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GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A QUASILINEAR PARABOLIC EQUATION WITH ABSORPTION AND NONLINEAR BOUNDARY CONDITION

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Abstract. This paper deals with the evolution r-Laplacian equation with absorption and nonlinear boundary condition. By using differential inequality techniques, global existence and blow-up criteria of nonnegative solutions are determined. Moreover, upper bound of the blow-up time for the blow-up solution is obtained.

1. Introduction

In this paper, we investigate the global existence and finite time blow-up of nonnegative solutions for the following initial-boundary value problem

$$(1.1)\qquad \begin{cases} u_t = \operatorname{div}(|\nabla u|^{r-2}\nabla u) - f(u), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{r-2}\dfrac{\partial u}{\partial n} = g(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u\,(x,0) = u_0\,(x) > 0, & x \in \Omega, \end{cases}$$

where $r \geq 2$, $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on the boundary $\partial\Omega$ assumed sufficiently smooth, Ω is a bounded star-shaped region in $\mathbb{R}^N$ $(N \geq 2)$ and t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$. It is well known that the functions f and g may greatly affect the behavior of the solution $u(x,t)$ with the development of time. From the physical standpoint, $-f$ is the cold source function, g is the heat-conduction function transmitting into interior of Ω from the boundary of Ω.

The global existence and blow-up for nonlinear parabolic equations have been extensively investigated by many authors in the last decades (see [1–6] and the references therein). In recent years, many authors have also studied bounds for the blow-up time in nonlinear parabolic problems by using differential inequality techniques (see [7–12]). In particular, Payne et al. [13] considered the following semilinear heat equation with nonlinear boundary condition

$$(1.2)\qquad \begin{cases} u_t = \Delta u - f(u), & (x,t) \in \Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial n} = g(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u\,(x,0) = u_0\,(x)\,, & x \in \Omega, \end{cases}$$

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and established sufficient conditions on the nonlinearities to guarantee that the solution $u(x,t)$ exists for all time $t>0$ or blows up in finite time t^*. Moreover, an upper bound for t^* was derived. Under more restrictive conditions, a lower bound for t^* was also obtained.

Moreover, in [14], Payne et al. also studied the following initial-boundary problem

$$\begin{cases} u_t = \nabla(|\nabla u|^{2p}\nabla u), & (x,t)\in\Omega\times(0,t^*), \\ |\nabla u|^{2p}\dfrac{\partial u}{\partial n} = f(u), & (x,t)\in\partial\Omega\times(0,t^*), \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

and obtained upper and lower bounds for the blow-up time under some conditions when blow-up does occur at some finite time.

In the present work, by using differential inequality techniques, we give some sufficient conditions on the functions f and g for the global existence and blow-up of nonnegative solutions to problem (1.1). Our main results are stated as follows.

Theorem 1.1. *(Conditions for global existence). Let $u(x,t)$ be the solution of problem (1.1) and assume that the non-negative functions f and g satisfy the following conditions*

$$f(\xi)\ge k_1\xi^p, \quad \xi\ge 0, \tag{1.4}$$

$$g(\xi)\le k_2\xi^q, \quad \xi\ge 0, \tag{1.5}$$

for some non-negative constants k_1 and k_2. Moreover suppose that the positive constants p and q satisfy the following conditions

$$p>q>r-1 \ \text{ and } \ rq<(r-1)(p+1). \tag{1.6}$$

Then the non-negative solution $u(x,t)$ of problem (1.1) exists globally for all time $t>0$.

Theorem 1.2. *(Conditions for blow-up in finite time). Let $u(x,t)$ be the solution of problem (1.1) and assume that the non-negative functions f and g satisfy the following conditions*

$$\xi f(\xi)\le rF(\xi), \quad \xi\ge 0, \tag{1.7}$$

$$\xi g(\xi)\ge rG(\xi), \quad \xi\ge 0, \tag{1.8}$$

with

$$F(\xi)=\int_0^\xi f(\eta)d\eta, \quad G(\xi)=\int_0^\xi g(\eta)d\eta. \tag{1.9}$$

Moreover suppose that $\Psi(0)>0$, where

$$\Psi(t)=r\int_{\partial\Omega}G(u)ds-\int_\Omega|\nabla u|^r dx-r\int_\Omega F(u)dx. \tag{1.10}$$

Then the solution $u(x,t)$ of problem (1.1) blows up at time $t^*<T$ with

$$T=\frac{\Phi(0)}{(r-2)\Psi(0)}, \quad \text{for } r>2, \tag{1.11}$$

where $\Phi(t)=\int_\Omega u^2dx$. *If* $r=2$, *we have* $T=\infty$.

