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Exploring Profit Opportunities in Intuitionistic Fuzzy Metric Spaces via Edelstein Type Mappings

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Abstract. This paper establishes a break-even point theorem concerning a set of mappings adhering to an Edelstein-type contractive criterian within intuitionistic fuzzy metric domains. It explores the break-even analysis within a straightforward total cost-revenue model applicable to dynamic businesses. Utilizing the coincident point theorem within intuitionistic fuzzy metric space, the study demonstrates the inclination of profit-sensitive (or loss-sensitive) dynamic businesses towards their respective break-even points.

1. Introduction

The idea of hesitation was proposed by Atanassov [1] in the form of intuitionistic fuzzy sets, which are expansions of classical fuzzy sets. The study conducted by Abbas et al. [2] investigated the optimum coincidence point outcomes in partially ordered non-Archimedean fuzzy metric spaces. This research shed light on the relationship between order structure and metric features.

The influential research conducted by Banach in 1922 [3] focused on the examination of operations inside abstract sets and their utilisation in integral equations, so establishing fundamental concepts for contemporary functional analysis. The supply function auction model for linear asymmetric oligopoly was presented by Dolmatova et al. [4], therefore making a valuable contribution to the comprehension of market dynamics and balance results.

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Edelstein's [5] investigation of fixed and periodic points under contractive mappings laid the groundwork for the development of fixed point theory, which has applications across various branches of mathematics and beyond. George and Veeramani's [6] exploration of fuzzy metric spaces expanded the scope of metric spaces by incorporating fuzzy logic, enabling the modeling of imprecise and uncertain information. Hussain et. al [7] study on pentagonal controlled fuzzy metric spaces addressed dynamic market equilibrium, offering insights into stability and convergence properties in economic systems. Jeyaraman et al. [8] demonstrated the versatility of fixed point theory in various metric settings by discussing fixed point theorems in generalized $\mathfrak{IFMS}$ using contractive conditions of integral type. Jeyaraman et al. proposed common fixed point theorems for $(\phi - \psi)$-weak contractions in intuitionistic generalized fuzzy cone metric spaces, providing tools for analyzing fixed point properties in more general metric spaces.

Jungck et al. [10] discovered the common fixed point theorems for weakly compatible pairings on cone metric spaces. These theorems have implications for the research of metric spaces under geometric restrictions. The scholarly work of Jungck [11] has had a lasting impact on the field of dynamical systems and functional analysis, particularly in the areas of periodic and fixed points and commuting maps. The development of fuzzy metric and statistical metric spaces by Kramosil and Michalek [12] broadened the scope of metric spaces to include areas characterised by ambiguous or inaccurate data.

Meznik's work [13] underscored the interdisciplinary significance of fixed point theory by linking Banach's Fixed Point Theorem with market stability, thereby bridging mathematics and economics. Fuzzy metric spaces and Mihet's work [14] on fuzzy contractive mappings provide useful tools for studying stability and convergence in fuzzy systems. Mustafa et al. [15] into coincidence point results for generalised (ψ, ϕ) -weakly contractive mappings in ordered G-metric spaces contributed significantly to the comprehension of fixed point theory within ordered metric spaces. Park's introduction [16] of $\mathfrak{IFMS}$ presented a framework for effectively modelling uncertainty and vagueness within metric spaces, with practical applications in decision-making and pattern recognition. The proposal by Shukla et al. [17] of a new class of fuzzy contractive mappings and associated fixed point theorems expanded the analytical toolkit for studying fixed point properties within fuzzy metric spaces. Shukla et al. [18] investigate coincidence points of Edelstein type mappings in fuzzy metric spaces and their application. Gupta et al. [20] presented a common fixed point theorem for pair of self mappingsin partially ordered fuzzy metric spaces. Recently Mani et al. [19] have produced some fixed point results in fuzzy b- metric spaces by using two different t- norms, see also [21,22]. Zadeh's seminal work [23] on fuzzy sets marked a revolutionary advancement in the field of fuzzy logic, opening avenues for applications in artificial intelligence and decision-making under conditions of uncertainty.

2. Preliminaries

This part provides essential background information for the rest of the argument by outlining a number of definitions and a major finding.

