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D^*-Local Functions in Ideal Spaces

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Abstract. In this study, a novel operation is introduced, which creates a local function of A regard to $\mathcal{I}$ and τ respectively denoted as $A_{D^*}(\mathcal{I},\tau) = \{y \in Y \mid V \cap A \notin \mathcal{I}$, for each $V \in \tau^D(y)\}$ where $\tau^D(y) = \{V \in \tau^D \mid y \in V\}$. We then look into some of the fundamental characteristics and attributes of $A_{D^*}(\mathcal{I},\tau)$. Additionally, we look into an operator $\eta : P(Y) \to \tau$ provides $\eta(E) = Y - [Y - E]_{D^*}$ for all $E \in P(Y)$. Then the closure operator $cl_{D^*}(E) = E_{D^*} \cup E$ which forms the topology and the relation $\tau_{D^*} = \{V \subseteq Y \mid cl_{D^*}(Y - V) = Y - V\}$.

1. Introduction and Preliminaries

Ideals in a topological space (Y,τ) were studied by Kuratowski in [5]. He had also defined local function for each subset of Y with regards to an ideal $\mathcal{I}$ and τ. In [9], Vaidyanathaswamy extended this study of ideals and local functions. In 1990, Jankovic and Hamlett [2] found more characteristics of ideal topological spaces. Assume that, (Y,τ) is a space without separation axioms. Then $cl(E); int(E)$ indicate the closure and interior of E respectively, in an ideal space $(Y,\tau,\mathcal{I})$. A nonempty set of Y that satisfies the given conditions in (Y,τ) is defined ideal [2];

(a) $E \in \mathcal{I}, F \subseteq E \Rightarrow F \in \mathcal{I}$

(b) $E \in \mathcal{I}$ and $F \in \mathcal{I} \Rightarrow$ union of E and F belongs to $\mathcal{I}$.

In 1960, Vaidyanathaswamy [9] gave the new local function which is defined by $P(Y)$ of Y with a set operator $(.)^*$: $P(Y) \to P(Y)$. For a set A in Y, $A^*(\mathcal{I},\tau) = \{y \in Y \mid V \cap A \notin \mathcal{I}$, where all $V \in \tau(y)\}$ where in $\tau(y) = \{V \in \tau \mid y \in V\}$. Moreover, we will just denote $A^*(\mathcal{I},\tau)$ by A^* and $\tau^*(\mathcal{I},\tau)$ by τ^*.

A Kuratowski closure operator [10] denoted by $cl^*(E)$ for τ^* finer than τ is defined as $cl^*(E) = E \cup E^*(\mathcal{I},\tau)$. In 2013, Ahmad Al-Omairi et. al [1] defined local closure functions in ideal spaces

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and studied their various properties. δ^*-local closure functions were analyzed and their various characterizations were studied in 2020 by P.Periyasamy and P.Rock Ramesh [7]. In 1966 N. Velicko [8] studied about θ-open sets and defined $cl_\theta(E)$ as $cl_\theta(E) = \{y \in Y : cl(V) \cap E = \phi$, for each $V \in \tau(y)\}$ also, a set E of Y is a θ-closed set if $cl_\theta(E) = E$. Similarly, many authors have defined local functions using various open sets and have studied them.

Analogous to that in this paper, we have defined local function using Δ-open sets which was first introduced by M. Veera Kumar in [4]. If a set E of Y in (Y, τ) equals to $(B - C) \cup (C - B)$, where B and C are sets of Y which are open if so E is defined as Δ-open. All Δ-open set collection satisfies the topology criterion and is given by τ^D for Y.

We generally, get Δ-closed sets from the complement of Δ-open sets. The set E of Y is said to be Δ-closed if $E = cl_D(E)$ where $cl_D(E) = \{y \in Y : V \cap E \neq \phi$, for every $V \in \tau^D(y)\}$.

In $(Y, \tau, \mathcal{I})$, τ is defined as compatible with $\mathcal{I}$, expressed as $\tau \sim \mathcal{I}$, if the axioms given below is true for all $E \subseteq Y$, if for all $y \in E$, where $V \in \tau(y)$ in such a way that $V \cap E \in \mathcal{I}, \Rightarrow E \in \mathcal{I}$. T_o spaces were introduced by P.S. Alexandroff and H. Hopf in 1935. A space is a T_o space [3] if and only if for two points that are not equal, precisely one will be contained in an open set. In 2007 M.N.Mukherjee et al. [6] introduced free ideal and defined it in the following way: Consider the space (Y, cl^*) and $y \in Y$, $\mathcal{I}_{cl^*}(y) = \{A \subseteq Y : y \notin cl^*(A)\}$ is an ideal on Y called a free ideal on (Y, cl^*).