This paper is organized as follows. In Section 2, we establish the conditions on the functions f and g, which guarantee that $u(x,t)$ exists globally, and prove Theorem 1.1. In Section 3, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*.

2. Conditions for global existence

In this section, we establish the sufficient conditions on the functions f and g, which guarantee that $u(x,t)$ exists globally, and prove Theorem 1.1. To do this, we need the following Lemma.

Lemma 2.1. *Let* Ω *be a bounded star-shaped domain in* $\mathbb{R}^N$, $N\geq 2$. *Then for any non-negative* C^1 *function* u *and* $\gamma>0$, *we have*

$$\int_{\partial\Omega}u^\gamma ds\leq \frac{N}{\rho_0}\int_\Omega u^\gamma dx+\frac{\gamma d}{\rho_0}\int_\Omega u^{\gamma-1}|\nabla u|dx, \tag{2.1}$$

where

$$\rho_0=\min_{x\in\partial\Omega}(x\cdot n)\ \ \textit{and}\ \ d=\max_{x\in\partial\Omega}|x|. \tag{2.2}$$

Proof. As Ω is a bounded star-shaped domain, it is easy to see that $\rho_0>0$. Integrating the identity

$$\operatorname{div}(u^\gamma x)=Nu^\gamma+\gamma u^{\gamma-1}(x\cdot\nabla u) \tag{2.3}$$

over Ω, it follows from the divergence theorem that

$$\int_{\partial\Omega}u^\gamma(x\cdot n)ds=N\int_\Omega u^\gamma dx+\gamma\int_\Omega u^{\gamma-1}(x\cdot\nabla u)dx. \tag{2.4}$$

By the definition of ρ_0 and d, we obtain

$$\rho_0\int_{\partial\Omega}u^\gamma ds\leq\int_{\partial\Omega}u^\gamma(x\cdot n)ds\leq N\int_\Omega u^\gamma dx+\gamma d\int_\Omega u^{\gamma-1}|\nabla u|dx, \tag{2.5}$$

which implies the desired conclusion.

Proof of Theorem 1.1. Setting

$$\Phi(t)=\int_\Omega u^2dx, \tag{2.6}$$

then it follows from (1.1), (1.4) and (1.5) that

$$\begin{aligned}\Phi'(t)&=2\int_\Omega uu_tdx\\&=2\int_\Omega u[\operatorname{div}(|\nabla u|^{r-2}\nabla u)-f(u)]dx\\&=2\int_{\partial\Omega}u|\nabla u|^{r-2}\frac{\partial u}{\partial n}ds-2\int_\Omega|\nabla u|^rdx-2\int_\Omega uf(u)dx\\&=2\int_{\partial\Omega}ug(u)ds-2\int_\Omega|\nabla u|^rdx-2\int_\Omega uf(u)dx\\&\leq 2k_2\int_{\partial\Omega}u^{q+1}ds-2\int_\Omega|\nabla u|^rdx-2k_1\int_\Omega u^{p+1}dx.\end{aligned} \tag{2.7}$$

By Lemma 2.1, we have

$$\int_{\partial\Omega} u^{q+1} ds \le \frac{N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx, \tag{2.8}$$

where ρ_0 and d are given by (2.2). Combining (2.7) with (2.8), we obtain

$$\Phi'(t) \le \frac{2k_2N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{2k_2(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx - 2\int_\Omega |\nabla u|^r dx - 2k_1\int_\Omega u^{p+1}dx. \tag{2.9}$$

