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On Some Applications of P-Contraction Fixed Point Theorems in Bipolar Metric Spaces

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Abstract. In this paper, we have discussed the P-contraction type fixed point theorems of covariant mappings in bipolar metric spaces, and we have shown an example which supports our results. Also we discussed applications to integral equations and Homotopy theory.

1. Introduction

The concept of a metric space has numerous generalizations in the field of literature. A recent expansion in the field is the concept of a bipolar metric space, which was presented by Mutlu and Gurdal [1].

Mutlu extended certain coupled fixed-point theorems, which can be seen as a generalization of the Banach fixed-point theorem, to bipolar metric spaces. Kishore et al. recently proved the existence and uniqueness of common coupled fixed-point results for three covariant mappings in bipolar metric spaces.

Furthermore, with significant applications, Kishore et al. [3] proved a few common fixed point theorems in a bipolar metric space, whereas Mutlu et al.([1], [2]) The associated fixed point results and the principle of locally and weakly contractive mappings in bipolar metric spaces were demonstrated. Therefore, fixed point theory of bipolar metric space is a field of active research that is attracting a lot of interest for further investigation(see [1]- [15] and references therein).

In order to gain a new awareness of fixed point theory in bipolar metric space, we will explore the concepts of P-contraction mappings in this study. A few fixed point theorems that generalize

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earlier results found in the literature will be presented. Additionally, applications to Homotopy and integral equations are given with suitable and pertinent examples.

Definition 1.1 ([1]). *A mapping d is a Bipolar metric if $d : \mathfrak{S} \times \mathfrak{T} \to [0, \infty)$ such that*

(B_1) $d(\mathfrak{z}, \mathfrak{y}) = 0$ *implies that* $\mathfrak{z} = \mathfrak{y}$;
(B_2) $\mathfrak{z} = \mathfrak{y}$ *implies that* $d(\mathfrak{z}, \mathfrak{y}) = 0$;
(B_3) *if* $\mathfrak{z}, \mathfrak{y} \in \mathfrak{S} \cap \mathfrak{T}$, *then* $d(\mathfrak{z}, \mathfrak{y}) = d(\mathfrak{y}, \mathfrak{z})$;
(B_4) $d(\mathfrak{z}_1, \mathfrak{y}_2) \leq d(\mathfrak{z}_1, \mathfrak{y}_1) + d(\mathfrak{z}_2, \mathfrak{y}_1) + d(\mathfrak{z}_2, \mathfrak{y}_2)$,

for all $\mathfrak{z}, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{S}$ *and* $\mathfrak{y}, \mathfrak{y}_1, \mathfrak{y}_2 \in \mathfrak{T}$, *and the triple* $(\mathfrak{S}, \mathfrak{T}, d)$ *is called a Bipolar-metric space(BMS).*

Definition 1.2 ([1]). *Let* $\Omega : \mathfrak{S}_1 \cup \mathfrak{T}_1 \to \mathfrak{S}_2 \cup \mathfrak{T}_2$ *be a function defined on two pairs* $(\mathfrak{S}_1, \mathfrak{T}_1)$ *and* $(\mathfrak{S}_2, \mathfrak{T}_2)$ *is said to be*

(i) *covariant if* $\Omega(\mathfrak{S}_1) \subseteq \mathfrak{S}_2$ *and* $\Omega(\mathfrak{T}_1) \subseteq \mathfrak{T}_2$. *This is denoted as* $\Omega : (\mathfrak{S}_1, \mathfrak{T}_1) \rightrightarrows (\mathfrak{S}_2, \mathfrak{T}_2)$;
(ii) *contravariant if* $\Omega(\mathfrak{S}_1) \subseteq \mathfrak{T}_2$ *and* $\Omega(\mathfrak{T}_1) \subseteq \mathfrak{S}_2$. *It is denoted as* $\Omega : (\mathfrak{S}_1, \mathfrak{T}_1) \leftrightarrows (\mathfrak{S}_2, \mathfrak{T}_2)$.

Particularly, if d_1 is bipolar metrics on $(\mathfrak{S}_1, \mathfrak{T}_1)$ and d_2 is bipolar metrics on $(\mathfrak{S}_2, \mathfrak{T}_2)$, we often write $\Omega : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \rightrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ *and*
$\Omega : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \leftrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ *respectively.*

Definition 1.3 ([1]). *If $\xi \in \mathfrak{S} \cup \mathfrak{T}$, then $(\mathfrak{S}, \mathfrak{T}, d)$ is a BMS. If $\xi \in \mathfrak{S}$, then it is a left point; if $\xi \in \mathfrak{S}$, then it is a right point; if $\xi \in \mathfrak{S} \cap \mathfrak{T}$, then it is a central point. Furthermore, the sequences $\{_i\}$ in $\mathfrak{T}$ and $\{\mathfrak{Z}_i\}$ in $\mathfrak{S}$ are respectively left and right. We refer to a sequence in a BPMS as either left or right. When $\{\xi_i\}$ is a left sequence, ξ is a right point, and $\lim_{i\to\infty} d(\xi_i, \xi) = 0$, or when $\{\xi_i\}$ is a right sequence, ξ is a left point, and $\lim_{i\to\infty} d(\xi, \xi_i) = 0$. A sequence on $\mathfrak{S} \times \mathfrak{T}$ is the bisequence $(\{\mathfrak{Z}_i\}, \{_i\})$ on $(\mathfrak{S}, \mathfrak{T}, d)$. $(\{\mathfrak{Z}_i\}, \{_i\})$ is convergent in the scenario when $\{\mathfrak{Z}_i\}$ and $\{_i\}$ are both convergent.*
If $\lim_{i,j\to\infty} d(\mathfrak{Z}_i, _j) = 0$, then the bi-sequence $(\{\mathfrak{Z}_i\}, \{_i\})$ is a Cauchy bisequence.

Every convergent Cauchy bisequence is biconvergent, as you can see. If every Cauchy bisequence is convergent, then the BMS is complete (and so it is biconvergent).

Definition 1.4 ([3]). *Consider two BMS: $(\mathfrak{S}_1, \mathfrak{T}_1, d_1)$ and $(\mathfrak{S}_2, \mathfrak{T}_2, d_2)$.*

(a) *At a point $\xi_0 \in \mathfrak{S}_1$, a map $\mathfrak{P} : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \rightrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ is said to be left-continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1(\xi_0,) < \delta$ implies $d_2(\mathfrak{P}(\xi_0), \mathfrak{P}()) < \epsilon$ all $\in \mathfrak{T}_1$.*
(b) *At a point $_0 \in \mathfrak{T}_1$, a map $\mathfrak{P} : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \rightrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ is said to be right-continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1(\xi, _0) < \delta$ implies $d_2(\mathfrak{P}(\xi), \mathfrak{P}(_0)) < \epsilon$ for all $\xi \in \mathfrak{S}_1$.*
(c) *A map $\mathfrak{P}$ is called continuous, if it is left-continuous at each point $\xi \in \mathfrak{S}_1$ and right-continuous at each point $\in \mathfrak{T}_1$.*
(d) *A contravariant map $\mathfrak{P} : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \leftrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ is continuous if and only if it is continuous as a covariant map $\mathfrak{P} : (\mathfrak{S}_1, \mathfrak{T}_1, d_1) \rightrightarrows (\mathfrak{S}_2, \mathfrak{T}_2, d_2)$*

A covariant or contravariant map $\mathfrak{P}$ from $(\mathfrak{S}_1, \mathfrak{T}_1, d_1)$ to $(\mathfrak{S}_2, \mathfrak{T}_2, d_2)$ is considered continuous if and only if $(_z) \to$ on $(\mathfrak{S}_1, \mathfrak{T}_1, d_1)$ implies $(\mathfrak{P}(_z)) \to \mathfrak{P}()$ on $(\mathfrak{S}_2, \mathfrak{T}_2, d_2)$. This is evident from the Definition 1.4.