2. D^*-Local Functions

In this section, we introduce one new tool namely, D^*-local function and analyze its nature in ideal topological space.

Definition 2.1. *Assume that Y be an ideal space, $A \subseteq Y$. The operator $A_{D^*}(\mathcal{I}, \tau) = \{y \in Y \mid V \cap A \notin \mathcal{I}$, $\forall\ V \in \tau^D(y)\}$ where $\tau^D(y) = \{V \in \tau^D \mid y \in V\}$ is known as D^*-local function in A related to $\mathcal{I}$, τ.*

Lemma 2.1. *$A_{D^*}(\mathcal{I},\tau) \subseteq A^*(\mathcal{I},\tau)$ always holds where Y is in ideal topological space.*

Proof. Suppose $y \in A_{D^*}$. Then $V \cap A \notin \mathcal{I}\ \forall\ V \in \tau^D(y)$. However, we know that each open set is Δ-open which implies $V \cap A \notin \mathcal{I}\ \forall\ V \in \tau(y)$. Hence, $y \in A^*$.

Example 2.1. Consider, $Y = \{i, a, e, g\}$,

$$\tau = \{\phi, Y, \{a\}, \{e\}, \{a, e\}\}$$

and

$$\mathcal{I} = \{\phi, \{i\}\}.$$

Consider, $A = \{a\} \Rightarrow A^* = \{i, a\}$ and $A_{D^*} = \{a\}$.

Remark 2.1. *Generally, neither $A \subseteq A_{D^*}$ nor $A_{D^*} \subseteq A$.*

Example 2.2. Consider, $Y = \{k, i, e, f\}$,

$$\tau = \{\phi, Y, \{k\}, \{i\}, \{k, i\}, \{k, e, f\}\}$$

and

$$\mathcal{I} = \{\phi, \{k\}, \{f\}, \{k, f\}\}.$$

Take $A = \{k, e\}$. Then $A_{D^*} = \{e, f\}$.

Theorem 2.1. *Suppose E and F are subsets of Y, an ideal space. If so the given conditions can be proved:*

(1) *If $E \subseteq F$, $\Rightarrow E_{D^*} \subseteq F_{D^*}$.*
(2) *$(E_{D^*})_{D^*} \subseteq E_{D^*}$.*
(3) *$E_{D^*} = cl_D(E_{D^*}) \subseteq cl_\theta(E)$ also E_{D^*} is closed.*
(4) *If $E \in \mathcal{I}$, then $E_{D^*} = \phi$.*
(5) *$E_{D^*} \subseteq cl_D(E)$.*

Proof. (1) Assume that, $x \notin F_{D^*}$. Then we can find a $V \in \tau^D(x)$ provided $F \cap V \in \mathcal{I}$. E is a subset of $F \Rightarrow E \cap V \subseteq F \cap V$, if so $E \cap V \in \mathcal{I}$. Therefore, $x \notin E_{D^*} \Rightarrow E_{D^*} \subseteq F_{D^*}$.

(2) Let $x \in (E_{D^*})_{D^*}$. Then for all $V \in \tau^D(x), V \cap E_{D^*} \notin \mathcal{I} \Rightarrow V \cap E_{D^*} \neq \phi$. Assume $z \in V \cap E_{D^*}$. In this case, $V \in \tau^D(z)$ with $z \in E_{D^*}$. Thus, $V \cap E \notin \mathcal{I}$ and $x \in E_{D^*}$. This brings out $(E_{D^*})_{D^*} \subseteq A_{D^*}$.

(3) We know $E_{D^*} \subseteq cl_D(E_{D^*})$. We can find an element $x \in cl_D(E_{D^*}) \Rightarrow E_{D^*} \cap V \neq \phi$ for all $V \in \tau^D(x)$. Then we can choose one $t \in E_{D^*} \cap V$ with $V \in \tau^D(t)$. Perhaps $t \in E_{D^*}, E \cap V \notin \mathcal{I} \Rightarrow x \in E_{D^*} \Rightarrow cl_D(E_{D^*}) \subseteq E_{D^*}$ and we get $E_{D^*} = cl_D(E_{D^*})$. Again, let $x \in E_{D^*}$, then $E \cap V \notin \mathcal{I}$ for all V belongs to $\tau^D(x)$. Then for all V belonging to $\tau^D(x)$ we get $E \cap V \neq \phi$. Hence $E \cap cl(V)$ is non-empty for all open set V. Hence, $x \in cl_\theta(E)$. This proves that $E_{D^*} = cl(E_{D^*}) \subseteq cl_\theta(E)$.

