E}$. Additionally, this association is expressed as $f(z) = g(\omega(z))$ for $z \in \mathbb{E}$.

Received: Apr. 17, 2024.

2020 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic function; univalent function; Hadmard product (convolution); majorization; modified tremblay fractional operator.

https://doi.org/10.28924/2291-8639-22-2024-119

ISSN: 2291-8639

By amalgamating the concepts of subordination and majorization, we introduce the concept of quasi-subordination. In the realm of two functions, f and g, we state that f is quasi-subordinate to g (see [14]). This relationship is defined by the following expression:

$$f(z) \prec_q g(z) \quad (z \in \mathbb{E}) \tag{1.3}$$

If $\Psi(z)$ and $\omega(z)$ are analytic functions within the open unit disk $\mathbb{E}$, the ratio $(f(z)/\Psi(z))$ is also analytic in $\mathbb{E}$, fulfilling:

$$|\Psi(z)| \le 1 \quad and \quad \omega(0) = 0, \quad |\omega(z)| \le |z| \le 1 \quad (z \in \mathbb{E}),$$

$$f(z) = \Psi(z)g(\omega(z)) \quad (z \in \mathbb{E}) \tag{1.4}$$

The connection established through majorization (Equation (1.3)) is closely associated with the concept of quasi-subordination among analytic functions.

Let's represent by $\mathcal{A}$ the set encompassing all functions expressible as:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.5}$$

that exhibit analytic properties within the open unit disk $\mathbb{E}$.

This introduction serves as an entry point into the exploration of majorization properties in the realm of analytic functions, with a specific emphasis on the distinctive impact of the modified Tremblay operator within the framework of geometric function theory. As we commence this exploration, our aim is to uncover the intricate interplay between majorization principles and the modified Tremblay operator, contributing to a deeper comprehension of the geometric aspects inherent in the behavior of analytic functions. Through this exploration, we anticipate uncovering novel insights that enrich the broader landscape of complex analysis and geometric function theory.

Tremblay [17] explored a fractional calculus operator in his thesis, utilizing the Riemann-Liouville fractional differential operator. Ibrahim and Jahangiri [11], in a more recent study, expanded the application of the Tremblay operator to the complex plane.

Definition 1.1. *If a function f belongs to the specified class $\mathcal{A}$, the Tremblay fractional derivative operator $\mathcal{T}_z^{\tau,\xi}$ is defined for all points z within the open unit disk $\mathbb{E}$ as provided in [11]*

$$\mathcal{T}_z^{\tau,\xi} f(z) = \frac{\Gamma(\tau)}{\Gamma(\xi)} z^{1-\xi} \mathcal{D}_z^{\tau-\xi} z^{\tau-1} f(z), \tag{1.6}$$

$$where \quad 0 < \tau \le 1; \quad 0 < \xi \le 1; \quad 0 \le \tau - \xi < 1$$

For $\tau = \xi = 1$, it is evident that we obtain

$$\mathcal{T}_z^{1,1} f(z) = f(z)$$

Esa et al. in [5] introduced a modification of the Tremblay operator specifically designed for analytic functions within the complex domain.

Definition 1.2. *For a function f belonging to the set $\mathcal{A}$, the modified Tremblay operator, denoted as $\mathcal{T}^{\tau,\xi}$, acts from $\mathcal{A}$ to $\mathcal{A}$ and is defined in the following manner:*

$$\mathfrak{S}_z^{\tau,\xi} f(z) = \frac{\tau}{\xi}\mathcal{T}_z^{\tau,\xi} f(z) = \frac{\Gamma(\tau+1)}{\Gamma(\xi+1)} z^{1-\xi}\mathcal{D}_z^{\tau-\xi} z^{\tau-1} f(z)$$

$$\mathfrak{S}_z^{\tau,\xi} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\tau+1)\Gamma(\xi+k)}{\Gamma(\xi+1)\Gamma(\tau+k)} a_k z^k \quad z \in \mathbb{E} \tag{1.7}$$

$$\textit{where} \quad 0 < \tau \leq 1; \quad 0 < \xi \leq 1; \quad 0 \leq \tau - \xi < 1$$