Definition 1.5 ([3]). *Let $(\mathfrak{S}, \mathfrak{T}, d)$ be a BMS and $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}) \rightrightarrows (\mathfrak{S}, \mathfrak{T})$ be two covariant mappings. A pair $(\mathfrak{P}, \mathfrak{R})$ is called a compatible if and only if $\lim_{i\to\infty} d(\mathfrak{P}\mathfrak{R}\mathfrak{Z}_i, \mathfrak{R}\mathfrak{P}_i) = \lim_{i\to\infty} d(\mathfrak{R}\mathfrak{P}\mathfrak{Z}_i, \mathfrak{P}\mathfrak{R}_i) = 0$ whenever, $(\{\mathfrak{Z}_i\}, \{_i\})$ is a sequence in $(\mathfrak{S}, \mathfrak{T})$ such that $\lim_{i\to\infty} \mathfrak{P}\mathfrak{Z}_i = \lim_{i\to\infty} \mathfrak{P}_i = \lim_{i\to\infty} \mathfrak{R}\mathfrak{Z}_i = \lim_{i\to\infty} \mathfrak{R}_i =$ for some $\in \mathfrak{S} \cap \mathfrak{T}$.*

Definition 1.6 ([16]). *Let $(\mathfrak{S}, d)$ be an M-metric space and $\mathfrak{P} : \mathfrak{S} \to \mathfrak{S}$ be a mapping. Then, $\mathfrak{P}$ is said to be a P-contraction mapping, if there exists $L \in [0, 1)$ such that for all $, \in \mathfrak{S}$;*
$d(\mathfrak{P}, \mathfrak{P}) \leq L\,(d(,) + |d(, \mathfrak{P}) - d(, \mathfrak{P})|)$
and $\mathfrak{P}$ is said to be a P-contractive mapping, if it satisfies
$d(\mathfrak{P}, \mathfrak{P}) < d(,) + |d(, \mathfrak{P}) - d(, \mathfrak{P})|$ with $d(,) > 0$, also $\mathfrak{P}$ is said to have 0-property if $d(,) = 0 \Leftrightarrow d(\mathfrak{P}, \mathfrak{P}) = 0$ holds for all $\in \mathfrak{S}$
For more information about P-contraction in the literature, we suggest to readers the papers (see. [16]- [20]).

2. Main Results

Definition 2.1. *Let $(\mathfrak{S}, \mathfrak{T}, d)$ be a BMS.*
(i) A covariant mappings $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}) \rightrightarrows (\mathfrak{S}, \mathfrak{T})$ are called a P-covariant contraction mappings if there exists $\ell \in [0, \frac{1}{3})$ such that $\forall \in \mathfrak{S}, \in \mathfrak{T}$

$$d(\mathfrak{P}, \mathfrak{P}) \leq \ell\,[d(\mathfrak{R}, \mathfrak{R}) + |d(\mathfrak{R}, \mathfrak{P}) - d(\mathfrak{P}, \mathfrak{R})|] \tag{2.1}$$

(ii) A contravariant mappings $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}) \leftrightarrows (\mathfrak{S}, \mathfrak{T})$ are called a P-contravariant contraction mappings if there exists $\ell \in [0, \frac{1}{3})$ such that $\forall\ (,) \in (\mathfrak{S}, \mathfrak{T})$

$$d(\mathfrak{P}, \mathfrak{P}) \leq \ell\,[d(\mathfrak{R}, \mathfrak{R}) + |d(\mathfrak{R}, \mathfrak{P}) - d(\mathfrak{P}, \mathfrak{R})|] \tag{2.2}$$

Theorem 2.2. *Assume that the BMS $(\mathfrak{S}, \mathfrak{T}, d)$ is complete. It can be assumed that $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}) \rightrightarrows (\mathfrak{S}, \mathfrak{T})$ be two covariant mappings satiesfies P-covariant contraction and assume that*

- (i_0) *$\mathfrak{P}(\mathfrak{S} \cup \mathfrak{T}) \subseteq \mathfrak{R}(\mathfrak{S} \cup \mathfrak{T})$,*
- (i_1) *pair $(\mathfrak{P}, \mathfrak{R})$ is compatible,*
- (i_2) *$\mathfrak{R}$ is continuous.*

Next, in $\mathfrak{S} \cup \mathfrak{T}$, there is a UCFP (unique common fixed point) for $\mathfrak{P}$ and $\mathfrak{R}$.

Proof. Let $_0 \in \mathfrak{S}$ and $_0 \in \mathfrak{T}$ be arbitrary, and from (i_0), we construct the bisequences $(\{\mathfrak{Z}_\kappa\}, \{\mathfrak{Y}_\kappa\})$ in $(\mathfrak{S}, \mathfrak{T})$ as

$$\mathfrak{P}_\kappa = \mathfrak{R}_{\kappa+1} = \mathfrak{Z}_\kappa, \quad \mathfrak{P}_\kappa = \mathfrak{R}_{\kappa+1} = \mathfrak{Y}_\kappa \quad \text{where } \kappa = 0, 1, 2, \ldots.$$

Then from (2.1), we can get

$$\begin{aligned}
d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1}) &= d(\mathfrak{P}_\kappa, \mathfrak{P}_{\kappa+1}) \\
&\leq \ell\left[d(\mathfrak{R}_\kappa, \mathfrak{R}_{\kappa+1}) + |d(\mathfrak{R}_\kappa, \mathfrak{P}_{\kappa+1}) - d(\mathfrak{P}_\kappa, \mathfrak{R}_{\kappa+1})|\right] \\
&\leq \ell\left[d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) + |d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa+1}) - d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa)|\right] \\
&\leq \ell\left[d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) + |d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1}) - d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa)|\right] \\
&\leq \ell\left[2d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1})\right].
\end{aligned}$$

Therefore,

$$d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1}) \leq \frac{2\ell}{1-\ell} d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa).$$

letting $\xi = \frac{2\ell}{1-\ell} < 1$, then we have

$$\begin{aligned}
d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1}) &\leq \xi d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) \\
&\leq \xi^2 d(\mathfrak{Z}_{\kappa-2}, \mathfrak{Y}_{\kappa-1}) \\
&\vdots \\
&\leq \xi^\kappa d(\mathfrak{Z}_0, \mathfrak{Y}_1) \to 0 \text{ as } \kappa \to \infty. \qquad (2.3)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_\kappa) &= d(\mathfrak{P}_{\kappa+1}, \mathfrak{P}_\kappa) \\
&\leq \ell\left[d(\mathfrak{R}_{\kappa+1}, \mathfrak{R}_\kappa) + |d(\mathfrak{R}_{\kappa+1}, \mathfrak{P}_\kappa) - d(\mathfrak{P}_{\kappa+1}, \mathfrak{R}_\kappa)|\right] \\
&\leq \ell\left[d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) + |d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) - d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa-1})|\right] \\
&\leq \ell\left[d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) + |d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_\kappa) - d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa-1})|\right] \\
&\leq \ell\left[2d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_\kappa)\right].
\end{aligned}$$

Therefore,

$$d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_\kappa) \leq \frac{2\ell}{1-\ell} d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}).$$

letting $\xi = \frac{2\ell}{1-\ell} < 1$, then we have

$$\begin{aligned}
d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_\kappa) &\leq \xi d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) \\
&\leq \xi^2 d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-2}) \\
&\vdots \\
&\leq \xi^\kappa d(\mathfrak{Z}_1, \mathfrak{Y}_0) \to 0 \text{ as } \kappa \to \infty. \qquad (2.4)
\end{aligned}$$

