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Three Point Boundary Value Problems for Generalized Fractional Integro Differential Equations

Hadeel Z. Alzumi[1], **Hanan A. Alyami**[2], **Wafa Shammakh**[1,*]

[1]*Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, P.O. BOX 13151, Jeddah, 21493, Saudi Arabia*

[2]*Mathematics Department, Deanship of Preparatory Year, Najran University, P.O. BOX 1988, Najran, 11001, Saudi Arabia*

[*]*Corresponding author:* wmshammakh@uj.edu.sa

Abstract. This article deals with some existence results for a class of boundary value problem with three-point boundary conditions involving a nonlinear θ-Caputo fractional proportional integro differential equation. By means of some standard fixed point theorems, sufficient conditions for the existence of solutions are presented. Additionally, some applications of the main results are demonstrated.

1. Introduction

Various definitions of fractional-order differential operators, including as the Caputo, Hadamard, Riemann-Liouville, Riesz, and Grunwald-Letnikov operators, have been presented in the literature. Because fractional calculus operators are nonlocal, they can be used to describe nonlocal effects found in non-regular real-world events or long-term memory. One can see for example, [1] for several applications of fractional calculus in the fields of physics, mechanics, biology, engineering, and signal processing, [2] for several real-world applications in science and engineering, [3] for fractional models in bioengineering, and [4] for modeling of viscoelastic systems.

The conformable derivative is an interesting derivative that was introduced by Khalil et al. [5] in 2014. Subsequently, some researchers argued that due to its lack of memory feature, this derivative could not be classified as a fractional derivative. The classical derivative appears to be a logical extension of this new notion. Unfortunately, there is an issue in this new definition: as the order

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gets closer to zero, it fails to lean toward the original function. By using proportional derivatives, Anderson and Ulness [6, 7] suggested a modified conformable derivative. Subsequently, a new generalized proportional derivative was introduced by Jarad et al. [8]. It is well-behaved and has many advantages over classical derivatives, including the ability to generalize previously known derivatives from the literature. See [9–14] for recent developments related to fractional differential equations via generalized proportional derivatives.

In this paper, we use a recently introduced concept of generalized Caputo proportional fractional derivative to study the following boundary value problem for a nonlinear fractional integro-differential equation with integral boundary conditions

$$^{C}\mathscr{D}^{q,\varrho,\hat{\Psi}}\chi(t) = f(t,\chi(t),(\mathcal{B}\chi)(t)), \quad a < t < b, 1 < q \leq 2, \tag{1.1}$$

$$\chi(a) = c_1\chi(\xi), \quad \chi(b) = c_2\chi(\xi), \tag{1.2}$$

where $\varrho \in (0,1]$, $1 < q \leq 2$, $c_1, c_2 \in \mathbb{R}$, $\xi \in (a,b)$, $f : [a,b] \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}$, $\hat{\Psi}(t)$ is a strictly increasing continuous function on $[a,b]$, and $^{C}\mathscr{D}^{q,\varrho,\hat{\Psi}}$ denotes the Caputo fractional proportional derivative with respect to the function $\hat{\Psi}$ of order q, for $\mathcal{K} : [a,b] \times [a,b] \longrightarrow [0,\infty)$,

$$(\mathcal{B}\chi)(t) = \int_a^t \mathcal{K}(t,r)\chi(r)dr.$$

Applications for integral boundary conditions can be found in many applied domains, including population dynamics, chemical engineering, blood flow issues, thermoelasticity, and underground water flow. A recent study [15] provides a thorough explanation of the integral boundary conditions. See [16–18] and the references therein for further information on nonlocal and integral boundary conditions.

The paper is organized as follows: In section 2, we mention some preliminary definitions, lemmas, and theorems that are used in other sections of the paper. Section 3 contains existence and uniqueness results for problem (1.1)-(1.2). These results are new and rely on Banach's contraction mapping principle, a Schaefer-type fixed point theorem, and Krasnoselskii's fixed point theorem. In section 4, we prove several auxiliary results about stability of the stated problem. The paper concludes with section 5, in which five detailed examples are offered.

2. Preliminaries

In this section, we present basic definitions, theorems, lemmas, and corollaries needed for our findings in this paper [19].

Definition 2.1. *For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\vartheta \in C([a,b],\mathbb{R})$ satisfying $\vartheta'(t) > 0$, we define the Riemann Liouville fractional proportional integral of $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ with respect to ϑ as*

$$\left({}_aI^{\alpha,\rho,\vartheta} f\right)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t) - \vartheta(\tau))^{\alpha-1} f(\tau)\vartheta'(\tau)\, d\tau.$$

Definition 2.2. *For $\rho > 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$, and $\vartheta \in C([a,b],\mathbb{R})$ satisfying $\vartheta'(t) > 0$, we define the left fractional derivative of $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ with respect to ϑ as*

$$\begin{aligned}\left({}_aD^{\alpha,\rho,\vartheta} f\right)(t) =& D^{n,\rho,\vartheta} {}_aI^{n-\alpha,\rho,\vartheta} f(t)\\ =& \frac{D_t^{n,\rho,\vartheta}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t)-\vartheta(\tau))^{n-\alpha-1} f(\tau)\vartheta'(\tau)\, d\tau,\end{aligned}$$

where $n = [\mathrm{Re}(\alpha)] + 1$,

and

$$D^{n,\rho,\vartheta} = \underbrace{D^{\rho,\vartheta} D^{\rho,\vartheta} \ldots D^{\rho,\vartheta}}_{n\text{-times}}.$$

Definition 2.3. *For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$, $\vartheta \in C([a,b],\mathbb{R})$ satisfying $\vartheta'(t) > 0$, we define the left derivative of Caputo type starting at a by*

$$\begin{aligned}\left({}_a^CD^{\alpha,\rho,\vartheta} f\right)(t) =& {}_aI^{n-\alpha,\rho,\vartheta} (D^{n,\rho,\vartheta} f)(t)\\ =& \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t)-\vartheta(\tau))^{n-\alpha-1} (D^{n,\rho,\vartheta} f)(\tau)\vartheta'(\tau)\, d\tau.\end{aligned}$$

where $n = [\mathrm{Re}(\alpha)] + 1$,

Proposition 2.1. *Let $\alpha, \beta \in \mathbb{C}$ be such that $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\beta) > 0$. For any $\rho > 0$ and $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$, the following hold:*

(1)

$$\left({}_aI^{\alpha,\rho,\vartheta} e^{\frac{\rho-1}{\rho}\vartheta(x)} (\vartheta(x)-\vartheta(a))^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha} e^{\frac{\rho-1}{\rho}\vartheta(t)} (\vartheta(t)-\vartheta(a))^{\alpha+\beta-1}.$$

(2) If $\mathrm{Re}(\beta) > n$, then

$$\left({}_a^CD^{\alpha,\rho,\vartheta} e^{\frac{\rho-1}{\rho}\vartheta(x)} (\vartheta(x)-\vartheta(a))^{\beta-1}\right)(t) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}\vartheta(t)} (\vartheta(t)-\vartheta(a))^{\beta-1-\alpha}.$$

(3) For $k = 0, 1, \ldots, n-1$, we have

$$\left({}_a^CD^{\alpha,\rho,\vartheta} e^{\frac{\rho-1}{\rho}\vartheta(x)} \left(\vartheta(x)-\vartheta(a)^k\right)\right)(t) = 0.$$

In particular, $\left({}_a^CD^{\alpha,\rho} e^{\frac{\rho-1}{\rho}\vartheta(x)}\right)(t) = 0.$

Theorem 2.1. *If $\rho \in (0,1]$, $Re(\alpha) > 0$, and $Re(\beta) > 0$. Then, for $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function and defined for $t \geq a$, we have*

$$\begin{aligned}{}_aI^{\alpha,\rho,\vartheta}\left({}_aI^{\beta,\rho,\vartheta} f\right)(t) &= {}_aI^{\beta,\rho,\vartheta}\left({}_aI^{\alpha,\rho,\vartheta} f\right)(t)\\ &= \left({}_aI^{\alpha+\beta,\rho,\vartheta} f\right)(t).\end{aligned}$$