The modified Tremblay operator, developed by Esa et al. [5], is employed. Recurrence relations using equation 1.7 can be derived as follows:

$$z(\mathfrak{S}_z^{\tau,\xi} f(z))' = (\xi+1)\mathfrak{S}_z^{\tau,\xi+1} f(z) - \xi \mathfrak{S}_z^{\tau,\xi} f(z) \tag{1.8}$$

Definition 1.3. *A function f belonging to the set $\mathcal{A}$ is identified as a member of the class if and only if it adheres to the condition $\mathfrak{I}_{\Upsilon}[A,B]$*

$$1 + \frac{1}{\Upsilon}\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} - 1\right] \prec \frac{1+A\omega(z)}{1+B\omega(z)} \tag{1.9}$$

Here, the parameters A and B have the following: $-1 \leq B < A \leq 1 \quad \Upsilon \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Definition 1.4. *A function f in the set $\mathcal{A}$ is considered a member of the class $\mathfrak{Q}_{\Upsilon}[A,B]$ if it adheres to the specified criterion [10]:*

$$\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} - \nu\left|\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} - 1\right|\right] \prec e^z \quad (z \in \mathbb{E}) \tag{1.10}$$

Here, $\nu \geq 0, \quad -1 \leq B < A \leq 1$.

Definition 1.5. *A function f within the set $\mathcal{A}$ is recognised as part of the class $\mathfrak{P}_{\Upsilon}[A,B]$ if it meets the given condition [10]:*

$$e^{i\theta}\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)}\right] \prec e^{(z)(cos\theta + isin\theta)} \tag{1.11}$$

Here, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

MacGregor [12] initiated an exploration into a majorization problem specifically focused on the normalised subset of starlike functions. This line of inquiry was subsequently expanded upon by Altintas et al. [1]. Notably, recent research endeavours led by various scholars have extended the scope of majorization studies. These investigations encompass a diverse range of functions, encompassing both univalent and multivalent functions, as well as meromorphic and multivalent meromorphic functions. Moreover, these studies introduced different operators and classes, contributing to a richer understanding of majorization principles [[1]- [4], [6]- [9]].

Motivated by the advancements in this field, our current study aims to delve into majorization problems within specific classes, namely $\mathfrak{I}_{\Upsilon}[A,B]$, $\mathfrak{Q}_{\Upsilon}[A,B]$, and $\mathfrak{P}_{\Upsilon}[A,B]$. This exploration builds upon the existing body of knowledge, seeking to contribute novel insights and deepen our understanding of majorization phenomena within these distinct mathematical frameworks.

2. Exploring majorization results within the categories $\mathfrak{I}_{\Upsilon}[A,B]$,$\mathfrak{Q}_{\Upsilon}[A,B]$, and$\mathfrak{P}_{\Upsilon}[A,B]$

Theorem 2.1. *Consider a function f within the class $\mathcal{A}$, and suppose g belongs to the class $\mathfrak{I}_{\Upsilon}[A,B]$. If the function $\mathfrak{S}_z^{\tau,\xi}f(z)$ is majorized by $\mathfrak{S}_z^{\tau,\xi}g(z)$ in the open unit disk $\mathbb{E}$, then*

$$|\mathfrak{S}_z^{\tau,\xi}f(z)| \leq |\mathfrak{S}_z^{\tau,\xi}g(z)| \quad for \quad |z| \leq t_0 \tag{2.1}$$

where t_0 corresponds to the least positive solution of the equation

$$\Big|\Upsilon(A-B)+\xi B+B\Big|t^3 - \Big[|\xi+1|+2|B|\Big]t^2 - \Big[\Big|\Upsilon(A-B)+\xi B+B\Big|+2\Big]t + |\xi+1| = 0 \tag{2.2}$$

where $-1 \leq B < A \leq 1, \Upsilon \in \mathbb{C}^*, 0 < \xi \leq 1$.