Moreover,

$$\begin{aligned} d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) &= d(\mathfrak{P}_\kappa, \mathfrak{P}_\kappa) \\ &\leq \ell\left[d(\mathfrak{R}_\kappa, \mathfrak{R}_\kappa) + |d(\mathfrak{R}_\kappa, \mathfrak{P}_\kappa) - d(\mathfrak{P}_\kappa, \mathfrak{R}_\kappa)|\right] \\ &\leq \ell\left[d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}) + |d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_\kappa) - d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1})|\right] \\ &\leq \ell\left[d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}) + |d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) - d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa-1})|\right] \\ &\leq \ell\left[2d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa)\right]. \end{aligned}$$

Therefore,

$$d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) \leq \frac{2\ell}{1-\ell} d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}).$$

letting $\xi = \frac{2\ell}{1-\ell} < 1$, then we have

$$\begin{aligned} d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) &\leq \xi d(\mathfrak{Z}_{\kappa-1}, \mathfrak{Y}_{\kappa-1}) \\ &\leq \xi^2 d(\mathfrak{Z}_{\kappa-2}, \mathfrak{Y}_{\kappa-2}) \\ &\vdots \\ &\leq \xi^\kappa d(\mathfrak{Z}_0, \mathfrak{Y}_0) \to 0 \text{ as } \kappa \to \infty. \end{aligned} \tag{2.5}$$

For each $\kappa, \delta \in \mathbb{N}$ and $\Upsilon = d(\mathfrak{Z}_0, \mathfrak{Y}_1) + d(\mathfrak{Z}_0, \mathfrak{Y}_0)$, $\mathfrak{N} = d(\mathfrak{Z}_1, \mathfrak{Y}_0) + d(\mathfrak{Z}_0, \mathfrak{Y}_0)$ and $\mathfrak{K}_\kappa = \frac{\xi^\kappa}{1-\xi}\Upsilon$. Then, from (2.3), (2.4), (2.5) and using property (B_4), we have

$$\begin{aligned} d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_\kappa) &\leq d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa) \\ &\leq d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_{\kappa+1}) + \xi^\kappa \Upsilon \\ &\leq d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_{\kappa+2}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa+2}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa+1}) + \xi^\kappa \Upsilon \\ &\leq d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_{\kappa+2}) + (\xi^{\kappa+1} + \xi^\kappa)\Upsilon \\ &\vdots \\ &\leq d(\mathfrak{Z}_{\kappa+\delta}, \mathfrak{Y}_{\kappa+\delta}) + (\xi^{\kappa+\delta-1} + \cdots + \xi^{\kappa+1} + \xi^\kappa)\Upsilon \\ &\leq (\xi^{\kappa+\delta} + \xi^{\kappa+\delta-1} + \cdots + \xi^{\kappa+1} + \xi^\kappa)\Upsilon \\ &\leq \xi^\kappa \Upsilon \sum_{i=0}^{\infty} \xi^i = \frac{\xi^\kappa}{1-\xi}\Upsilon = \mathfrak{K}_\kappa \end{aligned}$$

and similarly, we can prove that

$$d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa+\delta}) \leq \mathfrak{K}_\kappa.$$

Let $\epsilon > 0$, since $\xi \in [0, 1)$, there exists an $\kappa_0 \in \mathbb{N}$ such that $\mathfrak{K}_{\kappa_0} = \frac{\xi^{\kappa_0}}{1-\xi}\Upsilon < \frac{\epsilon}{3}$. Then

$$d(\mathfrak{Z}_\kappa, \mathfrak{Y}_\delta) \leq d(\mathfrak{Z}_\kappa, \mathfrak{Y}_{\kappa_0}) + d(\mathfrak{Z}_{\kappa_0}, \mathfrak{Y}_{\kappa_0}) + d(\mathfrak{Z}_{\kappa_0}, \mathfrak{Y}_\delta) \leq 3\mathfrak{K}_{\kappa_0} < \epsilon$$

and the bisequences $(\{\mathfrak{Z}_\kappa\}, \{\mathfrak{Y}_\kappa\})$ are a Cauchy sequence in $(\mathfrak{S}, \mathfrak{T})$. In other words, $(\mathfrak{Z}_\kappa, \mathfrak{Y}_\kappa)$ is a Cauchy bi-sequence. $(\{\mathfrak{Z}_\kappa\}, \{\mathfrak{Y}_\kappa\})$ converges and biconverges to a point $\in \mathfrak{S} \cap \mathfrak{T}$ since $(\mathfrak{S}, \mathfrak{T}, d)$ is complete.So that

$$\lim_{\kappa\to\infty} \mathfrak{Z}_\kappa = \ = \lim_{\kappa\to\infty} \mathfrak{Y}_\kappa \tag{2.6}$$

That is

$$\lim_{\kappa\to\infty} \mathfrak{P}_\kappa = \lim_{\kappa\to\infty} \mathfrak{R}_{\kappa+1} = \lim_{\kappa\to\infty} \mathfrak{P}_\kappa = \lim_{\kappa\to\infty} \mathfrak{R}_{\kappa+1} =$$

Since $\mathfrak{R}$ is continuous function, we have

$$\begin{aligned} &\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_\kappa = \mathfrak{R} \quad \lim_{\kappa\to\infty} \mathfrak{R}^2{}_{\kappa+1} = \mathfrak{R} \\ &\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_\kappa = \mathfrak{R} \quad \lim_{\kappa\to\infty} \mathfrak{R}^2{}_{\kappa+1} = \mathfrak{R} \end{aligned} \tag{2.7}$$

Since the pair $\{\mathfrak{P}, \mathfrak{R}\}$ is compatible, we have
$\lim_{\kappa\to\infty} d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{R}\mathfrak{P}_\kappa) = \lim_{\kappa\to\infty} d(\mathfrak{R}\mathfrak{P}_\kappa, \mathfrak{P}\mathfrak{R}_{\kappa+1}) = 0.$
Therefore,

$$\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_\kappa = \lim_{\kappa\to\infty} \mathfrak{P}\mathfrak{R}_{\kappa+1} = \mathfrak{R} \quad \lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_\kappa = \lim_{\kappa\to\infty} \mathfrak{P}\mathfrak{R}_{\kappa+1} = \mathfrak{R}. \tag{2.8}$$

Taking $= \mathfrak{R}_{\kappa+1}$ and $= {}_\kappa$ in (2.1), we get

$$d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{P}_\kappa) \le \ell\Big[d(\mathfrak{R}^2{}_{\kappa+1}, \mathfrak{R}_\kappa) + |d(\mathfrak{R}^2{}_{\kappa+1}, \mathfrak{P}_\kappa) - d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{R}_\kappa)|\Big].$$

Letting $\kappa \to \infty$ in this inequality, by (2.6), (2.7),(2.9), we obtain

$$d(\mathfrak{R},) \le \ell\,[d(\mathfrak{R},) + |d(\mathfrak{R},) - d(\mathfrak{R},)|] \le \ell d(\mathfrak{R},).$$

This is possible only if $d(\mathfrak{R},) = 0$. That is $\mathfrak{R} = $. By using the condition (2.1) and (B_4), we obtain

$$\begin{aligned} d(\mathfrak{P},) &\le d(\mathfrak{P}, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_{\kappa+1},) \\ &\le d(\mathfrak{P}, \mathfrak{P}_{\kappa+1}) + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_{\kappa+1},) \\ &\le \ell\,[d(\mathfrak{R}, \mathfrak{R}_{\kappa+1}) + |d(\mathfrak{R}, \mathfrak{P}_{\kappa+1}) - d(\mathfrak{P}, \mathfrak{R}_{\kappa+1})|] + d(\mathfrak{Z}_{\kappa+1}, \mathfrak{Y}_{\kappa+1}) + d(\mathfrak{Z}_{\kappa+1},) \\ &\to 0 \text{ as } \kappa \to \infty. \end{aligned}$$