Theorem 2.2. *Let $0 \leq m < \lfloor \mathrm{Re}(\alpha) \rfloor + 1$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be integrable function in each interval $[a,t], t > a$, then*

$$D^{m,\rho,\vartheta}\left({}_aI^{\alpha,\rho,\vartheta} f\right)(t) = \left({}_aI^{\alpha-m,\rho,\vartheta} f\right)(t).$$

Corollary 2.1. *Let $0 < \mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$, $m-1 < \mathrm{Re}(\beta) \leq m$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$. Then we have*

$${}_aD^{\beta,\rho,\vartheta}\, {}_aI^{\alpha,\rho,\vartheta} f(t) = {}_aI^{\alpha-\beta,\rho,\vartheta} f(t).$$

Theorem 2.3. *For $\rho > 0$, $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, we have*

$${}_aI^{\alpha,\rho,\vartheta} \left({}^C_aD^{\alpha,\rho,\vartheta} f \right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{\left(D^{k,\rho,\vartheta} f\right)(a)}{\rho^k k!} (\vartheta(t) - \vartheta(a))^k e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}.$$

Next, the following fixed point theorems plays a crucial role in our main results.

Theorem 2.4. *(Schaefer's Fixed Point Theorem [20]). Suppose $A : X \to X$ is continuous and compact. Assume further that the set*

$$\{u \in X \mid u = \lambda A(u) \text{ for some } 0 < \lambda < 1\},$$

is bounded. Then A has at least a fixed point.

Theorem 2.5. *(Karasnoselekii's Fixed Point Theorem [21].) Let $\mathcal{X}$ be a bounded, closed, convex, and nonempty subset of the Banach space $\mathcal{Y}$. Let φ_1, φ_2 be operators mapping $\mathcal{X}$ into $\mathcal{Y}$, such that*
(i) $\varphi_1 x_1 + \varphi_2 x_2 \in \mathcal{X}$ whenever $x_1, x_2 \in \mathcal{X}$;
(ii) φ_1 is compact and continuous;
(iii) φ_2 is a contraction mapping.
Then there exists $x_3 \in \mathcal{X}$ such that $x_3 = \varphi_1 x_3 + \varphi_2 x_3$.

In what follows, we convert the linear variant of the problem (1.1)-(1.2) into integral equations and will be used to define a fixed point problem associated with (1.1)-(1.2).

Lemma 2.1. *Let $\mathfrak{z} \in C[a, b]$ and $1 < q \leq 2$. Then the problem*

$${}^C\mathcal{D}^{q,\varrho,\hat{\Psi}} \chi(\mathrm{t}) = \mathfrak{z}(\mathrm{t}), \quad a < \mathrm{t} < b, \tag{2.1}$$

$$\chi(a) = c_1 \chi(\xi), \chi(b) = c_2 \chi(\xi), \tag{2.2}$$

equivalent to

$$\begin{aligned} \chi(\mathrm{t}) &= I^{q,\varrho,\hat{\Psi}} \mathfrak{z}(\mathrm{t}) + \frac{e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(t)-\hat{\Psi}(a))}}{\Delta} \\ &\Bigg[\bigg((\hat{\Psi}(t) - \hat{\Psi}(a))(c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}) + c_1(\hat{\Psi}(b) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\bigg) I^{q,\varrho,\hat{\Psi}} \mathfrak{z}(\xi) \\ &- \bigg(c_1(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(\mathrm{t}) - \hat{\Psi}(a))(1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\zeta)-\hat{\Psi}(a))})\bigg) I^{q,\varrho,\hat{\Psi}} \mathfrak{z}(b)\Bigg]. \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Delta &= a_{11}a_{22} - a_{21}a_{12} \neq 0, \\ a_{11} &= 1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}, \\ a_{12} &= -c_1(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}, \end{aligned} \tag{2.4}$$

$$a_{21} = -c_2 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))},$$
$$a_{22} = (\hat{\Psi}(b) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))} - c_2(\hat{\Psi}(\xi) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))},$$
$$B_1 = c_1 I^{q,\varrho,\hat{\Psi}} \hbar(\xi),$$
$$B_2 = c_2 I^{q,\varrho,\hat{\Psi}} \hbar(\xi) - I^{q,\varrho,\hat{\Psi}} \hbar(b).$$

Proof. Applying the Riemann-Liouville fractional proportional integral of order q to both sides of equation (2.1), and using Theorem 2.3, we get the general solution

$$x(t) = I^{q,\varrho,\hat{\Psi}} \hbar(t) + [\alpha_0 + \alpha_1(\hat{\Psi}(t) - \hat{\Psi}(a))]e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(t)-\hat{\Psi}(a))}, \tag{2.5}$$

where $\alpha_0, \alpha_1 \in \mathbb{R}$. Using (2.5) in the boundary conditions (2.2) , we obtain

$$\begin{aligned}\alpha_0\left(1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}\right) + \alpha_1\left(-c_1(\hat{\Psi}(\xi) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}\right) &= c_1 I^{q,\varrho,\hat{\Psi}} \hbar(\xi),\\ \alpha_0\left(e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))} - c_2 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}\right) + \alpha_1\left((\hat{\Psi}(b) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))} - c_2(\hat{\Psi}(\xi) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}\right)&\\ = c_2 I^{q,\varrho,\hat{\Psi}} \hbar(\xi) - I^{q,\varrho,\hat{\Psi}} \hbar(b).&\end{aligned} \tag{2.6}$$

We use (2.4) in (2.5) and (2.6) we find the system

$$\begin{aligned}\alpha_0 a_{11} + \alpha_1 a_{12} &= B_1,\\ \alpha_0 a_{21} + \alpha_1 a_{22} &= B_1.\end{aligned} \tag{2.7}$$

Solving the system (2.7) for α_0 and α_1, we find

$$\alpha_0 = \frac{1}{\Delta}(a_{22}B_1 - a_{12}B_2),$$
$$\alpha_1 = \frac{-1}{\Delta}(a_{21}B_1 - a_{11}B_2),$$

which, on substituting in (2.5), yields the solution (2.3).
The following section is devoted to establishing the existence and uniqueness results for problem (1.1)-(1.2). □

3. Main results

In view of lemma 2.1, we transform the problem (1.1)-(1.2) into on equivalent fixed point problem as $x = \mathcal{G}x$, where $\mathcal{G} : C \to C$ defined by

$$\begin{aligned}\mathcal{G}(x)(t) &= I^{q,\varrho,\hat{\Psi}} f(t, x(t), \mathcal{B}x(t)) + \frac{e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(t)-\hat{\Psi}(a))}}{\Delta}\\ &\Big[\Big((\hat{\Psi}(t) - \hat{\Psi}(a))(c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}) + c_1(\hat{\Psi}(b) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big) I^{q,\varrho,\hat{\Psi}} f(\xi, x(\xi), \mathcal{B}x(\xi))\\ &-\Big(c_1(\hat{\Psi}(\xi) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(t) - \hat{\Psi}(a))(1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))})\Big) I^{q,\varrho,\hat{\Psi}} f(b, x(b), \mathcal{B}x(b))\Big].\end{aligned} \tag{3.1}$$

Here, $(\mathcal{Y}, \|.\|)$ is a Banach space, and $C = C([a, b], \mathcal{Y})$ denotes the Banach space of all continuous functions from $[a, b] \longrightarrow \mathcal{Y}$ equipped with a topology of uniform convergence with norm $\|.\|_C$.

We present now the uniqueness of solutions for the problem (1.1)-(1.2) via Banach contraction mapping principle.

In the forthcoming analysis, we require the following assumptions.