Proof: Given that g belongs to the class $\mathfrak{I}_{\Upsilon}[A,B]$, as indicated by equation (1.9) ,

$$1+\frac{1}{\Upsilon}\left[\frac{z(\mathfrak{S}_z^{\tau,\xi}g(z))'}{\mathfrak{S}_z^{\tau,\xi}g(z)}-1\right] = \frac{1+A\omega(z)}{1+B\omega(z)} \tag{2.3}$$

In this particular situation, ω denotes an analytic function defined within the open unit disk $\mathbb{E}$, adhering to the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{E}$.

$$\frac{z(\mathfrak{S}_z^{\tau,\xi}g(z))'}{\mathfrak{S}_z^{\tau,\xi}g(z)} = \frac{1+[\Upsilon(A-B)+B]\omega(z)}{1+B\omega(z)}$$

Utilizing the recurrence relation (1.8) at this point, we obtain

$$\frac{\mathfrak{S}_z^{\tau,\xi+1}g(z)}{\mathfrak{S}_z^{\tau,\xi}g(z)} = \frac{\xi+1+[\Upsilon(A-B)+\xi B+B]\omega(z)}{(\xi+1)(1+B\omega(z))}$$

This implies that

$$\left|\mathfrak{S}_z^{\tau,\xi}g(z)\right| \leq \frac{(\xi+1)(1+B\,|z|)}{(\xi+1)-\Big|\Upsilon(A-B)+\xi B+B\Big|\,|z|}\left|\mathfrak{S}_z^{\tau,\xi+1}g(z)\right| \tag{2.4}$$

Currently, given that $\mathfrak{S}_z^{\tau,\xi}f(z)$ is under the majorization of $\mathfrak{S}_z^{\tau,\xi}g(z)$ within the region defined by the open unit disk $\mathbb{E}$,

$$\mathfrak{S}_z^{\tau,\xi}f(z) = \Psi(z)\mathfrak{S}_z^{\tau,\xi}g(z)$$

Take the derivative of the final inequality with respect to z and multiply both sides by z.

$$z(\mathfrak{S}_z^{\tau,\xi}f(z))' = z\Psi'(z)\mathfrak{S}_z^{\tau,\xi}g(z) + z\Psi(z)(\mathfrak{S}_z^{\tau,\xi}g(z))'$$

By employing the relationship given in (1.8), we can observe that

$$(\xi+1)\mathfrak{S}_z^{\tau,\xi+1}f(z) = z\Psi'(z)\mathfrak{S}_z^{\tau,\xi}g(z) + \Psi(z)(\xi+1)\mathfrak{S}_z^{\tau,\xi+1}g(z)$$

This suggests that

$$(\xi+1)|\Im_z^{\tau,\xi+1} f(z)| \leq |z||\Psi'(z)||\Im_z^{\tau,\xi} g(z)| + (\xi+1)|\Psi(z)||\Im_z^{\tau,\xi+1} g(z)| \tag{2.5}$$

If the Schwartz function Ψ adheres to the inequality [13],

$$\left|\Psi'(z)\right| \leq \frac{1-\left|\Psi'(z)\right|^2}{1-|z|^2} \tag{2.6}$$

By applying equations (2.4) and (2.6) within (2.5), we obtain

$$\left|\Im_z^{\tau,\xi+1} f(z)\right| \leq \left(\Psi(z) + \frac{1-\left|\Psi'(z)\right|^2}{1-|z|^2} \frac{(1+B\,|z|)\,|z|}{(\xi+1) - \left|\Upsilon(A-B)+\xi B+B\right| |z|}\right) \left|\Im_z^{\tau,\xi+1} g(z)\right| \tag{2.7}$$

Let's examine $|z| = t$ and $\left|\Psi(z)\right| = \iota$, and the inequality (2.7) results in

$$\left|\Im_z^{\tau,\xi+1} f(z)\right| \leq \left(\iota + \frac{1-\iota^2}{1-t^2} \frac{(1+Bt)t}{|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t}\right) \left|\Im_z^{\tau,\xi+1} g(z)\right|$$

in which,

$$\varsigma(\iota,t) = (1-t^2)\iota\left[\,|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t\right] + t(1-\iota^2)(1+Bt)$$

$$\left|\Im_z^{\tau,\xi+1} f(z)\right| \leq \frac{\varsigma(\iota,t)}{(1-t^2)\left[\,|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t\right]} \left|\Im_z^{\tau,\xi+1} g(z)\right| \tag{2.8}$$

subsequently,

$$\ell(\iota,t) = \frac{\varsigma(\iota,t)}{(1-t^2)\left[\,|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t\right]}$$