Consequently, $\mathfrak{P} = $. $\mathfrak{P} = \mathfrak{R} = $ as a result. We start by supposing $\aleph$ to be an additional fixed point of the covariant maps $\mathfrak{P}$ and $\mathfrak{R}$. This allows us to demonstrate the uniqueness. Consequently, $\mathfrak{P}\aleph = \mathfrak{R}\aleph = \aleph$ implies that $\aleph \in \mathfrak{S} \cap \mathfrak{T}$. From this, we obtain

$$\begin{aligned} d(, \aleph) &= d(\mathfrak{P}, \mathfrak{P}\aleph) \\ &\le \ell\,[d(\mathfrak{R}, \mathfrak{R}\aleph) + |d(\mathfrak{R}, \mathfrak{P}\aleph) - d(\mathfrak{P}, \mathfrak{R}\aleph)|] \le \ell d(, \aleph) \end{aligned}$$

a contradiction. Consequently, we have $= \aleph$. This shows that is the UCFP of $\mathfrak{P}$ and $\mathfrak{R}$ in $\mathfrak{S} \cup \mathfrak{T}$. The proof is completed. □

Corollary 2.3. *Assume that the BMS $(\mathfrak{S}, \mathfrak{T}, d)$ is complete. Assume that a covariant mapping satisfying P-contraction is $\mathfrak{P} : (\mathfrak{S}, \mathfrak{T}) \rightrightarrows (\mathfrak{S}, \mathfrak{T})$.*
In $\mathfrak{S} \cup \mathfrak{T}$, there is a unique fixed point for $\mathfrak{P}$.

Proof. Theorem (2.2) provides the proof, which can be obtained by assuming
$\mathfrak{R} = I_{\mathfrak{S}\cup\mathfrak{T}}$. □

Theorem 2.4. *Assume that the BMS $(\mathfrak{S}, \mathfrak{T}, d)$ is complete. It can be assumed that $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}) \leftrightarrows (\mathfrak{S}, \mathfrak{T})$ be two contravariant mappings satisfying P-contravariant contraction and assume that*

- (i_0) $\mathfrak{P}(\mathfrak{S} \cup \mathfrak{T}) \subseteq \mathfrak{R}(\mathfrak{S} \cup \mathfrak{T})$,
- (i_1) *pair* $(\mathfrak{P}, \mathfrak{R})$ *is compatible,*
- (i_2) $\mathfrak{R}$ *is continuous.*

After that, $\mathfrak{S} \cup \mathfrak{T}$ has a UCFP for $\mathfrak{P}$ and $\mathfrak{R}$.

Proof. Let $_0 \in \mathfrak{S}$ and $_0 \in \mathfrak{T}$ be arbitrary, and from (i_0), we construct the bisequences $(\{_\kappa\}, \{_\kappa\})$ in $(\mathfrak{S}, \mathfrak{T})$ as

$$\mathfrak{P}_\kappa = \mathfrak{R}_{\kappa+1} = {}_\kappa, \quad \mathfrak{P}_\kappa = \mathfrak{R}_{\kappa+1} = {}_{\kappa+1} \quad \text{where } \kappa = 0, 1, 2, \ldots.$$

Then $(_\kappa, _\kappa)$ is a bisequence on $(\mathfrak{S}, \mathfrak{T}, d)$. Say $\mathfrak{K}_\kappa = \frac{\xi^\kappa}{1-\xi}\Upsilon$ where $\Upsilon = d(_0, _0) + d(_1, _0)$. Then for all $\kappa, \delta \in \mathbb{N}$,

$$\begin{aligned}
d(_\kappa, _\kappa) &= d(\mathfrak{P}_{\kappa-1}, \mathfrak{P}_\kappa) \\
&\leq \ell\left[d(\mathfrak{R}_{\kappa-1}, \mathfrak{R}_\kappa) + |d(\mathfrak{R}_{\kappa-1}, \mathfrak{P}_\kappa) - d(\mathfrak{P}_{\kappa-1}, \mathfrak{R}_\kappa)|\right] \\
&\leq \ell\left[d(_{\kappa-1}, _{\kappa-1}) + |d(_{\kappa-1}, _\kappa) - d(_\kappa, _{\kappa-1})|\right] \\
&\leq \ell\left[d(_{\kappa-1}, _{\kappa-1}) + |d(_{\kappa-1}, _{\kappa-1}) + d(_\kappa, _{\kappa-1}) + d(_\kappa, _\kappa) - d(_\kappa, _{\kappa-1})|\right] \\
&\leq \ell\left[2d(_{\kappa-1}, _{\kappa-1}) + d(_\kappa, _\kappa)\right].
\end{aligned}$$

Therefore, letting $\xi = \frac{2\ell}{1-\ell} < 1$, then we have

$$\begin{aligned}
d(_\kappa, _\kappa) &\leq \xi d(_{\kappa-1}, _{\kappa-1}) = \xi d(\mathfrak{P}_{\kappa-2}, \mathfrak{P}_{\kappa-1}) \\
&\leq \xi^2 d(_{\kappa-2}, _{\kappa-2}) \\
&\vdots \\
&\leq \xi^\kappa d(\mathfrak{Z}_0, \mathfrak{Y}_0) \to 0 \text{ as } \kappa \to \infty.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d(_{\kappa+1}, _\kappa) &= d(\mathfrak{P}_\kappa, \mathfrak{P}_\kappa) \\
&\leq \ell\left[d(\mathfrak{R}_\kappa, \mathfrak{R}_\kappa) + |d(\mathfrak{R}_\kappa, \mathfrak{P}_\kappa) - d(\mathfrak{P}_\kappa, \mathfrak{R}_\kappa)|\right] \\
&\leq \ell\left[d(_\kappa, _{\kappa-1}) + |d(_\kappa, _\kappa) - d(_{\kappa+1}, _{\kappa-1})|\right] \\
&\leq \ell\left[d(_\kappa, _{\kappa-1}) + |d(_\kappa, _{\kappa-1}) + d(_{\kappa+1}, _{\kappa-1}) + d(_{\kappa+1}, _\kappa) - d(_{\kappa+1}, _{\kappa-1})|\right] \\
&\leq \ell\left[2d(_\kappa, _{\kappa-1}) + d(_{\kappa+1}, _\kappa)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned} d(_{\kappa+1},_{\kappa}) &\leq \xi d(_{\kappa},_{\kappa-1}) = \xi d(\mathfrak{P}_{\kappa-1}, \mathfrak{P}_{\kappa}) \\ &\leq \xi^2 d(_{\kappa-1},_{\kappa-2}) \\ &\vdots \\ &\leq \xi^{\kappa} d(_{1},_{0}) \to 0 \text{ as } \kappa \to \infty. \end{aligned}$$

$$\begin{aligned} d(_{\kappa+\delta},_{\kappa}) &\leq d(_{\kappa+\delta},_{\kappa+1}) + d(_{\kappa+1},_{\kappa+1}) + d(_{\kappa+1},_{\kappa}) \\ &\leq d(_{\kappa+\delta},_{\kappa+1}) + \xi^{\kappa}\Upsilon \\ &\leq d(_{\kappa+\delta},_{\kappa+2}) + d(_{\kappa+2},_{\kappa+2}) + d(_{\kappa+2},_{\kappa+1}) + \xi^{\kappa}\Upsilon \\ &\leq d(_{\kappa+\delta},_{\kappa+2}) + (\xi^{\kappa+1} + \xi^{\kappa})\Upsilon \\ &\vdots \\ &\leq d(_{\kappa+\delta},_{\kappa+\delta}) + (\xi^{\kappa+\delta-1} + \cdots + \xi^{\kappa+1} + \xi^{\kappa})\Upsilon \\ &\leq (\xi^{\kappa+\delta} + \xi^{\kappa+\delta-1} + \cdots + \xi^{\kappa+1} + \xi^{\kappa})\Upsilon \\ &\leq \xi^{\kappa}\Upsilon \sum_{i=0}^{\infty} \xi^{i} = \frac{\xi^{\kappa}}{1-\xi}\Upsilon = \mathfrak{K}_{\kappa} \end{aligned}$$