(4) Choose $x \in E_{D^*}$. Then we can find any $V \in \tau^D(x)$, $E \cap V$ does not belong to $\mathcal{I}$. But we have E belongs to $\mathcal{I} \Rightarrow E \cap V \in \mathcal{I}\ \forall\ V \in \tau^D(x)$ which is absurd. Therefore, $E_{D^*} = \phi$.

(5) Let $x \in E_{D^*}$. Consequently, for each $V \in \tau^D(x), V \cap E \notin \mathcal{I}$ with $\forall\ V \in \tau^D(x)$, $V \cap E \neq \phi$. Thus, $x \in cl_D(E)$. □

Theorem 2.2. *The ideal space Y containing ideals J_1, J_2 as well as and $C \subseteq Y$. If so the following conditions are true:*

(1) *If J_1 is a subset of J_2 then $C_{D^*}(J_2)$ is a subset of $C_{D^*}(J_1)$.*
(2) *$C_{D^*}(J_1 \cap J_2)$ equals to $C_{D^*}(J_1) \cup C_{D^*}(J_2)$.*

Proof. (1) Let $x \in C_{D^*}(J_2)$ and $J_1 \subseteq J_2$. For each $C \cap U \notin J_2$ follows so does $C \cap U \notin J_1$. Therefore $x \in C_{D^*}(J_1)$.

(2) We know $C_{D^*}(J_1)$ is a subset of $C_{D^*}(J_1 \cap J_2)$ and $C_{D^*}(J_2) \subseteq C_{D^*}(J_1 \cap J_2)$ by (1). Hence, union of $C_{D^*}(J_1)$ and $C_{D^*}(J_2)$ is a subset of $C_{D^*}(J_1 \cap J_2)$. We take $x \in C_{D^*}(J_1 \cap J_2)$, for each Δ-open U, $U \cap C \notin J_1 \cap J_2 \Rightarrow U \cap C \notin J_1$ or $U \cap C \notin J_2$. Then $x \in C_{D^*}(J_1)$ or $x \in C_{D^*}(J_2)$. So $x \in C_{D^*}(J_1) \cup C_{D^*}(J_2)$. □

Lemma 2.2. *Assume Y be of $(Y, \tau, \mathcal{I})$ with Δ-open set V, If so $V \cap C_{D^*} = V \cap (V \cap C)_{D^*} \subseteq (V \cap C)_{D^*}$ for each subset C of Y.*

Proof. Lets assume that when V is Δ-open set with $y \in V \cap C_{D^*} \Rightarrow y \in V$ and $y \in C_{D^*}$ then $V \cap C \notin \mathcal{I}$ for each Δ-open set containing y. Assume W to be any Δ-open set comprising y that is $W \in \tau^D(y)$. So $W \cap V \in \tau^D(y)$ and $W \cap (V \cap C) = (W \cap V) \cap C \notin \mathcal{I}$. This expresses that $y \in (V \cap C)_{D^*}$ which leads to the conclusion that $V \cap C_{D^*}$ is a subset of $(V \cap C)_{D^*}$. Additionally, $V \cap C_{D^*} \subseteq V \cap (V \cap C)_{D^*}$ and by Theorem [2.2] $(V \cap C)_{D^*} \subseteq C_{D^*}$ and $V \cap (V \cap C)_{D^*} \subseteq V \cap C_{D^*}$. Hence, $V \cap C_{D^*} = V \cap (V \cap C)_{D^*}$. □

Theorem 2.3. *Assume that Y is an ideal space with E, G are any two sets of Y. If so, the given observations are true:*

(1) $\phi_{D^*} = \phi$.
(2) $E_{D^*} \cup G_{D^*} = (E \cup G)_{D^*}$.
(3) $(E \cap G)_{D^*} \subseteq E_{D^*} \cap G_{D^*}$.

Proof. (1) It is obvious.

(2) From Theorem [2.2] it is evident that $(E \cup G)_{D^*} \supseteq E_{D^*} \cup G_{D^*}$. To get the reverse part, assume $y \in (E \cup G)_{D^*}$. Then for all Δ-open set containing y, $V \cap (E \cup G) \notin \mathcal{I}$. From this $(V \cap E) \notin \mathcal{I}$ or $(V \cap G) \notin \mathcal{I}$ for each Δ-open set containing y. Hence $y \in E_{D^*} \cup G_{D^*} \Rightarrow (E \cup G)_{D^*} \subseteq E_{D^*} \cup G_{D^*}$.