Derived from (2.8),

$$\left|\Im_z^{\tau,\xi+1} f(z)\right| \leq \ell(\iota,t) \left|\Im_z^{\tau,\xi+1} g(z)\right| \tag{2.9}$$

To establish our conclusion, we must ascertain the expression from equation (2.9)

$$\begin{aligned} t_0 &= max\{t \in [0,1); \quad \ell(\iota,t) \leq 1 \quad \forall \iota \in [0,1]\} \\ &= max\{t \in [0,1); \quad \mathbb{G}(\iota,t) \geq 0 \quad \forall \iota \in [0,1]\} \end{aligned} \tag{2.10}$$

in which,

$$\mathbb{G}(\iota,t) = (1-t^2)(1-\iota)\left[\,|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t\right] - (1-\iota^2)t(1+|B|t)$$

Demonstrating the equivalence of the inequality $\mathbb{G}(\iota,t) \geq 0$ is equivalent to

$$\mathbb{S}(\iota,t) = \left[\,|\xi+1| - \left|\Upsilon(A-B)+\xi B+B\right| t\right](1-t^2) - t(1+\iota)(1+|B|t) \geq 0 \tag{2.11}$$

The function $\mathbb{S}(\iota,t)$ attains its minimum value at $\iota = 1$, indicating that

$$min\{\mathbb{S}(\iota,t); \iota \in [0,1]\} = \mathbb{S}(1,t) = \mathbb{V}(t)$$

in which,

$$\mathbb{S}(1,t) = \Big|\Upsilon(A-B)+\xi B+B\Big|t^3 - \Big[|\xi+1|+2|B|\Big]t^2 - \Big[\Big|\Upsilon(A-B)+\xi B+B\Big|+2\Big]t + |\xi+1|$$

is the least positive solution of the equation.

Corollary 2.1. *Consider a function f within the class $\mathcal{A}$, and assume g is a member of the class $\mathfrak{I}_{\Upsilon}[A,B]$. If $\mathfrak{S}_z^{\tau,\xi} f(z)$ majorized by $\mathfrak{S}_z^{\tau,\xi} g(z)$ within the open unit disc $\mathbb{E}$, then*

$$|\mathfrak{S}_z^{\tau,\xi} f(z)| \le |\mathfrak{S}_z^{\tau,\xi} g(z)| \quad for \quad |z| \le t_1$$

where t_1 corresponds to the smallest positive solution of the equation

$$t_1 = \frac{E - \sqrt{E^2 - 4\Big|2\Upsilon-(\xi+1)\Big|(\xi+1)}}{2\Big|2\Upsilon-(\xi+1)\Big|}$$

setting A=1; B=-1 in(13) with $E = \Big[\Big|2\Upsilon-(\xi+1)\Big| + (2\Upsilon+(\xi+1))\Big] -1 \le B < A \le 1, \Upsilon \in \mathbb{C}^*, 0 < \xi \le 1$.

Theorem 2.2. *Consider a function f within the class $\mathcal{A}$, and assume g is a member of the class $\mathfrak{Q}_{\Upsilon}[A,B]$. If $\mathfrak{S}_z^{\tau,\xi} f(z)$ majorized by $\mathfrak{S}_z^{\tau,\xi} g(z)$ within the open unit disk $\mathbb{E}$, then*

$$|\mathfrak{S}_z^{\tau,\xi} f(z)| \le |\mathfrak{S}_z^{\tau,\xi} g(z)| \quad for \quad |z| \le t_* \tag{2.12}$$

where t_* corresponds to the least positive solution of the equation

$$\Big[e^t + \nu(\xi+1) - |\tau|\Big]t^2 - 2t(1+\nu) + \Big[|\xi| - \nu(\xi+1) - e^t\Big] = 0 \tag{2.13}$$

where $\nu \ge 0$.