and similarly, we can prove that

$$d(_{\kappa},_{\kappa+\delta}) \leq \mathfrak{K}_{\kappa}.$$

Let $\epsilon > 0$, since $\xi \in [0, 1)$, there exists an $\kappa_0 \in \mathbb{N}$ such that $\mathfrak{K}_{\kappa_0} = \frac{\xi^{\kappa_0}}{1-\xi}\Upsilon < \frac{\epsilon}{3}$. Then

$$d(_{\kappa},_{\delta}) \leq d(_{\kappa},_{\kappa_0}) + d(_{\kappa_0},_{\kappa_0}) + d(_{\kappa_0},_{\delta}) \leq 3\mathfrak{K}_{\kappa_0} < \epsilon$$

and $(_{\kappa},_{\kappa})$ is a Cauchy bi-sequence, that is, the bisequences $(\{_{\kappa}\},\{_{\kappa}\})$ is a Cauchy sequence in $(\mathfrak{S}, \mathfrak{T})$. Since $(\mathfrak{S}, \mathfrak{T}, d)$ is complete, $(\{_{\kappa}\},\{_{\kappa}\})$ converges and thus it biconverges to a point $\in \mathfrak{S} \cap \mathfrak{T}$ such that

$$\lim_{\kappa\to\infty} {}_{\kappa+1} = \; = \lim_{\kappa\to\infty} {}_{\kappa}$$

That is

$$\lim_{\kappa\to\infty} \mathfrak{P}_{\kappa} = \lim_{\kappa\to\infty} \mathfrak{R}_{\kappa+1} = \lim_{\kappa\to\infty} \mathfrak{P}_{\kappa} = \lim_{\kappa\to\infty} \mathfrak{R}_{\kappa+1} =$$

Since $\mathfrak{R}$ is continuous function, we have

$$\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_{\kappa} = \mathfrak{R} \qquad \lim_{\kappa\to\infty} \mathfrak{R}^2{}_{\kappa+1} = \mathfrak{R}$$
$$\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_{\kappa} = \mathfrak{R} \qquad \lim_{\kappa\to\infty} \mathfrak{R}^2{}_{\kappa+1} = \mathfrak{R}$$

Since the pair $\{\mathfrak{P}, \mathfrak{R}\}$ is compatible, we have
$\lim_{\kappa\to\infty} d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{R}\mathfrak{P}_{\kappa}) = \lim_{\kappa\to\infty} d(\mathfrak{R}\mathfrak{P}_{\kappa}, \mathfrak{P}\mathfrak{R}_{\kappa+1}) = 0.$
Therefore,

$$\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_{\kappa} = \lim_{\kappa\to\infty} \mathfrak{P}\mathfrak{R}_{\kappa+1} = \mathfrak{R} \qquad \lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}_{\kappa} = \lim_{\kappa\to\infty} \mathfrak{P}\mathfrak{R}_{\kappa+1} = \mathfrak{R}. \tag{2.9}$$

From (2.2), we get

$$d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{P}_\kappa) \leq \ell\Big[d(\mathfrak{R}^2{}_{\kappa+1}, \mathfrak{R}_\kappa) + |d(\mathfrak{R}^2{}_{\kappa+1}, \mathfrak{P}_\kappa) - d(\mathfrak{P}\mathfrak{R}_{\kappa+1}, \mathfrak{R}_\kappa)|\Big].$$

Letting $\kappa \to \infty$ in this inequality, we obtain

$$d(\mathfrak{R},) \leq \ell\,[d(\mathfrak{R},) + |d(\mathfrak{R},) - d(\mathfrak{R},)|] \leq \ell d(\mathfrak{R},).$$

This is possible only if $d(\mathfrak{R},) = 0$. That is $\mathfrak{R} = .$ By using the condition (2.2) and (B_4), we obtain

$$\begin{aligned} d(\mathfrak{P},) &\leq d(\mathfrak{P}, {}_{\kappa+1}) + d({}_{\kappa+1}, {}_{\kappa+1}) + d({}_{\kappa+1},) \\ &\leq d(\mathfrak{P}, \mathfrak{P}_{\kappa+1}) + d({}_{\kappa+1}, {}_{\kappa+1}) + d({}_{\kappa+1},) \\ &\leq \ell\,[d(\mathfrak{R}, \mathfrak{R}_{\kappa+1}) + |d(\mathfrak{R}, \mathfrak{P}_{\kappa+1}) - d(\mathfrak{P}, \mathfrak{R}_{\kappa+1})|] + d({}_{\kappa+1}, {}_{\kappa+1}) + d({}_{\kappa+1},) \\ &\to 0 \text{ as } \kappa \to \infty. \end{aligned}$$

Thus $\mathfrak{P} = .$ Hence $\mathfrak{P} = \mathfrak{R} = .$ As in the proof of the Theorem (2.2), uniqueness of the fixed point of $\mathfrak{P}$ and $\mathfrak{R}$ can be shown easily. □

Corollary 2.5. *Assume that the BMS $(\mathfrak{S}, \mathfrak{T}, d)$ is complete. Assume that there exists a contravariant mapping satisfying P-contravariant contraction*
$\mathfrak{P} : (\mathfrak{S}, \mathfrak{T}) \leftrightharpoons (\mathfrak{S}, \mathfrak{T})$. In $\mathfrak{S} \cup \mathfrak{T}$, there is a unique fixed point for $\mathfrak{P}$.

Proof. By using a contravariant mapping $\mathfrak{R} = I_{\mathfrak{S}\cup\mathfrak{T}}$, the proof follows from Theorem (2.4).the expressions $I() =$ and $I() = .$ □

Example 2.6. *Let $\mathfrak{S} = \mathfrak{U}_n(\mathbb{R}^2)$ and $\mathfrak{T} = \mathfrak{L}_n(\mathbb{R} \times \{0\})$ be the set of all $n \times n$ upper and lower triangular matrices. Define $d : \mathfrak{S} \times \mathfrak{T} \to [0, \infty)$ as*
$d(X, Y) = \sum_{i,j=1}^{n} |(\mathfrak{Z}_{ij} + {}_{ij}) - ({}_{ij} + 0_{ij})|$ for all $X = \Big((\mathfrak{Z}_{ij}, {}_{ij})\Big)_{n\times n} \in \mathfrak{U}_n(\mathbb{R}^2)$ and $Y = \Big(({}_{ij}, 0_{ij})\Big)_{n\times n} \in \mathfrak{L}_n(\mathbb{R} \times \{0\})$. Then obviously $(\mathfrak{S}, \mathfrak{T}, d)$ is a complete BMS. And define $\mathfrak{P}, \mathfrak{R} : (\mathfrak{S}, \mathfrak{T}, d) \rightrightarrows (\mathfrak{S}, \mathfrak{T}, d)$ as $\mathfrak{P}(X) = (\frac{\mathfrak{Z}_{ij}}{20}, \frac{{}_{ij}}{20})_{n\times n}$ and $\mathfrak{R}(X) = (\frac{\mathfrak{Z}_{ij}}{4}, \frac{{}_{ij}}{4})_{n\times n}$. Then obviously, $\mathfrak{P}(\mathfrak{S} \cup \mathfrak{T}) \subseteq \mathfrak{R}(\mathfrak{S} \cup \mathfrak{T})$ and observe that the pairs $(\mathfrak{P}, \mathfrak{R})$ is a compatible. Let (X_κ, Y_κ) be a bisequence in $(\mathfrak{S}, \mathfrak{T})$ such that, for some $= \Big((\xi_{ij}, 0_{ij})\Big)_{n\times n} \in \mathfrak{S} \cap \mathfrak{T}$, $\lim_{\kappa\to\infty} d(\mathfrak{R}X_\kappa,) = 0$, $\lim_{\kappa\to\infty} d(, \mathfrak{R}Y_\kappa) = 0$, $\lim_{\kappa\to\infty} d(\mathfrak{P}X_\kappa,) = 0$, $\lim_{\kappa\to\infty} d(, \mathfrak{P}Y_\kappa) = 0$.
Since $\mathfrak{P}$ and $\mathfrak{R}$ are continuous, we have