(3) $E \cap G$ is a subset of $E \Rightarrow (E \cap G)_{D^*} \subseteq E_{D^*}$. Similarly, $(E \cap G)_{D^*}$ is a subset of G_{D^*} by Theorem [2.2]. So $(E \cap G)_{D^*}$ is a set of $E_{D^*} \cap G_{D^*}$. □

Remark 2.2. *The reverse implication of Theorem [2.5] (3) doesn't always apply, as may be seen by an example.*

Example 2.3. *Consider, $Y = \{l, c, e, g\}$,*

$$\tau = \{\phi, Y, \{l, c\}\}$$

and

$$\mathcal{I} = \{\phi, \{l\}, \{c\}, \{l, c\}\}.$$

Suppose $C = \{e\}$ and $D = \{c, g\}$. Then $C_{D^*} \cap D_{D^*} = \{e, g\}$ and $(C \cap D)_{D^*} = \phi$.

Lemma 2.3. *Suppose Y be an ideal space with E, $H \subseteq Y$. If so $E_{D^*} - H_{D^*}$ equals to $(E - H)_{D^*} - H_{D^*}$ is a set of $(E - H)_{D^*}$.*

Proof. We have by Theorem [2.5], $E_{D^*} = [(E - H) \cup (E \cap H)]_{D^*} = (E - H)_{D^*} \cup (E \cap H)_{D^*} \subseteq (E - H)_{D^*} \cup H_{D^*}$. Thus $E_{D^*} - H_{D^*} \subseteq (E - H)_{D^*} - H_{D^*}$.We have by Theorem [2.2], $(E - H)_{D^*} \subseteq E_{D^*}$ it follows $(E - H)_{D^*} - H_{D^*}$ is a subset of $E_{D^*} - H_{D^*}$. Henceforth, $E_{D^*} - H_{D^*}$ equals $(E - H)_{D^*} - H_{D^*} \subseteq (E - H)_{D^*}$. □

Corollary 2.1. *Assuming Y is an ideal topological space with two subsets G and H in Y, $H \in \mathcal{I}$. Then $(G \cup H)_{D^*} = G_{D^*} = (G - H)_{D^*}$ holds.*

Proof. Though $H \in \mathcal{I}$, by Theorem [2.2], we get $H_{D^*} = \phi$. By Previous Lemma 2.3, $G_{D^*} = (G - H)_{D^*}$ and by Theorem [2.5], we get $(G \cup H)_{D^*} = G_{D^*} \cup H_{D^*} = G_{D^*}$. □

Theorem 2.4. $E_{D^*} \subseteq \Gamma(E)$

Proof. By lemma 2.1, $E_{D^*} \subseteq E^*$ and we know that, $E^* \subseteq \Gamma(E)$ [1]. Therefore, $E_{D^*} \subseteq \Gamma(E)$. □

Theorem 2.5. $E_{D^*} \subseteq E_{\delta^*}$

Proof. By lemma 2.1, $E_{D^*} \subseteq E^*$ and we know that, $E^* \subseteq E_{\delta^*}$ [7]. Therefore, $E_{D^*} \subseteq E_{\delta^*}$. □

3. The Open Sets of τ_{D*}

A closure operator $cl_{D^*}(E) = E \cup E_{D^*}$ is defined in this section and we prove the Kuratowski closure operator.

Theorem 3.1. *Assume Y be an ideal space, $cl_{D^*}(B)$=$B \cup B_{D^*}$ where B and C are sets of Y. If so the given observations hold:*

1. *$cl_{D^*}(\phi) = \phi$ and $cl_{D^*}(Y) = Y$.*
2. *$B \subseteq cl_{D^*}(B)$.*
3. *$cl_{D^*}(B \cup C) = cl_{D^*}(B) \cup cl_{D^*}(C)$.*
4. *$cl_{D^*}(cl_{D^*}(B)) = cl_{D^*}(B)$.*

Proof. (1) Since $\phi \in Y$ $cl_{D^*}(\phi) = \phi_{D^*} \cup \phi = \phi$, $cl_{D^*}(Y) = Y_{D^*} \cup Y = Y$.

(2) We know that $B \subseteq B \cup B_{D^*} = cl_{D^*}(B)$.

(3) As a result of Theorem [2.5], $cl_{D^*}(B \cup C) = (B \cup C)_{D^*} \cup (B \cup C) = (B_{D^*} \cup C_{D^*}) \cup (B \cup C) = (B_{D^*} \cup B) \cup (C_{D^*} \cup C) = cl_{D^*}(B) \cup cl_{D^*}(C)$.