Proof: Given that $g \in \mathfrak{Q}_{\Upsilon}[A,B]$, the inference can be drawn from both (1.10) and the subordination relation that

$$\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} - \nu\left|\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} - 1\right|\right] = e^{\omega(z)} \quad (z \in \mathbb{E}) \tag{2.14}$$

In this scenario, consider the analytic function ω defined within the open unit disk $\mathbb{E}$, with $\omega(0) = 0$ and $|\omega(z)| \le 1$ for all $z \in \mathbb{E}$.

$$\mathbb{I} = \frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} \tag{2.15}$$

Expressed in (2.14),

$$\mathbb{I} - \nu|\mathbb{I} - 1| = e^{\omega(z)}$$

This suggests that,

$$\mathbb{I} - \nu(\mathbb{I} - 1)e^{i\psi} = e^{\omega(z)}$$

As a result, we obtain

$$\mathbb{I} = \frac{e^{\omega(z)} - \nu e^{i\psi}}{1 - \nu e^{i\psi}} \tag{2.16}$$

Derived from ((2.15) & (2.16)), we obtain

$$\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)} = \frac{e^{\omega(z)} - \nu e^{i\psi}}{1 - \nu e^{i\psi}}$$

Now, applying the recurrence relation (1.8) to (2.16), we obtain

$$\frac{\mathfrak{S}_z^{\tau,\xi+1} g(z)}{\mathfrak{S}_z^{\tau,\xi} g(z)} = \frac{e^{\omega(z)} + \xi - (\xi+1)\nu e^{i\psi}}{(\xi+1)(1 - \nu e^{i\psi})}$$

This implies that

$$\left|\mathfrak{S}_z^{\tau,\xi} g(z)\right| \leq \frac{(\xi+1)(1+\nu)}{|\xi| - \nu(\xi+1) - e^{|z|}} \left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right| \tag{2.17}$$

Now, considering that $\mathfrak{S}_z^{\tau,\xi} f(z)$ is dominated by $\mathfrak{S}_z^{\tau,\xi} g(z)$ within the open unit disk $\mathbb{E}$,

$$\mathfrak{S}_z^{\tau,\xi} f(z) = \Psi(z) \mathfrak{S}_z^{\tau,\xi} g(z)$$

Differentiate the final inequality with respect to z and subsequently multiply both sides by z.

$$z(\mathfrak{S}_z^{\tau,\xi} f(z))' = z\Psi'(z) \mathfrak{S}_z^{\tau,\xi} g(z) + z\Psi(z)(\mathfrak{S}_z^{\tau,\xi} g(z))'$$

By utilizing relation (1.8), we can express the statement in the following manner,

$$(\xi+1)\mathfrak{S}_z^{\tau,\xi+1} f(z) = z\Psi'(z)\mathfrak{S}_z^{\tau,\xi} g(z) + \Psi(z)(\xi+1)\mathfrak{S}_z^{\tau,\xi+1} g(z)$$

This indicates that,

$$(\xi+1)|\mathfrak{S}_z^{\tau,\xi+1} f(z)| \leq |z||\Psi'(z)||\mathfrak{S}_z^{\tau,\xi} g(z)| + (\xi+1)|\Psi(z)||\mathfrak{S}_z^{\tau,\xi+1} g(z)| \tag{2.18}$$

The Schwartz function Ψ adheres to the inequality [13],

$$\left|\Psi'(z)\right| \leq \frac{1 - \left|\Psi'(z)\right|^2}{1 - |z|^2}; \quad (z \in \mathbb{E}) \tag{2.19}$$

By incorporating (2.17) and (2.19) into (2.18), we obtain

$$\left|\mathfrak{S}_z^{\tau,\xi+1} f(z)\right| \leq \left(\Psi(z) + \frac{1 - \left|\Psi'(z)\right|^2}{1 - |z|^2} \frac{(1+\nu)|z|}{|\xi| - \nu(\xi+1) - e^{|z|}}\right) \left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right| \tag{2.20}$$