By using Young's inequality with $\varepsilon > 0$, we derive

$$\int_\Omega u^q|\nabla u|dx \le \frac{1}{r\varepsilon}\int_\Omega |\nabla u|^r dx + \frac{r-1}{r}\varepsilon^{\frac{1}{r-1}}\int_\Omega u^{\frac{rq}{r-1}}dx, \tag{2.10}$$

where $\varepsilon = \frac{k_2(q+1)d}{r\rho_0} > 0$. It follows from (2.9) and (2.10) that

$$\Phi'(t) \le \frac{2k_2N}{\rho_0}\int_\Omega u^{q+1}dx + 2(r-1)\left(\frac{k_2(q+1)d}{r\rho_0}\right)^{\frac{r}{r-1}}\int_\Omega u^{\frac{rq}{r-1}}dx - 2k_1\int_\Omega u^{p+1}dx. \tag{2.11}$$

By Hölder's inequality , we have

$$\int_\Omega u^{\frac{rq}{r-1}}dx \le \left(\int_\Omega u^{q+1}dx\right)^{\alpha}\left(\int_\Omega u^{p+1}dx\right)^{1-\alpha}, \tag{2.12}$$

where $\alpha = \frac{(r-1)(p+1)-rq}{(r-1)(p-q)} \in (0,1)$, due to (1.6). By using the fundamental inequality

$$a_1^{r_1}a_2^{r_2} \le r_1a_1 + r_2a_2, \quad a_1, a_2 > 0, r_1, r_2 \ge 0 \text{ and } r_1 + r_2 = 1, \tag{2.13}$$

it follows from (2.12) that

$$\begin{aligned}\int_\Omega u^{\frac{rq}{r-1}}dx &\le \left(\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx\right)^{\alpha}\left(\kappa\int_\Omega u^{p+1}dx\right)^{1-\alpha}\\ &\le \alpha\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx + (1-\alpha)\kappa\int_\Omega u^{p+1}dx,\end{aligned} \tag{2.14}$$

where

$$0 < \kappa < \frac{k_1}{(r-1)(1-\alpha)}\left(\frac{k_2(q+1)d}{r\rho_0}\right)^{\frac{-r}{r-1}}. \tag{2.15}$$

Combining (2.11) with (2.14), we obtain

$$\Phi'(t) \le K_1\int_\Omega u^{q+1}dx - K_2\int_\Omega u^{p+1}dx, \tag{2.16}$$

where

$$K_1 = \frac{2k_2N}{\rho_0} + 2(r-1)\alpha\kappa^{\frac{\alpha-1}{\alpha}}\left(\frac{k_2(q+1)d}{r\rho_0}\right)^{\frac{r}{r-1}} > 0, \tag{2.17}$$

and

$$K_2 = 2k_1 - 2(r-1)(1-\alpha)\kappa\left(\frac{k_2(q+1)d}{r\rho_0}\right)^{\frac{r}{r-1}} > 0, \tag{2.18}$$

due to (2.15). According to Hölder's inequality, we derive

$$\int_\Omega u^{q+1}dx \le \left(\int_\Omega u^{p+1}dx\right)^{\frac{q+1}{p+1}}|\Omega|^{\frac{p-q}{p+1}}, \tag{2.19}$$

where $|\Omega| = \int_\Omega dx$ is the N-volume of Ω. It follows from (2.16) and (2.19) that

$$\Phi'(t) \le \left(\int_\Omega u^{p+1}dx\right)^{\frac{q+1}{p+1}} \left[K_1|\Omega|^{\frac{p-q}{p+1}} - K_2\left(\int_\Omega u^{p+1}dx\right)^{\frac{p-q}{p+1}}\right]. \tag{2.20}$$

By Hölder's inequality again, we have

$$\Phi(t) = \int_\Omega u^2 dx \le \left(\int_\Omega u^{p+1}dx\right)^{\frac{2}{p+1}} |\Omega|^{\frac{p-1}{p+1}}. \tag{2.21}$$

Therefore, we deduce from (2.20) and (2.21) that

$$\Phi'(t) \le \left(\int_\Omega u^{p+1}dx\right)^{\frac{q+1}{p+1}} \left[K_1|\Omega|^{\frac{p-q}{p+1}} - K_2|\Omega|^{\frac{(1-p)(p-q)}{2(p+1)}}\Phi^{\frac{p-q}{2}}\right]. \tag{2.22}$$

Hence, we infer from (2.22) that $\Phi(t)$ is decreasing in each time interval on which we have

$$\Phi(t) > \left(\frac{K_1}{K_2}\right)^{\frac{2}{p-q}} |\Omega|, \tag{2.23}$$

so that $\Phi(t)$ remains bounded for all time under the conditions stated in Theorem 1.1, which completes the proof. □

3. Conditions for blow-up in finite time

In this section, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*.