(4) $cl_{D^*}(cl_{D^*}(B)) = cl_{D^*}(B_{D^*} \cup B) = (B_{D^*} \cup B)_{D^*} \cup (B_{D^*} \cup B) = ((B_{D^*})_{D^*} \cup B_{D^*}) \cup (B_{D^*} \cup B) = B_{D^*} \cup (B_{D^*} \cup B) = B_{D^*} \cup B = cl_{D^*}(B)$, by Theorem [2.5]. □

Remark 3.1. *According to Theorem [3.1], $cl_{D^*}(E) = E \cup E_{D^*}$ is a Kuratoswski closure operator. The open sets in Δ are referred to as τ_{D^*}-open sets while its complement is referred to as τ_{D^*}-closed sets. This topology is represented by τ_{D^*} and defined as $\tau_{D^*} = \{V \subseteq Y / cl_{D^*}(Y - V) = Y - V\}$.*

Theorem 3.2. *Suppose Y is an ideal space with R, C as sets of Y. If so the given observations are true:*

1. *If R is a set of $C \Rightarrow cl_{D^*}(R) \subseteq cl_{D^*}(C)$.*
2. *$cl_{D^*}(R \cap C) \subseteq cl_{D^*}(R) \cap Cl_{D^*}(C)$.*
3. *$cl_{D^*}(R) \subseteq cl^*(R)$.*

Proof. (1) Assume R is a set of C where $cl_{D^*}(R) = R \cup R_{D^*} \subseteq C \cup C_{D^*} = cl_{D^*}(C)$ by Theorem [2.2].

(2) We had, $R \cap C$ is a set of R and $R \cap C \subseteq C$ then by (1), $cl_{D^*}(R \cap C) \subseteq cl_{D^*}(R)$ and $cl_{D^*}(R \cap C) \subseteq cl_{D^*}(C) \Rightarrow cl_{D^*}(R \cap C) \subseteq cl_{D^*}(R) \cap Cl_{D^*}(C)$.

(3) $cl_{D^*}(R) = R \cup R_{D^*} \subseteq R \cup R^* = cl^*(R)$ since $R_{D^*} \subseteq R^*$. □

Definition 3.1. *Consider, the space (Y, cl_{D^*}) and for $y \in Y$, $\mathcal{I}_{cl_{D^*}}(y) = \{E \subseteq Y : y \notin cl_{D^*}(E)\}$ is an ideal on Y called a free ideal on (Y, cl_{D^*})*

Theorem 3.3. *A space (Y, cl_{D^*}) is a T_o space $\iff$ for any two points w, z of Y and $w \neq z$, $\mathcal{I}_{cl_{D^*}}(w) \neq \mathcal{I}_{cl_{D^*}}(z)$.*

Proof. Assume (Y, cl_{D^*}) is a T_o space. Suppose $w \notin cl_{D^*}(E)$ then $z \in cl_{D^*}(E)$. But if $w \notin cl_{D^*}(E)$ $\Rightarrow E \in \mathcal{I}_{cl_{D^*}}(w)$. Also $z \in cl_{D^*}(E) \Rightarrow E \notin \mathcal{I}_{cl_{D^*}}(z)$. Therefore, $\mathcal{I}_{cl_{D^*}}(w) \neq \mathcal{I}_{cl_{D^*}}(z)$. Conversely, let $\mathcal{I}_{cl_{D^*}}(w) \neq \mathcal{I}_{cl_{D^*}}(z)$. Suppose $E \subseteq Y$ such that $E \in \mathcal{I}_{cl_{D^*}}(w)$ but $E \notin \mathcal{I}_{cl_{D^*}}(z)$. Implies $w \notin cl_{D^*}(E)$ and $z \in cl_{D^*}(E)$ for some $E \subseteq Y$. Therefore, $w \in Y - cl_{D^*}(E)$ but $z \notin Y - cl_{D^*}(E) \Rightarrow (Y, cl_{D^*})$ is a T_o space. □

Remark 3.2. $\mathcal{I}_{cl^*}(y) \subseteq \mathcal{I}_{cl_{D^*}}(y)$

Example 3.1. Consider, $Y = \{p, b, s, d\}$ and

$$\tau = \{\phi, Y, \{p\}, \{s\}, \{d\}, \{p, s\}, \{p, d\}, \{s, d\}, \{p, s, d\}\}.$$

Then $\mathcal{I}_{cl^*}(b) = \{\phi, \{p\}\}$ and $\mathcal{I}_{cl_{D^*}}(b) = \{\phi, \{s\}, \{d\}, \{p, s\}, \{p, d\}, \{s, d\}, \{p, s, d\}\}$.