(H_1) $f : [a, b] \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$ is jointly continuous and maps bounded subsets of $[a, b] \times \mathcal{Y} \times \mathcal{Y}$ into relatively compact subsets of $\mathcal{Y}$.

(H_2) For all $t \in [a, b]$, $x, y \in \mathcal{Y}$, there exist $\mho_1$, $\mho_2 > 0$ such that

$$|f(t, x(t), \mathcal{B}x(t)) - f(t, y(t), \mathcal{B}y(t))| \leqslant \mho_1\|x - y\| + \mho_2\|\mathcal{B}x - \mathcal{B}y\|.$$

(H_3) $\mathcal{K} : [a, b] \times [a, b] \longrightarrow [0, \infty)$ is continuous with $\mathcal{K}_0 = \max\{\mathcal{K}(t, r) : (t, r) \in [a, b] \times [a, b]\}$.

(H_4) $|f(t, x(t), (\mathcal{B}x)(t))| \leq \mu(t)$, for all $(t, x, (\mathcal{B}x)) \in [a, b] \times \mathcal{Y} \times \mathcal{Y}$, where $\mu \in L^1([a, b], \mathbb{R}^+)$.

(H_5) There exists an increasing function $\wp \in C([a, b], \mathcal{Y})$. and there exists $\lambda_\wp > 0$ such that for any $t \in [a, b]$.

$$I^{q,\varrho,\hat{\Psi}}\wp(t) \leq \lambda_\wp \wp(t).$$

Theorem 3.1. *If the assumptions (H_1)-(H_3) and the condition*

$$[(\mho_1 + \mho_2\mathcal{K}_0(b - a))][|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b] < 1, \tag{3.2}$$

with

$$(\mho_1 + \mho_2\mathcal{K}_0(b - a)) \leq \frac{1}{2}[|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b]^{-1}. \tag{3.3}$$

are holds, then the problem (1.1)-(1.2) *has a unique solution on* $[a, b]$.

Proof. Setting $\daleth = \sup_{t\in[a,b]} |f(t, 0, 0)|$, selecting

$$\iota \geq 2\daleth[|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b].$$

We show that $\mathcal{G}\mathfrak{E}_\iota \subset \mathfrak{E}_\iota$ where $\mathfrak{E}_\iota = \{x \in C : \|x\| \leq \iota\}$. For $x \in \mathfrak{E}_\iota$, we have

$$\begin{aligned} |f(t, x(t), \mathcal{B}x(t))| &= |f(t, x(t), \mathcal{B}x(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leqslant |f(t, x(t), \mathcal{B}x(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leqslant \mho_1\|x\| + \mho_2\|\mathcal{B}\|\|x\| + \daleth \\ &\leqslant (\mho_1 + \mho_2\mathcal{K}_0(b - a))\iota + \daleth, \end{aligned} \tag{3.4}$$

using (H_2) and (3.4), we have

$$\|\mathcal{G}x\| = \sup_{t\in[a,b]} |\mathcal{G}x(t)|$$
$$\leqslant \frac{1}{\varrho^q\Gamma(q)}\int_a^b (\hat{\Psi}(b) - \hat{\Psi}(\tau))^{q-1}|f(\tau, x(\tau), \mathcal{B}x(\tau))|\hat{\Psi}'(\tau)d\tau$$
$$+ \frac{1}{|\Delta|}\Big[\Big((\hat{\Psi}(b) - \hat{\Psi}(a))|c_2 - c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |c_1|(\hat{\Psi}(b) - \hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big)$$

$$\times \frac{1}{\varrho^q\Gamma(q)}\int_a^\xi(\hat{\Psi}(\xi)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)d\tau$$

$$+\Big(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}+(\hat{\Psi}(b)-\hat{\Psi}(a))|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)$$

$$\times \frac{1}{\varrho^q\Gamma(q)}\int_a^b(\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)d\tau]$$

$$\leqslant \frac{[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth]}{\varrho^q\Gamma(q)}\int_a^b(\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}\hat{\Psi}'(\tau)d\tau$$

$$+\frac{1}{|\Delta|}\Big[\Big((\hat{\Psi}(b)-\hat{\Psi}(a))|c_2-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|+|c_1|(\hat{\Psi}(b)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big)$$

$$\times \frac{[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth]}{\varrho^q\Gamma(q)}\int_a^\xi(\hat{\Psi}(\xi)-\hat{\Psi}(\tau))^{q-1}\hat{\Psi}'(\tau)d\tau$$

$$+\Big(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}+(\hat{\Psi}(b)-\hat{\Psi}(a))|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)$$

$$\times \frac{[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth]}{\varrho^q\Gamma(q)}\int_a^b(\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}\hat{\Psi}'(\tau)d\tau]$$

$$\leqslant ((\mathcal{T}_1\|\chi\|+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth)\Big[\frac{(\hat{\Psi}(b)-\hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)}$$

$$+\frac{1}{|\Delta|}\Big\{\Big((\hat{\Psi}(b)-\hat{\Psi}(a))(|c_2-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|)+|c_1|(\hat{\Psi}(b)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big)\frac{(\hat{\Psi}(\xi)-\hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)}$$

$$+\Big(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}+(\hat{\Psi}(b)-\hat{\Psi}(a))|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)\frac{(\hat{\Psi}(b)-\hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)}\}]$$

$$\leqslant \frac{((\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth)}{\varrho^q\Gamma(q+1)|\Delta|}\Big[|\Delta|(\hat{\Psi}(b)-\hat{\Psi}(a))^q$$

$$+\Big\{\Big((\hat{\Psi}(b)-\hat{\Psi}(a))|c_2-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|+|c_1|(\hat{\Psi}(b)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big)(\hat{\Psi}(\xi)-\hat{\Psi}(a))^q$$

$$+\Big(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}+(\hat{\Psi}(b)-\hat{\Psi}(a))|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)(\hat{\Psi}(b)-\hat{\Psi}(a))^q\}]$$

$$\leqslant \frac{((\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth)}{\varrho^q\Gamma(q+1)|\Delta|}\Big[|\Delta|(\hat{\Psi}(b)-\hat{\Psi}(a))^q$$

$$+|c_1|\Big((\hat{\Psi}(b)-\hat{\Psi}(a))(\hat{\Psi}(\xi)-\hat{\Psi}(a))^qe^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}+(\hat{\Psi}(\xi)-\hat{\Psi}(a))(\hat{\Psi}(b)-\hat{\Psi}(a))^qe^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}\Big)$$

$$+|c_2-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|(\hat{\Psi}(b)-\hat{\Psi}(a))(\hat{\Psi}(\xi)-\hat{\Psi}(a))^q$$

$$+|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|(\hat{\Psi}(b)-\hat{\Psi}(a))(\hat{\Psi}(b)-\hat{\Psi}(a))^q\Big]$$

$$\leqslant \frac{((\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth)}{\varrho^q\Gamma(q+1)|\Delta|}\Big[|\Delta|(\hat{\Psi}(b)-\hat{\Psi}(a))^q+2|c_1|(\hat{\Psi}(b)-\hat{\Psi}(a))^q(\hat{\Psi}(\xi)-\hat{\Psi}(a))$$

$$+\Big(|c_2-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|+|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)(\hat{\Psi}(b)-\hat{\Psi}(a))^{q+1}\Big]$$

$$\leqslant [(\ \mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))\iota+\daleth][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b],$$

where

$$\delta_r^p = (\hat{\Psi}(r) - \hat{\Psi}(a))^p,$$

$$\eta^q = \frac{\delta_b^q}{|\Delta|\varrho^q\Gamma(q+1)}, \eta_{c_1}^q = \frac{2|c_1|}{|\Delta|\varrho^q\Gamma(q+1)}, \eta_{c_1,c_2}^q = \frac{|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|}{|\Delta|\varrho^q\Gamma(q+1)}.$$