Proof of Theorem 1.2. Using Green formula and the assumptions stated in Theorem 1.2, we have

$$\begin{aligned}
\Phi'(t) &= 2\int_\Omega uu_t dx \\
&= 2\int_\Omega u[\operatorname{div}(|\nabla u|^{r-2}\nabla u) - f(u)]dx \\
&= 2\int_{\partial\Omega} u|\nabla u|^{r-2}\frac{\partial u}{\partial n}ds - 2\int_\Omega |\nabla u|^r dx - 2\int_\Omega uf(u)dx \\
&= 2\int_{\partial\Omega} ug(u)ds - 2\int_\Omega |\nabla u|^r dx - 2\int_\Omega uf(u)dx \\
&\ge 2r\int_{\partial\Omega} G(u)ds - 2\int_\Omega |\nabla u|^r dx - 2r\int_\Omega F(u)dx \\
&\ge 2\Psi(t).
\end{aligned} \tag{3.1}$$

Differentiating (1.10), we obtain

$$\begin{aligned}
\Psi'(t) &= r\int_{\partial\Omega} u_t g(u)ds - \int_\Omega (|\nabla u|^r)_t dx - r\int_\Omega u_t f(u)dx \\
&= r\int_\Omega u_t \operatorname{div}(|\nabla u|^{r-2}\nabla u)dx - r\int_\Omega u_t f(u)dx \\
&= r\int_\Omega (u_t)^2 dx \ge 0.
\end{aligned} \tag{3.2}$$

As $\Psi(0) > 0$, then $\Psi(t) > 0$ for all $t \in (0, t^*)$. By using Hölder's inequality , we derive

$$(3.3) \qquad (\Phi'(t))^2 = 4\left(\int_\Omega u u_t dx\right)^2 \leq 4\int_\Omega u^2 dx \int_\Omega (u_t)^2 dx = \frac{4}{r}\Phi(t)\Psi'(t).$$

It follows from (3.1) and (3.3) that

$$(3.4) \qquad \Phi(t)\Psi'(t) \geq \frac{r}{4}(\Phi'(t))^2 \geq \frac{r}{2}\Phi'(t)\Psi(t),$$

that is

$$(3.5) \qquad (\Phi^{-\frac{r}{2}}\Psi)'(t) \geq 0.$$

Integrating from 0 to t, we have

$$(3.6) \qquad \Phi^{-\frac{r}{2}}(t)\Psi(t) \geq \Phi^{-\frac{r}{2}}(0)\Psi(0) =: K > 0.$$

Therefore, we deduce from (3.1) that

$$(3.7) \qquad \Phi'(t) \geq 2\Psi \geq 2K\Phi^{\frac{r}{2}}(t).$$

If $r > 2$, it follows from integrating over $(0, t)$ that

$$(3.8) \qquad \Phi(t) \geq \left[\Phi^{\frac{2-r}{2}}(0) - K(r-2)t\right]^{-\frac{2}{r-2}},$$

which implies $\Phi(t) \to +\infty$ as $t \to T = \frac{\Phi^{\frac{2-r}{2}}(0)}{K(r-2)} = \frac{\Phi(0)}{(r-2)\Psi(0)}$. Hence, for $r > 2$, we have

$$(3.9) \qquad t^* \leq \frac{\Phi(0)}{(r-2)\Psi(0)}.$$

If $r = 2$, we infer from (3.7) that

$$(3.10) \qquad \Phi(t) \geq \Phi(0)e^{2Kt}, \quad \text{for all} \quad t > 0,$$

which implies $t^* = \infty$, this completes the proof. $\square$

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