4. D^*-Compatibility

Definition 4.1. *Assume Y is an ideal space. If the given conditions are true for all $F \subseteq Y$ then $F \in \mathcal{I}$. If for each $y \in F$,then $\exists$ a Δ-open set in a way that $V \cap F \in \mathcal{I}$ then τ is D^*-compatible with the ideal $\mathcal{I}$, indicated by $\tau \sim_D \mathcal{I}$.*

Theorem 4.1. *Suppose Y be an ideal space, if so the observations given below can be equivalent:*

(1) $\tau \sim_D \mathcal{I}$,
(2) *$E \in \mathcal{I}$, if E a set of Y has a Δ-open cover, which intersects E is in $\mathcal{I}$,*
(3) *For all E set of Y, $E_{D^*} \cap E = \phi \Rightarrow E$ belongs to $\mathcal{I}$,*
(4) *For each $E \subseteq Y$, $E - E_{D^*} \in \mathcal{I}$,*
(5) *For all $E \subseteq Y$ does not contains nonempty subset D with $D \subseteq D_{D^*}$, then $E \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): The result is true obviously.

(2) $\Rightarrow$ (3): Suppose, $y \in E$ with $E \subseteq Y$. If $y \notin E_{D^*}$, we can find $V \in \tau^D(y)$ provided $V \cap E \in \mathcal{I}$. Hence $E \in \mathcal{I}$ since we have $E \subseteq \cup\{V_y : y \in E\}$ for V_y is in $\tau^D(x)$ containing y.

(3) $\Rightarrow$ (4): For each E which is a set of Y, $E - E_{D^*}$ is a set of E if so, $(E - E_{D^*}) \cap (E - E_{D^*})_{D^*}$ set of $(E - E_{D^*}) \cap E_{D^*} = \phi$, since $E - E_{D^*} \in \mathcal{I}$.

(4) $\Rightarrow$ (5): For each $E \subseteq Y, E - E_{D^*} \in \mathcal{I}$ by (4). Consider $E - E_{D^*} = J \in \mathcal{I}$, then $E = J \cup (E \cap E_{D^*})$ and according to Theorem [2.5] and [2.2], $E_{D^*} = J_{D^*} \cup (E \cap E_{D^*})_{D^*} = (E \cap E_{D^*})_{D^*}$. Then, we have $E \cap E_{D^*} = E \cap (E \cap E_{D^*})_{D^*}$ is a subset of $(E \cap E_{D^*})_{D^*}$ and $E \cap E_{D^*}$ is a set of E. On the basis of the assumption $E \cap E_{D^*} = \phi$ and hence $E = E - E_{D^*} \in \mathcal{I}$.

(5) $\Rightarrow$ (1): Let $E \subseteq Y$ with each $y \in E$, $\exists\ V \in \tau^D(y)$ in such a way that $V \cap E \in \mathcal{I} \Rightarrow E \cap E_{D^*} = \phi$. Also assume that E contains D such that $D \subseteq D_{D^*}$. Then $D = D \cap D_{D^*} \subseteq E \cap E_{D^*} = \phi$. Hence, E does not contains nonempty subset D with $D \subseteq D_{D^*}$. Hence, $E \in \mathcal{I}$. □

Theorem 4.2. *The given statements can be equivalent for an ideal space Y and if τ is D^* compatible in $\mathcal{I}$.*

1. *If for all $E \subseteq Y, E \cap E_{D^*} = \phi \Rightarrow E_{D^*} = \phi$,*
2. *If every $E \subseteq Y$, then $(E - E_{D^*})_{D^*} = \phi$,*
3. *If every $E \subseteq Y$, then $(E \cap E_{D^*})_{D^*} = E_{D^*}$.*

Proof. (1) $\Rightarrow$ (2): For each $E \subseteq Y$, $E \cap E_{D^*} = \phi \Rightarrow E_{D^*} = \phi$. Let $F = E - E_{D^*}$, so $F \cap F_{D^*} = (E \cap (Y - E_{D^*})) \cap (E \cap (Y - E_{D^*}))_{D^*} \subseteq [E \cap (Y - E_{D^*})] \cap [E_{D^*} \cap (Y - E_{D^*})_{D^*}] = \phi$. By (1), we have $F_{D^*} = \phi \Rightarrow (E - E_{D^*})_{D^*} = \phi$.

(2) $\Rightarrow$ (3): For each $E \subseteq Y, E = (E - E_{D^*}) \cup (E \cap E_{D^*})$. $E_{D^*} = (E - E_{D^*})_{D^*} \cup (E \cap E_{D^*})_{D^*} = (E \cap E_{D^*})_{D^*}$.