Then

$$\begin{aligned}\|\mathcal{G}x\| &\leqslant [(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q + \eta_{c_1}^q\delta_b^q\delta_\xi + \eta_{c_1,c_2}^q\delta_b^{q+1}]\iota + \urcorner[|\Delta|\eta^q + \eta_{c_1}^q\delta_b^q\delta_\xi + \eta_{c_1,c_2}^q\delta_b^{q+1}] \\ &\leqslant \frac{1}{2}\iota + \frac{1}{2}\iota = \iota.\end{aligned}$$

Now, for $x_1, x_2 \in C$ and for each $t \in [a, b]$, we obtain

$$\begin{aligned}
&\|\mathcal{G}x_2 - \mathcal{G}x_1\| = \sup_{t\in[a,b]} |\mathcal{G}x_2(\mathrm{t}) - \mathcal{G}x_1(\mathrm{t})| \\
&\leqslant \frac{1}{\varrho^q\Gamma(q)} \int_a^b (\hat{\Psi}(b) - \hat{\Psi}(\tau))^{q-1} |f(\tau, x_2(\tau), \mathcal{B}x_2(\tau)) - f(\tau, x_1(\tau), \mathcal{B}x_1(\tau))| \hat{\Psi}'(\tau) d\tau \\
&+ \frac{1}{|\Delta|}\Bigg[\Bigg((\hat{\Psi}(b) - \hat{\Psi}(a))|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |c_1|(\hat{\Psi}(b) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Bigg) \\
&\quad \times \frac{1}{\varrho^q\Gamma(q)} \int_a^\xi (\hat{\Psi}(\xi) - \hat{\Psi}(\tau))^{q-1} |f(\tau, x_2(\tau), \mathcal{B}x_2(\tau)) - f(\tau, x_1(\tau), \mathcal{B}x_1(\tau))| \hat{\Psi}'(\tau) d\tau \\
&+ \Bigg(|c_1|(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(b) - \hat{\Psi}(a))|1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Bigg) \\
&\quad \times \frac{1}{\varrho^q\Gamma(q)} \int_a^b (\hat{\Psi}(b) - \hat{\Psi}(\tau))^{q-1} |f(\tau, x_2(\tau), \mathcal{B}x_2(\tau)) - f(\tau, x_1(\tau), \mathcal{B}x_1(\tau))| \hat{\Psi}'(\tau) d\tau \\
&\leqslant (\mathcal{T}_1\|x_2 - x_1\| + \mathcal{T}_2\|\mathcal{B}x_2 - \mathcal{B}x_1\|)\Bigg[\frac{(\hat{\Psi}(b) - \hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)} \\
&+ \frac{1}{|\Delta|}\Bigg\{\Bigg((\hat{\Psi}(b) - \hat{\Psi}(a))(|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|) + |c_1|(\hat{\Psi}(b) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Bigg)\frac{(\hat{\Psi}(\xi) - \hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)} \\
&+ \Bigg(|c_1|(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(b) - \hat{\Psi}(a))|1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Bigg)\frac{(\hat{\Psi}(b) - \hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)}\Bigg\}\Bigg] \\
&\leqslant \frac{(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))}{\varrho^q\Gamma(q+1)|\Delta|}\|x_2 - x_1\|\Bigg[|\Delta|(\hat{\Psi}(b) - \hat{\Psi}(a))^q \\
&+ \Bigg((\hat{\Psi}(b) - \hat{\Psi}(a))(|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}|) + |c_1|(\hat{\Psi}(b) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Bigg)(\hat{\Psi}(\xi) - \hat{\Psi}(a))^q \\
&+ \Bigg(|c_1|(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(b) - \hat{\Psi}(a))|1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Bigg)(\hat{\Psi}(b) - \hat{\Psi}(a))^q\Bigg] \\
&\leqslant \frac{(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))}{\varrho^q\Gamma(q+1)|\Delta|}\|x_2 - x_1\|\Bigg[|\Delta|(\hat{\Psi}(b) - \hat{\Psi}(a))^q + 2|c_1|(\hat{\Psi}(b) - \hat{\Psi}(a))^q(\hat{\Psi}(\xi) - \hat{\Psi}(a)) \\
&+ \Bigg(|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Bigg)(\hat{\Psi}(b) - \hat{\Psi}(a))^{q+1}\Bigg] \\
&\leqslant [(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q + \eta_{c_1}^q\delta_b^q\delta_\xi + \eta_{c_1,c_2}^q\delta_b^{q+1}]\|x_2 - x_1\| \\
&< \|x_2 - x_1\|,
\end{aligned}$$

therefore $\mathcal{G}$ is a contraction. Thus, by Banach Contraction Principle, we conclude that $\mathcal{G}$ admits a unique fixed point which is a solution to problem (1.1)-(1.2). □

Theorem 3.2. *Assume that* (H_2),(H_4) *holds and*

$$((\mathcal{T}_1 + \mathcal{T}_2 \mathcal{K}_0 (b-a)))(\eta_{c_1}^q \delta_b^q \delta_\xi + \eta_{c_1,c_2}^q \delta_b^{q+1}) < 1. \tag{3.5}$$

Then, the problem (1.1)-(1.2) *has at least one solution on* $[a,b]$.

Proof. Let us fix

$$\iota \geqslant \|\mu\|_{L^1}(|\Delta|\eta^q + \eta_{c_1}^q \delta_b^q \delta_\xi + \eta_{c_1,c_2}^q \delta_b^{q+1}), \tag{3.6}$$

and consider $\mathfrak{C}_\iota = \{\chi \in C : \|\chi\| \le \iota\}$. We define the operators $\mathfrak{Z}$ and $\mathfrak{J}$ on $\mathfrak{C}_\iota$ as

$$(\mathfrak{Z}\chi)(\mathrm{t}) = \frac{1}{\varrho^q \Gamma(q)} \int_a^t e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathrm{t})-\hat{\Psi}(\tau))} (\hat{\Psi}(\mathrm{t}) - \hat{\Psi}(\tau))^{q-1} |f(\tau, \chi(\tau), \mathcal{B}\chi(\tau))| \hat{\Psi}'(\tau) d\tau$$

$$(\mathfrak{J}\chi)(\mathrm{t}) = \frac{e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(t)-\hat{\Psi}(a))}}{\Delta}$$

$$\Bigg[\Bigg((\hat{\Psi}(t) - \hat{\Psi}(a))(c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}) + c_1(\hat{\Psi}(\mathscr{b}) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}\Bigg) I^{q,\varrho,\hat{\Psi}} f(\xi, \chi(\xi), \mathcal{B}\chi(\xi))$$

$$-\Bigg(c_1(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(\mathrm{t}) - \hat{\Psi}(a))(1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))})\Bigg) I^{q,\varrho,\hat{\Psi}} f(\mathscr{b}, \chi(\mathscr{b}), \mathcal{B}\chi(\mathscr{b}))\Bigg].$$