$$\begin{aligned} \lim_{\kappa\to\infty} d(\mathfrak{R}\mathfrak{P}X_\kappa, \mathfrak{P}\mathfrak{R}Y_\kappa) &= d(\lim_{\kappa\to\infty} \mathfrak{R}\mathfrak{P}X_\kappa, \lim_{\kappa\to\infty} \mathfrak{P}\mathfrak{R}Y_\kappa) \\ &= d(\mathfrak{R}, \mathfrak{P}) \\ &= d\left((\frac{\xi_{ij}}{4}, 0_{ij})_{n\times n}, (\frac{\xi_{ij}}{20}, 0_{ij})_{n\times n}\right) \\ &= \sum_{i,j=1}^{n} |\frac{\xi_{ij}}{4} + \frac{\xi_{ij}}{20}| \\ &= \frac{3}{10} \sum_{i,j=1}^{n} |\xi_{ij}|. \end{aligned}$$

But $\frac{3}{10}\sum_{i,j=1}^{n}|\xi_{ij}| = 0 \Leftrightarrow \xi_{ij} = 0$. *Similarly, we prove* $\lim_{\kappa\to\infty} d(\mathfrak{P}\mathfrak{R}X_\kappa, \mathfrak{R}\mathfrak{P}Y_\kappa) = 0$.
In fact, we have for any elements $X, Y \in \mathfrak{S} \cup \mathfrak{T}$

$$\begin{aligned}
d(\mathfrak{P}X, \mathfrak{P}Y) &= d\left((\frac{\mathfrak{Z}_{ij}}{20}, \frac{ij}{20})_{n\times n}, (\frac{ij}{20}, 0_{ij})_{n\times n}\right) \\
&= \sum_{i,j=1}^{n} |(\frac{\mathfrak{Z}_{ij}}{20} + \frac{ij}{20}) - (\frac{ij}{20} + 0_{ij})| \\
&= \frac{1}{20}\sum_{i,j=1}^{n} |(\mathfrak{Z}_{ij} + {}_{ij}) - {}_{ij}|. \\
&\leq \frac{1}{16}\left(\sum_{i,j=1}^{n} |\mathfrak{Z}_{ij} + {}_{ij} - {}_{ij}| + \left|\sum_{i,j=1}^{n} |5\mathfrak{Z}_{ij} + {}_{ij} - 5_{ij}| - \sum_{i,j=1}^{n} |\mathfrak{Z}_{ij} + 5_{ij} - {}_{ij}|\right|\right) \\
&\leq \ell\,[d(\mathfrak{R}X, \mathfrak{R}Y) + |d(\mathfrak{R}X, \mathfrak{P}Y) - d(\mathfrak{P}X, \mathfrak{R}Y)|]
\end{aligned}$$

As a result, $\ell = \frac{1}{4} \in [0, \frac{1}{3})$ *and all the requirements of the Theorem (2.2) are met. Therefore, there must be a UCFP between* $\mathfrak{P}$ *and* $\mathfrak{R}$. *Actually, the UCFP of* $\mathfrak{P}$ *and* $\mathfrak{R}$ *is* $(0_{ij}, 0_{ij})_{n\times n}$.

3. Application to the Existence of Solutions of Integral Equations

This section uses Corollary 2.3 to demonstrate that a non-linear Fredholm integral equation has a solution and that it is unique.

Suppose that $\mathfrak{S} = C(L^\infty(\mathfrak{E}_1))$ and $\mathfrak{T} = C(L^\infty(\mathfrak{E}_2))$ be the set of all essential bounded measurable continuous functions on $\mathfrak{E}_1$ and $\mathfrak{E}_2$, where $m(\mathfrak{E}_1 \cup \mathfrak{E}_2) < \infty$ denotes the set of two Lebesgue measurable sets.

Define $d : \mathfrak{S} \times \mathfrak{T} \to R^+$ as $d(\ell, \sigma) = \|\ell - \sigma\|$ for all $\ell \in \mathfrak{S}, \sigma \in \mathfrak{T}$. Therefore, $(\mathfrak{S}, \mathfrak{T}, d)$ is a complete BMS.

The non-linear Fredholm integral equation given below is now taken into consideration:

$$\varsigma(t) = \varphi(t) + \kappa \int_{\mathfrak{E}_1\cup\mathfrak{E}_2} \Omega(t, \varsigma(s))ds. \tag{3.1}$$

where $\varsigma \in C(L^\infty(\mathfrak{E}_1) \cup L^\infty(\mathfrak{E}_2))$, $\kappa \in R$ and $t, s \in \mathfrak{E}_1 \cup \mathfrak{E}_2$,
$\Omega : (\mathfrak{E}_1 \cup \mathfrak{E}_2) \times (L^\infty(\mathfrak{E}_1) \cup L^\infty(\mathfrak{E}_2)) \to R$ and $\varphi : \mathfrak{E}_1 \cup \mathfrak{E}_2 \to R$ are given continuous functions.
Define the operator $\mathfrak{P} : \mathfrak{S} \cup \mathfrak{T} \to \mathfrak{S} \cup \mathfrak{T}$ by

$$\mathfrak{P}(\varsigma)(t) = \varphi(t) + \kappa \int_{\mathfrak{E}_1\cup\mathfrak{E}_2} \Omega(t, \varsigma(s))ds \tag{3.2}$$

Note that Eq. (3.1) has a solution if and only if $\mathfrak{P}$ has a fixed point

Theorem 3.1. *Let* Ω *be a continuous function satisfying*

(*i*) *There exists a continuous function* $\chi : \mathfrak{E}_1 \cup \mathfrak{E}_2 \to R^+$ *such that for all* $\varrho, v \in \mathfrak{S} \cup \mathfrak{T}$, $\kappa \in R$ *and* $t, s \in \mathfrak{E}_1 \cup \mathfrak{E}_2$, *we get that*

$\|\Omega(t,\varsigma(s)) - \Omega(t,\upsilon(s))\| \leq \chi(t,s)M(\varsigma,\upsilon)$ *where,*
$M(\varsigma,\upsilon) = d(\Re\varsigma(s),\Re\upsilon(s)) + |d(\Re\varsigma(s),\mathfrak{P}\upsilon(s)) - d(\mathfrak{P}\varsigma(s),\Re\upsilon(s))|$
(ii) $\|\kappa\| \int_{\mathfrak{E}_1\cup\mathfrak{E}_2} \chi(t,s)ds \leq \ell$ *where* $\Re \in [0,\frac{1}{3})$

For the initial value problem 3.1, there is then a unique solution in $C\left(L^\infty(\mathfrak{E}_1) \cup L^\infty(\mathfrak{E}_2)\right)$.