Considering $|z| = t$ and $\left|\Psi(z)\right| = \iota$, the inequality (2.20) results in

$$\left|\mathfrak{S}_z^{\tau,\xi+1} f(z)\right| \leq \left(\iota + \frac{1 - \iota^2}{1 - t^2} \frac{(1+\nu)t}{|\xi| - \nu(\xi+1) - e^{t}}\right) \left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right| \tag{2.21}$$

in which,

$\varsigma_*(\iota, t) = \iota(1-t^2)\Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg] + t(1-\iota^2)(1+\nu)$

from (2.21),

$$\left|\mathfrak{S}_z^{\tau,\xi+1} f(z)\right| \leq \frac{\varsigma_*(\iota, t)}{(1-t^2)\Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg]} \left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right| \tag{2.22}$$

in which,

$$\ell_*(\iota, t) = \frac{\varsigma_*(\iota, t)}{(1-t^2)\Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg]}$$

Hence, based on (2.22),

$$\left|\mathfrak{S}_z^{\tau,\xi+1} f(z)\right| \leq \ell_*(\iota, t)\left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right| \tag{2.23}$$

Given relation (2.23), to establish our result, it is necessary to elucidate

$$\begin{aligned} t_* &= max\{t \in [0,1); \quad \ell_*(\iota, t) \leq 1 \quad \forall \iota \in [0,1]\} \\ &= max\{t \in [0,1); \quad \mathbb{G}_*(\iota, t) \geq 0 \quad \forall \iota \in [0,1]\} \end{aligned} \tag{2.24}$$

in which,

$$\mathbb{G}_*(\iota, t) = (1-t^2)(1-\iota)\Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg] - (1-\iota^2)t(1+\nu)$$

Demonstrating $\mathbb{G}_*(\iota, t) \geq 0$ *is tantamount to*

$$\mathbb{S}_*(\iota, t) = \Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg](1-t^2) - t(1+\iota)(1+\nu) \geq 0$$

The function $\mathbb{S}_*(\iota, t)$ *reaches its minimum value at* $\iota = 1$, *suggesting that*

$$min\{\mathbb{S}_*(\iota, t); \iota \in [0,1]\} = \mathbb{S}_*(1, t) = \mathbb{V}_*(t)$$

where,

$$\mathbb{S}_*(1, t) = (1-t^2)\Bigg[|\xi| - \nu(\xi+1) - e^t\Bigg] - 2t(1+\nu)$$

is the least positive solution of (2.12) which prove conclusion (2.13).

Corollary 2.2. *Consider a function* f *within the class* $\mathcal{A}$, *and assume* g *is a member of the class* $\mathfrak{Q}_\Upsilon[A, B]$. *If* $\mathfrak{S}_z^{\tau,\xi} f(z)$ *majorized by* $\mathfrak{S}_z^{\tau,\xi} g(z)$ *within the open unit disk* $\mathbb{E}$, *then*

$$|\mathfrak{S}_z^{\tau,\xi} f(z)| \leq |\mathfrak{S}_z^{\tau,\xi} g(z)| \quad for \quad |z| \leq t_*$$

where t_* *corresponds to the least positive solution of the equation*

$$\Bigg[e^t + 2\nu - 1\Bigg]t^2 - 2t(1+\nu) + \Bigg[1 - 2\nu - e^t\Bigg] = 0$$

where $\nu \geq 0$.

Theorem 2.3. *Consider a function f within the class $\mathcal{A}$, and assume g is a member of the class $\mathfrak{P}_{\Upsilon}[A,B]$. If $\mathfrak{S}_z^{\tau,\xi} f(z)$ majorized by $\mathfrak{S}_z^{\tau,\xi} g(z)$ within the open unit disk $\mathbb{E}$, then*

$$|\mathfrak{S}_z^{\tau,\xi} f(z)| \leq |\mathfrak{S}_z^{\tau,\xi} g(z)| \quad for \quad |z| \leq t_{**} \tag{2.25}$$

where t_{**} corresponds to the least positive solution of the equation

$$\left[e^t + |\xi+1||tan\theta| - |\xi|\right] t^2 - 2t|sec\theta| + \left[|\xi| - |\xi+1||tan\theta| - e^t\right] = 0 \tag{2.26}$$

where $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

Proof: Given that g belongs to $\mathfrak{P}_{\Upsilon}[A,B]$, it follows from (1.11) and the subordination relation that,

$$e^{i\theta}\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)}\right] = e^{\omega(z)(cos\theta + isin\theta)} \tag{2.27}$$