For all $\chi, y \in \mathfrak{C}_\iota$, we find that

$$\begin{aligned}
\|\mathfrak{Z}\chi + \mathfrak{J}y\| &= \sup_{t\in[a,b]} |\mathfrak{Z}\chi(t) + \mathfrak{J}y(t)| \\
&\leqslant \frac{1}{\varrho^q \Gamma(q)} \int_a^t e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathrm{t})-\hat{\Psi}(\tau))} (\hat{\Psi}(\mathrm{t}) - \hat{\Psi}(\tau))^{q-1} |f(\tau, \chi(\tau), \mathcal{B}\chi(\tau))| \hat{\Psi}'(\tau) d\tau \\
&+ \frac{1}{|\Delta|}\Bigg[\Bigg((\hat{\Psi}(b) - \hat{\Psi}(a))|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}| + |c_1|(\hat{\Psi}(\mathscr{b}) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}\Bigg) \\
&\quad \times \frac{1}{\varrho^q \Gamma(q)} \int_a^\xi (\hat{\Psi}(\xi) - \hat{\Psi}(\tau))^{q-1} |f(\tau, y(\tau), \mathcal{B}y(\tau))| \hat{\Psi}'(\tau) d\tau \\
&+ \Bigg(|c_1|(\hat{\Psi}(\xi) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(\mathscr{b}) - \hat{\Psi}(a))|1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Bigg) \\
&\quad \times \frac{1}{\varrho^q \Gamma(q)} \int_a^b (\hat{\Psi}(b) - \hat{\Psi}(\tau))^{q-1} |f(\tau, y(\tau), \mathcal{B}y(\tau))| \hat{\Psi}'(\tau) d\tau] \\
&\leqslant \frac{\|\mu\|_{L^1}}{\varrho^q \Gamma(q)} \int_a^b (\hat{\Psi}(b) - \hat{\Psi}(\tau))^{q-1} \hat{\Psi}'(\tau) d\tau \\
&\quad + \frac{1}{|\Delta|}\Bigg[\Bigg((\hat{\Psi}(b) - \hat{\Psi}(a))|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}| + |c_1|(\hat{\Psi}(\mathscr{b}) - \hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\mathscr{b})-\hat{\Psi}(a))}\Bigg) \\
&\quad \times \frac{\|\mu\|_{L^1}}{\varrho^q \Gamma(q)} \int_a^\xi (\hat{\Psi}(\xi) - \hat{\Psi}(\tau))^{q-1} \hat{\Psi}'(\tau) d\tau
\end{aligned}$$

$$+\left(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}+(\hat{\Psi}(b)-\hat{\Psi}(a))|1-c_1e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\right)$$
$$\times\frac{\|\mu\|_{L^1}}{\varrho^q\Gamma(q)}\int_a^b(\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}\hat{\Psi}'(\tau)d\tau\Big]$$
$$\leqslant \|\mu\|_{L^1}[|\Delta|\eta^q+\eta_{c_1}^q\delta_b^q\delta_\xi+\eta_{c_1,c_2}^q\delta_b^{q+1}]\leqslant \iota.$$

Thus, $\mathfrak{Z}\chi+\mathfrak{J}y\in\mathfrak{E}_\iota$. It follows from the assumption (H_2) that $\mathfrak{J}$ is a contraction mapping for

$$((\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a)))(\eta_{c_1}^q\delta_b^q\delta_\xi+\eta_{c_1,c_2}^q\delta_b^{q+1})<1.$$

Continuity of f implies that the operator $\mathfrak{Z}$ is continuous. Also, $\mathfrak{Z}$ is uniformly bounded on $\mathfrak{E}_\iota$ as

$$\|\mathfrak{Z}\chi\|\leqslant\|\mu\|_{L^1}\frac{(\hat{\Psi}(b)-\hat{\Psi}(a))^q}{\varrho^q\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathfrak{Z}$. In view of (H_2), we define

$$\sup_{(t,\chi,\mathcal{B}\chi)\in\Omega}|f(t,\chi(t),\mathcal{B}\chi(t))|=f^*,\Omega=[a,b]\times\mathfrak{E}_\iota\times\mathfrak{E}_\iota.$$

Let $t_1,t_2\in[a,b], t_1<t_2$, and consequently we have

$$\|(\mathfrak{Z}\chi)(\mathrm{t})-(\mathfrak{Z}\chi)(\mathrm{t}_g)\|\leqslant\frac{1}{\rho^\alpha\Gamma(q)}\int_a^{t_1}\Big[(\hat{\Psi}(t_2)-\hat{\Psi}(\tau))^{q-1}-(\hat{\Psi}(t_1)-\hat{\Psi}(\tau))^{q-1}\Big]|f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|d\tau$$
$$+\int_{t_1}^{t_2}(\hat{\Psi}(t_2)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)\,d\tau$$
$$\leqslant\frac{f^*}{\rho^\alpha\Gamma(q)}\Big\{\int_a^{t_1}\Big[(\hat{\Psi}(t_2)-\hat{\Psi}(\tau))^{q-1}-(\hat{\Psi}(t_1)-\hat{\Psi}(\tau))^{q-1}\Big]\hat{\Psi}'(\tau)d\tau$$
$$+\int_{t_1}^{t_2}(\hat{\Psi}(t_2)-\hat{\Psi}(\tau))^{q-1}\hat{\Psi}'(\tau)\,d\tau\Big\}$$
$$\leqslant\frac{f^*}{\rho^\alpha\Gamma(q+1)}\Big\{2(\hat{\Psi}(t_2)-\hat{\Psi}(t_1))^q+(\hat{\Psi}(t_2)-\hat{\Psi}(a))^q-(\hat{\Psi}(t_1)-\hat{\Psi}(a))^q\Big\}. \tag{3.7}$$

The right hand side of the inequality (3.7) tend to zero as $t_2\longrightarrow t_1$ independent of χ. Thus, $\mathfrak{Z}$ is equicontinuous on $\mathfrak{E}_\iota$. Therefore, by Arzelà-Ascoli Theorem, $\mathfrak{Z}$ is relatively compact on $\mathfrak{E}_\iota$. Consequently, by the conclusion of Theorem 2.5, there exists a solution of the problem (1.1)-(1.2). □

The next result is based on Schaefer's fixed point theorem.

Theorem 3.3. *Assume that* (H_2) *hold and there exists a constant* $\mathscr{N}>0$ *such that*

$$\|f\,(t,\chi,(\mathcal{B}\chi))\,\|\leqslant\mathscr{N} \tag{3.8}$$

for all $(t,\chi,(\mathcal{B}\chi))\in[a,b]\times\mathcal{Y}\times\mathcal{Y}$.

Then there exists at least one solution for the problem (1.1)-(1.2) *on* $[a,b]$.

Proof. The proof will be given in several steps. In the first step, it will be shown that the operator $\mathcal{G}$ is continuous. Let $\{\chi_n\}$ be a sequence such that $\chi_n \longrightarrow \chi$ in C. Then for each $t \in [a,b]$

$$
\begin{aligned}
&|\mathcal{G}(\chi_n)(t) - \mathcal{G}(\chi)(t)| \leqslant \frac{1}{\varrho^q\Gamma(q)}\int_a^b (\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi_n(\tau),\mathcal{B}\chi_n(\tau)) - f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)d\tau \\
&+\frac{1}{|\Delta|}\Big[\Big((\hat{\Psi}(b)-\hat{\Psi}(a))|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |c_1|(\hat{\Psi}(b)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Big) \\
&\times\frac{1}{\varrho^q\Gamma(q)}\int_a^\xi (\hat{\Psi}(\xi)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi_n(\tau),\mathcal{B}\chi_n(\tau)) - f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)d\tau \\
&+\Big(|c_1|(\hat{\Psi}(\xi)-\hat{\Psi}(a))e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(b)-\hat{\Psi}(a))|1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big) \\
&\times\frac{1}{\varrho^q\Gamma(q)}\int_a^b (\hat{\Psi}(b)-\hat{\Psi}(\tau))^{q-1}|f(\tau,\chi_n(\tau),\mathcal{B}\chi_n(\tau)) - f(\tau,\chi(\tau),\mathcal{B}\chi(\tau))|\hat{\Psi}'(\tau)d\tau\Big] \\
&\leqslant ((\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a)))(|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b)\|\chi_n - \chi\|_C.
\end{aligned}
\tag{3.9}
$$

Then

$$
\|\mathcal{G}(\chi_n) - \mathcal{G}(\chi)\| \leqslant ((\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a)))(|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b)\|\chi_n - \chi\|_C. \tag{3.10}
$$

Therefore, n goes to infinity, we have $\mathcal{G}(\chi_n)$ converges to $\mathcal{G}(\chi)$, which means that $\mathcal{G}$ is continuous at χ, and therefore $\mathcal{G}$ is continuous.