Proof. The existence of a fixed point of $\mathfrak{P}$ is equal to the existence of a solution of (3.1). With the help of the inequality, (i), (ii), we have

$$\begin{aligned} d(\mathfrak{P}\varsigma(t),\mathfrak{P}\upsilon(t)) &= \|\kappa\int_{\mathfrak{E}_1\cup\mathfrak{E}_2}(\Omega(t,\varsigma(s)))ds - \kappa\int_{\mathfrak{E}_1\cup\mathfrak{E}_2}(\Omega(t,\upsilon(s)))ds\| \\ &\leq \|\kappa\|\int_{\mathfrak{E}_1\cup\mathfrak{E}_2}\|\Omega(t,\varsigma(s)) - \Omega(t,\upsilon(s))\|ds \\ &\leq \|\kappa\|\int_{\mathfrak{E}_1\cup\mathfrak{E}_2}\chi(t,s)M(\varsigma,\upsilon)ds \\ &\leq \|\kappa\|\left(\int_{\mathfrak{E}_1\cup\mathfrak{E}_2}\chi(t,s)ds\right)M(\varsigma,\upsilon) \\ &\leq \ell M(\varsigma,\upsilon) \\ &\leq \ell\left(d(\Re\varsigma(s),\Re\upsilon(s)) + |d(\Re\varsigma(s),\mathfrak{P}\upsilon(s)) - d(\mathfrak{P}\varsigma(s),\Re\upsilon(s))|\right) \end{aligned}$$

Since $\mathfrak{P}$ has a unique solution in $\mathfrak{S}\cup\mathfrak{T}$ to the integral equation (3.1), all the criteria of Corollary 2.3 hold. □

4. Application to the Existence of Solutions of Homotopy

In this section, we investigate the possibility of a single solution to homotopy theory.

Theorem 4.1. *Given a complete BMS* $(\mathfrak{S},\mathfrak{T},d)$, $(\overline{\mathfrak{U}},\overline{\mathfrak{V}})$ *and* $(\mathfrak{U},\mathfrak{V})$ *be an open and closed subset of* $(\mathfrak{S},\mathfrak{T})$ *such that* $(\mathfrak{U},\mathfrak{V}) \subseteq (\overline{\mathfrak{U}},\overline{\mathfrak{V}})$. *Assume that* $\mathfrak{H} : \left(\overline{\mathfrak{U}}\cup\overline{\mathfrak{V}}\right)\times[0,1] \to \mathfrak{S}\cup\mathfrak{T}$ *is an operator that satisfies the following requirements: i)* $\neq \mathfrak{H}(,s)$ *for each* $\in \partial\mathfrak{U}\cup\partial\mathfrak{V}$ *and* $s \in [0,1]$ *(here* $\partial\mathfrak{U}\cup\partial\mathfrak{V}$ *is boundary of* $\mathfrak{U}\cup\mathfrak{V}$ *in* $\mathfrak{S}\cup\mathfrak{T}$*);*
ii) for all $\in \overline{\mathfrak{U}}, \imath \in \overline{\mathfrak{V}}$, $s \in [0,1]$ *and* $\ell \in [0,\frac{1}{3})$ *such that*

$$d\left(\mathfrak{H}(,s),\mathfrak{H}(\imath,s)\right) \leq \ell\left(d(,\imath) + |d(,\mathfrak{H}(\imath,s)) - d(\mathfrak{H}(,s),\imath)|\right)$$

iii) $\exists\, M \geq 0 \ni d(\mathfrak{H}(,s),\mathfrak{H}(\imath,t)) \leq M|s-t|$ *for every* $\in \overline{\mathfrak{U}}$, $\imath \in \overline{\mathfrak{V}}$ *and* $s,t \in [0,1]$.
Then $\mathfrak{H}(.,0)$ *has a fixed point* $\iff$ $\mathfrak{H}(.,1)$ *has a fixed point.*

Proof. Let the sets

$$\Theta = \left\{\ s \in [0,1] : \mathfrak{H}(,s) = \text{ for some } \in \mathfrak{U}\ \right\}.$$
$$\Upsilon = \left\{\ t \in [0,1] : \mathfrak{H}(,t) = \text{ for some } \in \mathfrak{V}\ \right\}.$$

Assuming that there is a stable point for $\mathfrak{H}(.,0)$ in $\mathfrak{U} \cup \mathfrak{V}$, we obtain that $0 \in \Theta \cap \Upsilon$. in order for $\Theta \cap \Upsilon \neq \phi$. We now demonstrate that, given the connectedness $\Theta = \Upsilon = [0,1]$, $\Theta \cap \Upsilon$ is both closed and open in $[0,1]$. Consequently, there is a fixed point for $\mathfrak{H}(.,0)$ in $\Theta \cap \Upsilon$. We first demonstrate the closure of $\Theta \cap \Upsilon$ in $[0,1]$. Using $(a_p, x_p) \to (\mathfrak{Z}, \mathfrak{Z}) \in [0,1]$ as $p \to \infty$, let $(\left\{a_p\right\}_{p=1}^{\infty}, \left\{x_p\right\}_{p=1}^{\infty}) \subseteq (\Theta, \Upsilon)$. We must show that $\mathfrak{Z} \in \Theta \cap \Upsilon$. Since $(a_p, x_p) \in (\Theta, \Upsilon)$ for $p = 0, 1, 2, 3, \cdots$, there exists sequences $\left(\left\{{}_p\right\}, \left\{{}_p\right\}\right)$ with ${}_{p+1} = \mathcal{H}({}_p, a_p)$, ${}_{p+1} = \mathcal{H}({}_p, x_p)$.
Consider

$$\begin{aligned}
d({}_p, {}_{p+1}) &= d\left(\mathfrak{H}({}_{p-1}, a_{p-1}), \mathcal{H}({}_p, x_p)\right) \\
&\leq d\left(\mathfrak{H}({}_{p-1}, a_{p-1}), \mathcal{H}({}_p, a_{p-1})\right) + d\left(\mathfrak{H}({}_p, a_p), \mathcal{H}({}_p, a_{p-1})\right) + d\left(\mathfrak{H}({}_p, a_p), \mathcal{H}({}_p, x_p)\right) \\
&\leq \ell\left(d\left({}_{p-1}, {}_p\right) + |d\left({}_{p-1}, \mathcal{H}({}_p, x_p)\right) - d\left(\mathcal{H}({}_{p-1}, a_{p-1}), {}_p\right)|\right) \\
&\quad + M|a_p - a_{p-1}| + M|a_p - x_p| \\
&\leq \ell\left(d\left({}_{p-1}, {}_p\right) + |d\left({}_{p-1}, {}_p\right) + d\left({}_p, {}_p\right) + d\left({}_p, {}_{p+1}\right) - d\left({}_p, {}_p\right)|\right) \\
&\quad + M|a_p - a_{p-1}| + M|a_p - x_p|.
\end{aligned}$$

Letting $p \to \infty$ and $\frac{2\ell}{1-\ell} < 1$, we have

$$\begin{aligned}
\lim_{p\to\infty} d({}_p, {}_{p+1}) &\leq \lim_{p\to\infty} \frac{2\ell}{1-\ell} d\left({}_{p-1}, {}_p\right) \\
&\leq \lim_{p\to\infty} \left(\frac{2\ell}{1-\ell}\right)^2 d\left({}_{p-2}, {}_{p-1}\right) \\
&\;\;\vdots \\
&\leq \lim_{p\to\infty} \left(\frac{2\ell}{1-\ell}\right)^p d\left({}_0, {}_1\right) = 0
\end{aligned}$$

Therefore,

$$\lim_{p\to\infty} d({}_p, {}_{p+1}) = 0 \tag{4.1}$$

We shall show that the Cauchy bisequence $(\{{}_p\}, \{{}_p\})$ exists. Let $\epsilon > 0$ and $\{q_k\}, \{p_k\}$ exist, such that for $p_k > q_k > k$,

$$d({}_{p_k}, {}_{q_k}) \geq \epsilon \quad d({}_{p_k - 1}, {}_{q_k}) < \epsilon \tag{4.2}$$

and

$$d({}_{q_k}, {}_{p_k}) \geq \epsilon \quad d({}_{q_k}, {}_{p_k - 1}) < \epsilon \tag{4.3}$$