(3) $\Rightarrow$ (1): For all $E \subseteq Y, E \cap E_{D^*} = \phi$ and $(E \cap E_{D^*})_{D^*} = E_{D^*} \Rightarrow \phi = \phi_{D^*} = E_{D^*}$. □

Corollary 4.1. *Suppose Y be an ideal space with E set of Y and τ is D^*-compatible in ideal $\mathcal{I} \Rightarrow E_{D^*} = (E_{D^*})_{D^*}$.*

Proof. Assume $E \subseteq Y$, using Theorem [4.2] and the result in Theorem [2.2], we get $E_{D^*} = (E \cap E_{D^*})_{D^*} \subseteq E_{D^*} \cap (E_{D^*})_{D^*} = (E_{D^*})_{D^*}$. Hence we could have $E_{D^*} \subseteq (E_{D^*})_{D^*}$. Then by the result of Theorem [2.3], $(E_{D^*})_{D^*}$ is a subset of $E_{D^*} \Rightarrow E_{D^*} = (E_{D^*})_{D^*}$. □

Theorem 4.3. *The results can be equivalent to an ideal space Y*

1. *$\tau^D(y) \cap \mathcal{I} = \phi$, such that $\tau^D(y) = \{V \in \tau^D / y \in V\}$,*
2. *When $J \in \mathcal{I} \Rightarrow int_\theta(J) = \phi$,*
3. *$F \subseteq F_{D^*}$, For each clopen F,*
4. *$Y = Y_{D^*}$.*

Proof. (1) $\Rightarrow$ (2): Assume $J \in \mathcal{I}$ and $\tau^D(y) \cap \mathcal{I} = \phi$. Also, assume $y \in int_\theta(I)$. Then $\exists\ W$ belongs to $\tau^D(y)$ in such a way that $y \in W \subseteq cl(W) \subseteq I$. Since $J \in \mathcal{I} \Rightarrow \phi \neq \{y\} \subseteq cl(W) \in \tau^D(y) \cap \mathcal{I}$. This is a contradiction to $\tau^D(y) \cap \mathcal{I} = \phi$. Hence, $int_\theta(J) = \phi$.

(2) $\Rightarrow$ (3): Let $y \in F$. Assume $y \notin F_{D^*}$, such that $V \in \tau^D(y)$ in a way that $F \cap V \in \mathcal{I}$. As F is clopen, by [1], $y \in F \cap V = int(F \cap V) = int_\theta(F \cap (V)) = \phi$. This becomes a contradiction $\Rightarrow y \in F_{D^*}$, Hence $F \subseteq F_{D^*}$.

(3) $\Rightarrow$ (4): When Y is clopen $\Rightarrow Y \subseteq Y_{D^*}$ then by (3) $Y = Y_{D^*}$.

(4) $\Rightarrow$ (1): Assume $Y = Y_{D^*} = \{y \in Y \mid V \cap Y = V \notin \mathcal{I}$ where $V \in \tau^D(y)\}$. So $\tau^D(y) \cap \mathcal{I} = \phi$. □

Theorem 4.4. *Let Y be in $(Y, \tau, \mathcal{I})$ and τ is of D^*-compatible of $\mathcal{I}$. If so for each $S \in \tau^D$, any set T of Y, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} = cl_D(S \cap T_{D^*})$.*

Proof. Let $S \in \tau^D$. Then by Lemma [2.4] and by Theorem[2.2] $(S \cap T_{D^*})_{D^*} \subseteq ((S \cap T)_{D^*})_{D^*} \subseteq (S \cap T)_{D^*}$. Also, $(S \cap (T - T_{D^*}))_{D^*} \subseteq S_{D^*} \cap (T - T_{D^*})_{D^*} = S_{D^*} \cap \phi = \phi$ by Theorem [2.2] and [4.2]. Moreover, $(S \cap T)_{D^*} - (S \cap T_{D^*})_{D^*} \subseteq ((S \cap T) - (S \cap T_{D^*}))_{D^*} = (S \cap (T - T_{D^*}))_{D^*} = \phi \Rightarrow (S \cap T)_{D^*} \subseteq (S \cap T_{D^*})_{D^*}$. Hence, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*}$. Also, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} \subseteq cl_D(S \cap T_{D^*})$, by Theorem [2.2]. By Lemma [2.4], $S \cap T_{D^*} \subseteq (S \cap T)_{D^*} \Rightarrow cl_D(S \cap T_{D^*}) \subseteq cl_D((S \cap T)_{D^*}) = (S \cap T)_{D^*}$. Thus $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} = cl_D(S \cap T_{D^*})$. □

5. η-Operator

Definition 5.1. *An operator η: $\mathcal{P}(Y) \to \tau$ in an ideal space Y is given by the following; for each E belongs to Y, $\eta(E) = \{y \in Y$: $\exists$ a Δ-open set V in a way that $V - E \in \mathcal{I}\}$ and also we have $\eta(E) = Y - (Y - E)_{D^*}$.*