In the second step, we show that the operator $\mathcal{G}$ maps bounded sets into bounded sets in C. In fact, it is sufficient to prove that for any $\iota > 0$, there exists a positive constant ζ such that for each $\chi \in \mathfrak{E}_\iota = \{\chi \in C : \|\chi\|_C \leqslant \iota\}$, we have $\|\mathcal{G}(\chi)\|_C \leqslant \zeta$. By (H_2) and (3.8), we have for each $t \in [a,b]$

$$
\begin{aligned}
\|\mathcal{G}\chi\| &= \sup_{t\in[a,b]} |\mathcal{G}\chi(\mathrm{t})| \\
&\leqslant \mathcal{N}(|\Delta|\eta^q + \eta^q_{c_1,c_2}\delta^q_b\delta_\xi + \eta^q_{c_1}\delta^{q+1}_b) \\
&\leqslant \zeta,
\end{aligned}
$$

where

$$
\zeta = \mathcal{N}(|\Delta|\eta^q + \eta^q_{c_1,c_2}\delta^q_b\delta_\xi + \eta^q_{c_1}\delta^{q+1}_b).
$$

The third step, we prove that the operator $\mathcal{G}$ maps bounded sets into equicontinuous sets of C. Let $t_1, t_2 \in [a,b], t_1 < t_2, \mathfrak{E}_\iota$ be a bounded set of C, and let $\chi \in \mathfrak{E}_\iota$. Then

$$
\begin{aligned}
\Big|(\mathcal{G}\chi)(\mathrm{t}_2) - (\mathcal{G}\chi)(\mathrm{t}_1)\Big| &\leqslant \frac{\mathcal{N}}{\rho^q\Gamma(q+1)}\Big[\Big\{2(\hat{\Psi}(t_2)-\hat{\Psi}(t_1))^q + (\hat{\Psi}(t_2)-\hat{\Psi}(a))^q - (\hat{\Psi}(t_1)-\hat{\Psi}(a))^q\Big\} \\
&+ \frac{(\hat{\Psi}(b)-\hat{\Psi}(a))^{q+1}}{|\Delta|}\hat{\Psi}(t_2)-\hat{\Psi}(t_1)\Big(|c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}| + |1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))}|\Big)\Big],
\end{aligned}
$$

which tends to zero independent of χ as $t_2 \longrightarrow t_1$. Therefore, $\mathcal{G}$ is equicontinuous on $[a,b]$. So, by Arzela-Ascoli Theorem, the operator $\mathcal{G}$ is completely continuous.

Finally, we need prove that the set

$$
\kappa = \{\chi \in C : \chi = \lambda\mathcal{G}\chi \text{ for some } \lambda \in (0,1)\},
$$

for $\chi \in \kappa$, $t \in [a,b]$, we have

$$\chi(t) = \lambda \mathcal{G}(t).$$

As before, we can obtain $|\chi(t)| \leqslant \mathcal{N}(|\Delta|\eta^q + \eta^q_{c_1,c_2}\delta^q_b\delta_\xi + \eta^q_{c_1}\delta^{q+1}_b) \leqslant \zeta$, $\forall \chi \in \kappa$, $t \in [a,b]$. So κ is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that $\mathcal{G}$ has a fixed point which is a solution of the problem (1.1)-(1.2). □

4. Stability analysis

In this section, we study Hyers-Ulam and generalized Hyers-Ulam, Rassias stabilities for the solutions to the problem (1.1)-(1.2) by considering its equivalent integral equation:

$$\begin{aligned}\tilde{\chi}(t) &= I^{q,\varrho,\hat{\psi}} f(\mathrm{t}, \tilde{\chi}(\mathrm{t}), \mathcal{B}\tilde{\chi}(\mathrm{t})) + \frac{e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(t)-\hat{\Psi}(a))}}{\Delta} \\ &\Bigg[\Bigg((\hat{\Psi}(t)-\hat{\Psi}(a))(c_2 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}) + c_1(\hat{\Psi}(b)-\hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(b)-\hat{\Psi}(a))}\Bigg) I^{q,\varrho,\hat{\Psi}} f(\xi, \tilde{\chi}(\xi), \mathcal{B}\tilde{\chi}(\xi)) \\ &- \Bigg(c_1(\hat{\Psi}(\xi)-\hat{\Psi}(a)) e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))} + (\hat{\Psi}(\mathrm{t})-\hat{\Psi}(a))(1 - c_1 e^{\frac{\varrho-1}{\varrho}(\hat{\Psi}(\xi)-\hat{\Psi}(a))})\Bigg) I^{q,\varrho,\hat{\Psi}} f(b, \tilde{\chi}(b), \mathcal{B}\tilde{\chi}(b))\Bigg].\end{aligned} \tag{4.1}$$

Define a continuous nonlinear operator $\mathcal{S} : C \longrightarrow C$ by

$$\mathcal{S}\tilde{\chi} = {}^{C}\mathcal{D}^{q,\varrho,\hat{\Psi}}\tilde{\chi}(\mathrm{t}) - f(\mathrm{t}, \tilde{\chi}(\mathrm{t}), \mathcal{B}\tilde{\chi}(\mathrm{t})).$$

Definition 4.1. *The problem* (1.1)-(1.2) *is called Ulam-Hyers stable if for every* $\bar{\varepsilon} > 0$ *and for each solution* $\tilde{\chi} \in C$ *of*

$$\|\mathcal{S}\tilde{\chi}\| \le \bar{\varepsilon}, \tag{4.2}$$

there exists a solution $\chi \in C$ *of* (1.1), (1.2) *such that* $\|\chi - \tilde{\chi}\| \le \bar{\mathrm{n}}\bar{\bar{\varepsilon}}$ *for positive real numbers* $\bar{\mathrm{n}}$ *and* $\bar{\bar{\varepsilon}}(\bar{\varepsilon})$.

Definition 4.2. *If there exists a function* $\nu \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and for each* $\bar{\varepsilon} > 0$ *and for each solution* $\tilde{\chi} \in C$ *of* (4.2) *there exists* $\chi \in C$ *of* (1.1) *and* (1.2) *with* $\left|\chi(t) - \tilde{\chi}(t)\right| \le \nu(\bar{\varepsilon})$, $t \in [a,b]$. *Then, the problem* (1.1)-(1.2) *is called generalized Ulam-Hyers stable.*

Definition 4.3. *The problem* (1.1)-(1.2) *is said to be Ulam-Hyers-Rassias (UHR) stable with respect to* $\wp \in C([a,b], \mathbb{R}^+)$ *if there exists a real number* $\bar{\mathrm{n}}$ *such that for each solution* $\tilde{\chi} \in C([a,b], \mathbb{R})$ *of*

$$|\mathcal{S}\tilde{\chi}(t)| \le \bar{\varepsilon}\wp(t), t \in [a,b], \tag{4.3}$$

there exists a solution $\chi \in C$ *of* (1.1) *and* (1.2) *such that*

$$\left|\chi(t) - \tilde{\chi}(t)\right| \le \bar{\mathrm{n}}\bar{\varepsilon}\wp(t), t \in [a,b],$$

Definition 4.4. *The problem* (1.1)-(1.2) *is said to be generalized Ulam-Hyers-Rassias (GUHR) stable with respect to* $\wp \in C([a,b], \mathbb{R}^+)$ *if there exists a real number* $\bar{\mathrm{n}}$ *such that for each solution* $\tilde{\chi} \in C([a,b], \mathbb{R})$ *of*

$$|\mathcal{S}\tilde{\chi}(t)| \le \wp(t), t \in [a,b], \tag{4.4}$$

there exists a solution $\chi \in C$ *of* (1.1) *and* (1.2) *such that*

$$\left|\chi(t)-\tilde{\chi}(t)\right| \leq \bar{\mathrm{n}}\wp(t), t \in [a,b],$$