Now, considering ω as the analytic function in $\mathbb{E}$, with $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for all z in $\mathbb{E}$, we can express the sentence as follows:

$$\left[\frac{z(\mathfrak{S}_z^{\tau,\xi} g(z))'}{\mathfrak{S}_z^{\tau,\xi} g(z)}\right] = \frac{e^{\omega(z) + itan\theta}}{1 + itan\theta} \tag{2.28}$$

Utilizing the recurrence relation (1.8) in equation (2.28), we can express the sentence as follows:

$$\frac{\mathfrak{S}_z^{\tau,\xi+1} g(z)}{\mathfrak{S}_z^{\tau,\xi} g(z)} = \frac{e^{\omega(z)} + \xi + (\xi+1)itan\theta}{(\xi+1)sec\theta}$$

This suggests that

$$\left|\mathfrak{S}_z^{\tau,\xi} g(z)\right| \leq \frac{(\xi+1)|sec\theta|}{|\xi| - |\xi+1||tan\theta| - e^{|z|}} \left|\mathfrak{S}_z^{\tau,\xi+1} g(z)\right|$$

Now, given that $\mathfrak{S}_z^{\tau,\xi} f(z)$ is dominated by $\mathfrak{S}_z^{\tau,\xi} g(z)$ in the open unit disk $\mathbb{E}$,

$$\mathfrak{S}_z^{\tau,\xi} f(z) = \Psi(z)\mathfrak{S}_z^{\tau,\xi} g(z)$$

Differentiate the final inequality with respect to z and subsequently multiply both sides by z

$$z(\mathfrak{S}_z^{\tau,\xi} f(z))' = z\Psi'(z)\mathfrak{S}_z^{\tau,\xi} g(z) + z\Psi(z)(\mathfrak{S}_z^{\tau,\xi} g(z))'$$

By utilizing the relationship expressed in equation (1.8),

$$(\xi+1)\mathfrak{S}_z^{\tau,\xi+1} f(z) = z\Psi'(z)\mathfrak{S}_z^{\tau,\xi} g(z) + \Psi(z)(\xi+1)\mathfrak{S}_z^{\tau,\xi+1} g(z)$$

This leads to,

$$(\xi+1)|\mathfrak{S}_z^{\tau,\xi+1} f(z)| \leq |z||\Psi'(z)||\mathfrak{S}_z^{\tau,\xi} g(z)| + (\xi+1)|\Psi(z)||\mathfrak{S}_z^{\tau,\xi+1} g(z)| \tag{2.29}$$

The Schwartz function Ψ fulfills the inequality [13],

$$\left|\Psi'(z)\right| \leq \frac{1 - \left|\Psi'(z)\right|^2}{1 - |z|^2} (z \in \mathbb{E}) \tag{2.30}$$

Applying (2.28) and (2.30) to (2.29), we obtain

$$\left|\mathfrak{S}_z^{\tau,\xi+1}f(z)\right| \leq \left(\Psi(z) + \frac{1-\left|\Psi'(z)\right|^2}{1-|z|^2}\frac{|sec\theta||z|}{|\xi|-|\xi+1||tan\theta|-e^{|z|}}\right)\left|\mathfrak{S}_z^{\tau,\xi+1}g(z)\right| \tag{2.31}$$

Considering $|z| = t$ *and* $\left|\Psi(z)\right| = \iota$, *the inequality (2.31) results in*

$$\left|\mathfrak{S}_z^{\tau,\xi+1}f(z)\right| \leq \left(\iota + \frac{1-\iota^2}{1-t^2}\frac{(sec\theta)t}{|\xi|-|\xi+1||tan\theta|-e^t}\right)\left|\mathfrak{S}_z^{\tau,\xi+1}g(z)\right|$$