By view of (4.2) and triangle inequality, we get

$$\begin{aligned}
\epsilon &\leq d({}_{p_k}, {}_{q_k}) \\
&\leq d({}_{p_k}, {}_{p_k - 1}) + d({}_{p_k - 1}, {}_{p_k - 1}) + d({}_{p_k - 1}, {}_{q_k}) \\
&< d({}_{p_k}, {}_{p_k - 1}) + d\left(\mathfrak{H}({}_{p_{k-2}}, a_{p_{k-2}}), \mathfrak{H}({}_{p_{k-2}}, x_{p_{k-2}})\right) + \epsilon \\
&< d({}_{p_k}, {}_{p_k - 1}) + M|a_{p_{k-2}} - x_{p_{k-2}}| + \epsilon.
\end{aligned}$$

Letting $k \to \infty$, and using (4.1), we obtain

$$\lim_{p\to\infty} d(_{p_k}, _{q_k}) = \epsilon \tag{4.4}$$

Using (4.3), one can prove

$$\lim_{p\to\infty} d(_{q_k}, _{p_k}) = \epsilon \tag{4.5}$$

For all $k \in \mathbb{N}$, by (ii) we have

$$d(_{p_k+1}, _{q_k+1}) \leq \frac{2\ell}{1-\ell} d(_{p_k}, _{q_k})$$

and

$$d(_{q_k+1}, _{p_k+1}) \leq \frac{2\ell}{1-\ell} d(_{q_k}, _{p_k}).$$

By utilizing (4.4) and (4.5), we arrive at the limit, which is paradoxical:
$\epsilon \leq \frac{2\ell}{1-\ell}\epsilon < \epsilon$. Therefore In $(\mathfrak{U}, \mathfrak{V})$, there is a Cauchy bi-sequence $(\{_p\}, \{_p\})$. To be thorough, $\tau \in \mathfrak{U} \cap \mathfrak{V}$ exists with

$$\lim_{p\to\infty} {}_{p+1} = \tau = \lim_{p\to\infty} {}_{p+1} \tag{4.6}$$

we have

$$\begin{aligned} d\Big(\mathfrak{H}(\tau, \mathfrak{Z}), {}_{p+1}\Big) &= d\Big(\mathfrak{H}(\tau, \mathfrak{Z}), \mathfrak{H}(_p, x_p)\Big) \\ &\leq \ell\Big(d\Big(\tau, _p\Big) + |d\Big(\tau, \mathfrak{H}(_p, x_p)\Big) - d\Big(\mathfrak{H}(\tau, \mathfrak{Z}), _p\Big)|\Big) \\ &\leq \ell\Big(d\Big(\tau, _p\Big) + |d\Big(\tau, _{p+1}\Big) - d\Big(\mathfrak{H}(\tau, \mathfrak{Z}), _p\Big)|\Big). \end{aligned}$$

Based on non-decreasing and applying the limsup on both sides, we have
$d(\mathfrak{H}(\tau, \mathfrak{Z}), \tau) = 0$.The statement $\mathfrak{H}(\tau, \mathfrak{Z}) = \tau$ is implied. $\mathfrak{Z} \in \Theta \cap \Upsilon$ as a result. In $[0, 1]$, $\Theta \cap \Upsilon$ clearly closed. Given $(a_0, x_0) \in \Theta \times \Upsilon$, $(_0, _0)$ bisequences exist, $_0 = \mathfrak{H}(_0, x_0)$, and $_0 = \mathfrak{H}(_0, a_0)$. Given that $\mathfrak{U} \cup \mathfrak{V}$ is open, $\delta > 0$ exists such that $B_d(_0, \delta) \subseteq \mathfrak{U} \cup \mathfrak{V}$ and $B_d(\delta, _0) \subseteq \mathfrak{U} \cup \mathfrak{V}$.
Choose $a \in (a_0 - \epsilon, a_0 + \epsilon)$, $x \in (x_0 - \epsilon, x_0 + \epsilon)$ such that $|a - x_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$, $|x - a_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$, $|a_0 - a| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$, $|x - x_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$ and $|a_0 - x_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$.
Then for, $\in \overline{B}_{\mathfrak{U}\cup\mathfrak{V}}(_0, \delta) = \{, _0 \in \mathfrak{V} / d(_0,) + d(\mathfrak{H}(_0, x),) \leq d(_0, _0) + \delta\}$,
$\in \overline{B}_{\mathfrak{U}\cup\mathfrak{V}}(\delta, _0) = \{, _0 \in \mathfrak{U} / d(, _0) + d(, \mathfrak{H}(_0, a)) \leq d(_0, _0) + \delta\}$

$$\begin{aligned} d(\mathfrak{H}(, a), _0)) &= d(\mathfrak{H}(, a), \mathfrak{H}(_0, x_0)) \\ &\leq d(\mathfrak{H}(, a), \mathfrak{H}(_0, a)) + d(\mathfrak{H}(_0, a_0), \mathfrak{H}(_0, a)) \\ &\qquad + d(\mathfrak{H}(_0, a_0), \mathfrak{H}(_0, x_0)) \\ &\leq d(\mathfrak{H}(, a), \mathfrak{H}(_0, a)) + M|a_0 - a| + M|a_0 - x_0| \\ &\leq \frac{2}{M^{p-1}} + d(\mathfrak{H}(, a), \mathfrak{H}(_0, a)). \end{aligned}$$

Letting $p \to \infty$, then we have

$$\begin{aligned} d\left(\mathfrak{H}(,a),_0)\right) &\leq d\left(\mathfrak{H}(,a),\mathfrak{H}(_0,a)\right) \\ &\leq \ell\left(d\left(,_0\right)+\left|d\left(\mathfrak{H}(,a),_0\right)-d\left(,\mathfrak{H}(_0,a)\right)\right|\right) \\ &\leq \ell\left(d\left(,_0\right)+\left|d\left(\mathfrak{H}(,a),_0\right)\right|+\left|d\left(,\mathfrak{H}(_0,a)\right)\right|\right) \end{aligned}$$

Therefore,

$$\begin{aligned} d\left(\mathfrak{H}(,a),_0)\right) &\leq \frac{\ell}{1-\ell}\left(d\left(,_0\right)+d\left(,\mathfrak{H}(_0,a)\right)\right) \\ &< d\left(,_0\right)+d\left(,\mathfrak{H}(_0,a)\right) \\ &\leq d(_0,_0)+\delta \end{aligned}$$

Similarly we can prove

$$d\left(,\mathfrak{H}(,x)\right)\right) \leq d(_0,_0)+\delta$$

On the other hand,

$$d(_0,_0)=d\left(\mathcal{H}(_0,a_0),\mathcal{H}(_0,x_0)\right) \leq M|a_0-x_0| < \frac{1}{M^{p-1}} \to 0 \text{ as } p \to \infty.$$

Consequently, $_0 = _0$, and $a = x$. Thus, $\mathcal{H}(.,a) : \overline{B}_{\Theta\cup\Upsilon}(_0,\delta) \to \overline{B}_{\Theta\cup\Upsilon}(_0,\delta)$ for any fixed $a \in (a_0-\epsilon, a_0+\epsilon)$. Consequently, we deduce that there is a fixed point for $\mathfrak{H}(.,a)$ in $\overline{\mathfrak{U}} \cap \overline{\mathfrak{V}}$. However, $\mathfrak{U} \cup \mathfrak{V}$ is where this has to be. Consequently, for
$a \in (a_0-\epsilon, a_0+\epsilon)$, $a \in \Theta \cap \Upsilon$. For this reason, $(a_0-\epsilon, a_0+\epsilon) \subseteq \Theta \cap \Upsilon$. It is obvious that in $[0,1]$, $\Theta \cap \Upsilon$ is open. We employ the identical method for the opposite inference. □

5. Conclusion

This paper gives several fixed point conclusions by using P-contractive conditions specified on complete BMS, along with relevant examples that support the main findings. Homotopy theory and integral equation applications are also provided.

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