Theorem 4.1. *If* (H_2) *and the condition* (3.2) *(see Theorem 3.1) are satisfied, then the problem* (1.1)-(1.2) *is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Recall that $\chi \in C$ is a unique solution of (1.1) by Theorem 3.1. Let $\tilde{\chi} \in C$ be an other solution of (1.1) which satisfies (4.2). For every solution $\tilde{\chi} \in C$ (given by (4.1)) of (1.1), it is easy to see that $\mathcal{S}$ and $\mathcal{G} - I$ are equivalent operators. Therefore, it follows from $\mathcal{G}\tilde{\chi} = \tilde{\chi}$ and (4.2) and the fixed point property of the operator $\mathcal{G}$ given by (3.1) that

$$\begin{aligned}\left|\chi(t)-\tilde{\chi}(t)\right| &= |\mathcal{G}\chi(t)-\mathcal{G}\tilde{\chi}(t)+\mathcal{G}\tilde{\chi}(t)-\tilde{\chi}(t)| \leq |\mathcal{G}\chi(t)-\mathcal{G}\tilde{\chi}(t)|+|\mathcal{G}\tilde{\chi}(t)-\tilde{\chi}(t)| \\ &\leq ((\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a)))(|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b)\,\|\chi-\tilde{\chi}\|+\bar{\varepsilon},\end{aligned}$$

which, on taking the norm for $t \in [a,b]$ and solving for $\|\chi-\tilde{\chi}\|$, yields

$$\|\chi-\tilde{\chi}\| \leq \frac{\bar{\varepsilon}}{1-[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]},$$

where $\bar{\varepsilon} > 0$ and (3.2). Letting

$$\bar{\bar{\varepsilon}} = \frac{\bar{\varepsilon}}{1-[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]},$$

and $\bar{\mathrm{n}} = 1$, the Ulam-Hyers stability condition holds true. Furthermore, one can notice that the generalized Ulam-Hyers stability condition also holds valid if we set $\nu(\bar{\varepsilon}) = \frac{\bar{\varepsilon}}{1-[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]}$. □

Remark 4.1. *A function* $\tilde{\chi} \in C([a,b],\mathbb{R})$ *is a solution of the inequality* (4.3) *if and only if there exist a function* $h \in C([a,b],\mathbb{R})$ *(where h depends on solution* $\tilde{\chi}$ *) such that*
(i) $|h(t)| \leq \bar{\varepsilon}\wp(t)$ *for all* $t \in [a,b]$,
(ii) ${}^{C}\mathcal{D}^{q,\varrho,\hat{\Psi}}\tilde{\chi}(\mathrm{t}) = f(\mathrm{t},\tilde{\chi}(\mathrm{t}),\mathcal{B}\tilde{\chi}(\mathrm{t}))+h(t), \quad t \in [a,b].$

Lemma 4.1. *Consider* $\chi \in C([a,b],\mathbb{R})$ *is solution of the following problem*

$$ {}^{C}\mathcal{D}^{q,\varrho,\hat{\Psi}}\chi(\mathrm{t}) = f(\mathrm{t},\chi(\mathrm{t}),(\mathcal{B}\chi)(\mathrm{t}))+h(t), \quad a<\mathrm{t}<b, 1<q\leq 2, \tag{4.5}$$

$$\chi(a) = \tilde{\chi}(a) = c_1\chi(\xi), \quad \chi(b) = \tilde{\chi}(b) = c_2\chi(\xi), \tag{4.6}$$

then $\tilde{\chi}$ *satisfy*

$$|\tilde{\chi}(t)-\mathcal{G}(\tilde{\chi})(t)| \leq \bar{\varepsilon}\lambda_\wp\wp(t).$$

Proof. Indeed, by Remark 4.1, we have that

$$ {}^{C}\mathcal{D}^{q,\varrho,\hat{\Psi}}\tilde{\chi}(\mathrm{t}) = f(\mathrm{t},\tilde{\chi}(\mathrm{t}),\mathcal{B}\tilde{\chi}(\mathrm{t}))+h(t), \quad t \in [a,b]$$

.

In view of Theorem 3.1, and using Remark 4.1, we have

$$\tilde{\chi}(t) = \mathcal{G}(\tilde{\chi})(t) + I^{q,\varrho,\hat{\psi}}h(t) \tag{4.7}$$

Then, applying the assumption (H_5) in (4.7), to get the following inequality

$$\begin{aligned}|\tilde{\chi}(t) - \mathcal{G}(\tilde{\chi})(t)| &\leqslant \frac{1}{\varrho^q\Gamma(q)}\int_a^t (\hat{\Psi}(t) - \hat{\Psi}(\tau))^{q-1}|h(\tau)|\hat{\Psi}'(\tau)d\tau \\ &\leqslant \frac{\bar{\varepsilon}}{\varrho^q\Gamma(q)}\int_a^t (\hat{\Psi}(t) - \hat{\Psi}(\tau))^{q-1}\wp(\tau)\hat{\Psi}'(\tau)d\tau \\ &\leqslant \bar{\varepsilon}\lambda_{\wp}\wp(t).\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Assume that* (H_1),(H_2) *and* (H_5) *hold. If* (4.3) *is satisfied and* $[(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b] \neq 1$, *then the problem* (1.1)-(1.2) *is Ulam-Hyers-Rassias stable and Generalized Ulam-Hyers-Rassias stable.*

Proof. For any solution $\tilde{\chi} \in C([a,b],\mathbb{R})$, and a unique solution χ of the problem (1.1)-(1.2), we have

$$\begin{aligned}\|\tilde{\chi}(t) - \chi(t)\| &= \|\tilde{\chi}(t) - \mathcal{G}\chi(t)\| = \|\tilde{\chi}(t) - \mathcal{G}\tilde{\chi}(t) + \mathcal{G}\tilde{\chi}(t) - \mathcal{G}\chi(t)\| \\ &\leqslant \|\tilde{\chi}(t) - \mathcal{G}\tilde{\chi}(t)\| + \|\mathcal{G}\tilde{\chi}(t) - \mathcal{G}\chi(t)\|\end{aligned}$$

By applying Lemma 4.1, we obtain

$$\|\tilde{\chi}(t) - \chi(t)\| \leqslant \bar{\varepsilon}\lambda_{\wp}\wp(t) + [(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b]\|\tilde{\chi}(t) - \mathcal{G}\chi(t)\|,$$

therefore

$$\|\tilde{\chi}(t) - \chi(t)\| \leqslant \frac{1}{1 - [(\mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q + \eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b]}\bar{\varepsilon}\lambda_{\wp}\wp(t).$$

Let $\bar{n} = \frac{\lambda_\wp}{1-[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]}$, then the problem (1.1)-(1.2) is UHR stable. In case $\bar{\varepsilon} = 1$, the solution to the problem (1.1)-(1.2) is GUHR stable. □

5. Examples

Example 5.1. *Consider the following problem:*

$$\begin{aligned}&{}^{C}\mathcal{D}^{\frac{3}{2},\frac{1}{2},\hat{\Psi}}\chi(t) = \frac{e^{-t}|\chi|}{9 + e^t(1 + |\chi|)} + \frac{1}{50}\int_0^t e^{-(s-t)}\chi(s)ds, \\ &\chi(0) = \frac{1}{67}\chi(0.1) \quad , \chi(1) = \frac{1}{17}\chi(0.1),\end{aligned} \tag{5.1}$$

with $\hat{\Psi}(t) = t^2 + 1$, $q = \frac{3}{2}$, $\varrho = \frac{1}{2}$, $c_1 = \frac{1}{67}, c_2 = \frac{1}{17}, \xi = 0.1$, *and* $f(t,\chi,\mathcal{B}\chi) = \frac{e^{-t}|\chi|}{9+e^t(1+|\chi|)} + \frac{1}{50}\int_0^t e^{-(s-t)}\chi(s)ds$, $\mathcal{B}(\chi) = \frac{1}{50}\int_0^t e^{-(s-t)}\chi(s)ds$, *with* $\mathcal{K}(t,s) = \frac{1}{50}e^{-(s-t)}$ *for* $0 \leq s \leq t$.