Theorem 5.1. *Suppose Y is an ideal space. If so then the given statements are true:*

(1) *If $M \subseteq Y$, then $\eta(M)$ is open.*
(2) *If M is a subset of $E \Rightarrow \eta(M) \subseteq \eta(E)$.*
(3) *If $M, E \in \mathcal{P}(Y)$, then $\eta(M \cap E)$ equals $\eta(M) \cap \eta(E)$.*
(4) *If $M \subseteq Y$, then $\eta(M)$ equals $\eta(\eta(M))$ if and only if $(Y - M)_{D^*} = ((Y - M)_{D^*})_{D^*}$.*
(5) *If $M \in \mathcal{I}$, then $\eta(M)$ equals $Y - Y_{D^*}$.*
(6) *If $M \subseteq Y and J \in \mathcal{I} \Rightarrow \eta(M - J) = \eta(M)$.*
(7) *If M set of Y, $J \in \mathcal{I} \Rightarrow \eta(M \cup J) = \eta(M)$.*
(8) *If $(M - E) \cup (E - M) \in \mathcal{I}$, then $\eta(M) = \eta(E)$.*

Proof. (1) By Theorem [2.2], M_{D^*} is closed $\Rightarrow (Y - M)_{D^*}$ is closed $\Rightarrow \eta(M)$ is open.

(2) If $M \subseteq E$ in such a way that $(Y - E)$ is a set of $(Y - M) \Rightarrow (Y - E)_{D^*} \subseteq (Y - M)_{D^*}$. Hence $Y - (Y - M)_{D^*}$ is a set of $Y - (Y - E)_{D^*}$, that is $\eta(M) \subseteq \eta(E)$.

(3) $\eta(M \cap E) = Y - (Y - (M \cap E))_{D^*} = Y - [(Y - M) \cup (Y - E)]_{D^*} = Y - [(Y - M)_{D^*} \cup (Y - E)_{D^*}] = [Y - (Y - M)_{D^*}] \cap [Y - (Y - E)_{D^*}] = \eta(M) \cap \eta(E)$.

(4) If $(Y-M)_{D^*} = ((Y-M)_{D^*})_{D^*}$ Then, $\eta(\eta(M)) = \eta(Y-(Y-M)_{D^*}) = Y-(Y-(Y-(Y-M)_{D^*}))_{D^*} = Y-((Y-M)_{D^*})_{D^*} = Y-(Y-M)_{D^*} = \eta(M)$.

(5) By Corollary [2.6.1], we get $(Y-M)_{D^*} = Y_{D^*}$. Therefore, $\eta(M) = Y - Y_{D^*}$.

(6) $\eta(M-J) = Y-[Y-(M-J)]_{D^*} = Y-[(Y-M)-J]_{D^*} = Y-(Y-M)_{D^*} = \eta(M)$, by using Corollary [2.6.1].

(7) $\eta(M\cup J) = Y-[Y-(M\cup J)]_{D^*} = Y-[(Y-M)\cap(Y-J)]_{D^*} = Y-[(Y-M)-J]_{D^*} = Y-(Y-M)_{D^*} = \eta(M)$, by using Corollary [2.6.1].

(8) Assume that, union of $(M-E)$ and $(E-M)$ belongs to $\mathcal{I}$. Let $M-E = I$ and $E-M = J$. We observed that I, J belongs to $\mathcal{I}$ by the property heredity. Also, we can have $E = (M-I)\cup J$. Thus, $\eta(M) = \eta(M-I) = \eta[(M-I)\cup J] = \eta(E)$ by (6) and (7). □

Corollary 5.1. *Suppose Y is in $(Y,\tau,\mathcal{I})$. For each θ-open W set of Y, $W \subseteq \eta(W)$.*

Proof. We have $\eta(W) = Y-(Y-W)_{D^*}$. Then $(Y-W)_{D^*} \subseteq cl_\theta(Y-W) = Y-W$, though $Y-W$ is θ-closed. We can have $W = Y-(Y-W) \subseteq Y-(Y-W)_{D^*} = \eta(W)$. □

6. Conclusion

In this paper, we have defined the local function using Δ-open sets and studied its various properties in ideal topological space. We defined a closure operator and determined whether it was a Kuratowski closure operator. Also, the compatibility property of the topology with the ideal was verified. We then defined a new operator η and its characterizations were studied. Future work in this topic would be to generalize the Δ-closed set, compare it with the already existing generalized closed sets, and verify some of its properties.

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