In the given context,

$$\varsigma_{**}(\iota,t) = \iota(1-t^2)\left[|\xi|-|\xi+1||tan\theta|-e^t\right] + t(1-\iota^2)sec\theta$$

$$\left|\mathfrak{S}_z^{\tau,\xi+1}f(z)\right| \leq \frac{\varsigma_{**}(\iota,t)}{(1-t^2)\left[|\xi|-|\xi+1||tan\theta|-e^t\right]}\left|\mathfrak{S}_z^{\tau,\xi+1}g(z)\right| \tag{2.32}$$

In the given context,

$$\ell_{**}(\iota,t) = \frac{\varsigma_{**}(\iota,t)}{(1-t^2)\left[|\xi|-|\xi+1||tan\theta|-e^t\right]}$$

The information conveyed by equation (2.32) can be reformulated as,

$$\left|\mathfrak{S}_z^{\tau,\xi+1}f(z)\right| \leq \ell_{**}(\iota,t)\left|\mathfrak{S}_z^{\tau,\xi+1}g(z)\right|$$

To demonstrate our result based on equation (2.32), we must ascertain

$$\begin{aligned} t_{**} &= max\{t \in [0,1); \quad \ell_*(\iota,t) \leq 1 \quad \forall \iota \in [0,1]\} \\ &= max\{t \in [0,1); \quad \mathbb{G}_*(\iota,t) \geq 0 \quad \forall \iota \in [0,1]\} \end{aligned} \tag{2.33}$$

In the given context,

$$\mathbb{G}_{**}(\iota,t) = (1-t^2)(1-\iota)\left[|\xi|-|\xi+1||tan\theta|-e^t\right] - (1-\iota^2)t|sec\theta|$$

To establish the equivalence, we need to demonstrate that $\mathbb{G}_*(\iota,t) \geq 0$.

$$\mathbb{S}_{**}(\iota,t) = \left[|\xi|-|\xi+1||tan\theta|-e^t\right](1-t^2) - t(1+\iota)|sec\theta| \geq 0$$

The function $\mathbb{S}_*(\iota,t)$ *reaches its minimum value at* $\iota = 1$, *suggesting that*

$$min\{\mathbb{S}_{**}(\iota,t); \iota \in [0,1]\} = \mathbb{S}_{**}(1,t) = \mathbb{V}_{**}(t)$$

$$\mathbb{S}_{**}(1,t) = (1-t^2)\left[|\xi|-|\xi+1||tan\theta|-e^t\right] - 2t|sec\theta|$$

is the least positive solution of equation (2.25), thus establishing the validity of statement (2.27).

Corollary 2.3. *Consider a function f within the class, and assume g is a member of the class $\mathfrak{P}_{\Upsilon}[A,B]$. If $\mathfrak{S}_z^{\tau,\xi} f(z)$ majorized by $\mathfrak{S}_z^{\tau,\xi} g(z)$ within the open unit disk $\mathbb{E}$, then*

$$|\mathfrak{S}_z^{\tau,\xi} f(z)| \leq |\mathfrak{S}_z^{\tau,\xi} g(z)| \quad for \quad |z| \leq t_{**}$$

where t_{**} corresponds to the least positive solution of the equation

$$\left[e^t + 2|tan\theta| - 1\right]t^2 - 2t|sec\theta| + \left[1 - 2|tan\theta| - e^t\right] = 0$$

where $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

3. Conclusion

In conclusion, our exploration of the introduced classes $\mathfrak{I}_{\Upsilon}[A,B]$,$\mathfrak{Q}_{\Upsilon}[A,B]$ and $\mathfrak{P}_{\Upsilon}[A,B]$ defined by the Modified Tremblay operator has yielded valuable insights into the majorization problem within the domain of analytic functions. The examination of specialized parameters has further enriched our findings, demonstrating the flexibility and applicability of these classes in diverse scenarios. This study contributes to the evolving landscape of analytic functions, providing a foundation for future research and applications in related areas.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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