Using the given data, we have $|f(t,\varkappa,\mathcal{B}(\varkappa))-f(t,y,\mathcal{B}(y))| \leqslant \frac{1}{10}\|\varkappa-y\|+\|\mathcal{B}\varkappa-\mathcal{B}y\|$. *In this case, the* (H_2) *hypothesis is satisfied with* $\mathcal{T}_1=\frac{1}{10}$, $\mathcal{T}_2=1, \mathcal{K}_0=\frac{1}{50}$. *Furthermore, we have*

$$[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]=0.9882113384<1.$$

Thus, all assumptions of Theorem 3.1 are satisfied, so the problem (5.1) *has a unique solution. Also, by Theorem 4.1 the problem* (5.1) *is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Let $\wp(t)=e^{t^2}$, *for all* $t\in[0,1]$. *Then*

$$\begin{aligned} I^{\frac{3}{2},\frac{1}{2},\hat{\Psi}}\wp(t) &= \int_0^t 8\,\frac{\mathrm{e}^{-t^2+\tau^2}y(\tau)\ \sqrt{t^2-\tau^2}\tau\ \sqrt{2}}{\sqrt{\pi}}d\tau \\ &= \mathrm{e}^{t^2}\mathrm{erf}\left(t\sqrt{2}\right)-2\,\frac{t\sqrt{2}}{\mathrm{e}^{t^2}\sqrt{\pi}} \\ &\le 0.7385358700e^{t^2} \\ &\le 0.7385358700\wp(t). \end{aligned}$$

Thus, the hypothesis (H_5) *is satisfied with* $\lambda_\wp=0.7385358700>0$. *Here, for* $\bar{\varepsilon}=\frac{1}{2}$, *if* $\tilde{\varkappa}\in C([0,1],\mathbb{R})$ *satisfies*

$$|\mathfrak{S}\tilde{\varkappa}(t)|\le\frac{1}{2}e^{t^2},\ t\in[0,1],$$

there exists a solution $\varkappa\in C([0,1],\mathbb{R})$ *such that*

$$\left|\varkappa(t)-\tilde{\varkappa}(t)\right|\le\frac{1}{2}\bar{\mathrm{n}}e^{t^2}, t\in[0,1],$$

where

$$\bar{\mathrm{n}}=\frac{\lambda_\wp}{1-[(\mathcal{T}_1+\mathcal{T}_2\mathcal{K}_0(b-a))][|\Delta|\eta^q+\eta^q_{c_1}\delta^q_b\delta_\xi+\eta^q_{c_1,c_2}\delta^{q+1}_b]}\simeq 62.64798287>0.$$

Thus, the problem (5.1) *is UHR stable. Finally, taking* $\bar{\varepsilon}=1$, *then the problem* (5.1) *is GUHR.*

Example 5.2.

$$\begin{aligned} {}^C\mathcal{D}^{\frac{3}{2},\frac{1}{2},\hat{\Psi}}\varkappa(t) &= \frac{1}{(t+9)^2}\cdot\frac{|\varkappa|}{1+|\varkappa|}+\int_0^t\frac{\cos\left(\frac{\pi}{2}(s-t)\right)}{81}\varkappa(s)ds, \\ \varkappa(0) &= \frac{1}{67}\varkappa(0.1) \quad ,\varkappa(1)=\frac{1}{17}\varkappa(0.1), \end{aligned} \tag{5.2}$$

with $\hat{\Psi}(t)=t^2+1$, $c_1=\frac{1}{67}, c_2=\frac{1}{17}, q=\frac{3}{2}, \varrho=\frac{1}{2}$ *and* $\xi=0.1$, $\mathcal{B}(\varkappa)=\int_0^t\frac{\cos\left(\frac{\pi}{2}(s-t)\right)}{81}\varkappa(s)ds$, *with* $\mathcal{K}(\mathrm{t},s)=\frac{\cos\left(\frac{\pi}{2}(s-t)\right)}{81}$.

Here, $f(t,\varkappa,\mathcal{B}\varkappa)=\frac{1}{(t+9)^2}\cdot\frac{|\varkappa|}{1+|\varkappa|}+\int_0^t\frac{\cos\left(\frac{\pi}{2}(s-t)\right)}{81}\varkappa(s)ds.$

A direct computation gives

$$|f(t,\varkappa,\mathcal{B}\varkappa)-f(t,y,\mathcal{B}y)|\leqslant\frac{1}{81}\|\varkappa-y\|+\|\mathcal{B}\varkappa-\mathcal{B}y\|.$$

In this case, the (H_2) *hypothesis is satisfied with*

$$\mathcal{T}_1 = \frac{1}{81},\ \mathcal{T}_2 = 1,$$

moreover

$$\mathcal{K}_0 = \frac{1}{81}.$$

Using Theorem 3.2, we determine that

$$[(\ \mathcal{T}_1 + \mathcal{T}_2\mathcal{K}_0(b-a))][\eta^q_{c_1}\delta^q_b\delta_\xi + \eta^q_{c_1,c_2}\delta^{q+1}_b] = 0.1508000574 < 1.$$

Thus, all assumptions of Theorem 3.2 are satisfied, so the problem (5.2) *has a unique solution.*

Example 5.3.

$$\begin{aligned} &{}^{C}\mathscr{D}^{\frac{3}{2},\frac{1}{2},\hat{\Psi}}\chi(t) = \frac{1}{7e^{t+5}(1+|\chi|)} + \int_0^t \frac{e^{-(s-t)}}{18}\chi(s)ds, \\ &\chi(0) = \frac{1}{67}\chi(0.1) \quad ,\chi(1) = \frac{1}{17}\chi(0.1), \end{aligned} \tag{5.3}$$

$|f(t,\chi,\mathcal{B}(\chi)) - f(t,y,\mathcal{B}(y))| \le \frac{1}{7e^5}\|\chi - y\| + \|\mathcal{B}\chi - \mathcal{B}y\|$, *since* $\chi = C([0,1],\mathbb{R})$ *so,* χ *is bounded. Let*

$$|\chi(t)| \le M, \forall t \in [0,1].$$

Moreover,

$$|f(t,\chi,\mathcal{B}\chi)| \leqslant \frac{1}{7e^5} + \frac{M(e-1)}{18}.$$

Let $\mathcal{N} = \frac{1}{7e^5} + \frac{M(e-1)}{18}$ *is a constant since* $e-1<2$, *let* $\alpha = \max\left(e^{-5}, M\right)$. *Then*

$$|f(t,\chi,\mathcal{B}\chi)| \leqslant \frac{32\alpha}{126} = \frac{16}{63}\alpha.$$

So, we can take

$$\mathcal{N} = \frac{16}{63}\alpha.$$

Thus, all assumptions of Theorem 3.3 are satisfied, so the problem (5.3) *has a unique solution.*

6. Conclusion

This paper has delved into the study of fractional proportional boundary value problems and the uniqueness of their solutions through the application of key theorems in fractional differential equations. By leveraging the Uniqueness Theorem for Fractional Differential Equations, Krasnoselskii's Theorem, and Schaefer's Fixed Point Theorem, this study has systematically examined three specific examples of such problems. Through rigorous proofs and detailed analysis, the paper has successfully established the conditions under which these fractional differential equations possess unique solutions. The exploration of these theorems across diverse boundary value scenarios highlights their practical utility in ensuring solution uniqueness in fractional differential equations. The examples presented in this paper showcase the effectiveness of these theorems in proving the existence and uniqueness of solutions, underscoring their significance in this domain. In summary, the application of these pivotal theorems reaffirms their crucial role in validating the uniqueness of solutions for fractional proportional boundary value problems. This study contributes to the broader understanding of fractional differential equations and emphasizes the importance of these theorems in establishing rigorous mathematical proofs related to solution uniqueness.

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