Definition 2.1. *(Jungck [10, 11]). Consider a nonempty set Λ with mappings $\eta, \varrho : \Lambda \to \Lambda$. We define a coincidence point of η and ϱ as an element $\alpha \in \Lambda$ where $\varrho(\alpha) = \eta(\alpha)$, and this shared value is termed the corresponding point of coincidence. The set comprising all coincidence points of η and ϱ is denoted by $\mathcal{CP}(\eta, \varrho)$, while $P(\eta, \varrho)$ represents the set of all points of coincidence.A common fixed point of η and ϱ is recognised as α if $\alpha \in \mathcal{CP}(\eta, \varrho)$ and $\eta(\alpha) = \varrho(\alpha) = \alpha$, where α is identical to itself. The maps between η and ϱ are considered weakly compatible if the equation $\eta(\varrho(\alpha)) = \varrho(\eta(\alpha))$ is satisfied for every $\alpha \in \mathcal{CP}(\eta, \varrho)$.*

Jungck [11], along with numerous other scholars, explored the periodic points of Picard sequences arising from a mapping. In this context, we introduce the concept of periodic points that pertain to a pair of mappings.

Definition 2.2. *A point ϖ_n is deemed a periodic point of mappings η and ϱ with period p if p is the minimal natural number such that $\varpi_{n+p} = \varpi_n$.*

Proposition 2.1. *[2,11] In the context of a nonempty set Λ and weakly compatible self-mappings η and ϱ on Λ, if η and ϱ share only one point of intersection $\varpi = \eta(\alpha) = \varrho(\alpha)$, then ϖ stands as the unique common fixed point of η and ϱ.*

Definition 2.3. *[9] A binary operation, denoted by $\hat{\otimes} : [0,1] \times [0,1] \to [0,1]$, qualifies as a continuous t-norm (CTN) if it meets the following criteria:*

(1) $\hat{\otimes}$ adheres to both associativity and commutativity properties.
(2) $\hat{\otimes}$ needs to be continuous.
(3) For every $\delta \in [0,1]$, it holds that $\delta \hat{\otimes} 1 = \delta$.
(4) For any $\delta, \alpha, \varsigma, \upsilon \in [0,1]$, if $\delta \leq \varsigma$ and $\alpha \leq \upsilon$, then $\delta \hat{\otimes} \alpha \leq \varsigma \hat{\otimes} \upsilon$.

Definition 2.4. *[9] A binary operation, denoted by $\hat{\oplus}$, is defined as follows: It maps the interval $[0,1] \times [0,1]$ to the interval $[0,1]$. A continuous t-conorm (CTCN) is specified by the following conditions:*

(1) $\hat{\oplus}$ adheres to both associativity and commutativity properties
(2) $\hat{\oplus}$ needs to be continuous
(3) For every $\delta \in [0,1]$, it holds that $\delta \hat{\oplus} 0 = \delta$.
(4) For any $\delta, \alpha, \varsigma, \upsilon \in [0,1]$, if $\delta \leq \varsigma$ and $\alpha \leq \upsilon$,, then $\delta \hat{\oplus} \alpha \leq \varsigma \hat{\oplus} \upsilon$.

Definition 2.5. *[16] A five-tuple $(\Lambda, \tilde{\mathfrak{G}}, \mathfrak{H}, \hat{\otimes}, \hat{\oplus})$ is termed an Intuitionistic Fuzzy Metric Space ($\mathfrak{IFMS}$) if Λ is a nonempty set, $\hat{\otimes}$ represents a CTN, $\hat{\oplus}$ denotes a CTCN, and $\tilde{\mathfrak{G}}, \tilde{\mathfrak{H}} : \Lambda \times \Lambda \times (0, \infty) \to [0,1]$ are the fuzzy sets that satisfy the following conditions:*

(i) $\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) + \tilde{\mathfrak{H}}(\alpha, \varpi, \xi) \leq 1$
(ii) $\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) > 0$;

(iii) $\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) = 1 \Leftrightarrow \alpha = \varpi$;
(iv) $\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) = \tilde{\mathfrak{G}}(\varpi, \alpha, \xi)$;
(v) $\tilde{\mathfrak{G}}(\alpha, \breve{o}, \xi + v) \geq \tilde{\mathfrak{G}}(\alpha, \varpi, \xi)\hat{\otimes}\tilde{\mathfrak{G}}(\varpi, \breve{o}, v)$;
(vi) $\tilde{\mathfrak{G}}(\alpha, \varpi,) : (0, \infty) \rightarrow [0, 1]$ *is continuous;*
(vii) $\tilde{\mathfrak{H}}(\alpha, \varpi, \xi) < 1$;
(viii) $\tilde{\mathfrak{H}}(\alpha, \varpi, \xi) = 0 \Leftrightarrow \alpha = \varpi$;
(ix) $\tilde{\mathfrak{H}}(\alpha, \varpi, \xi) = \tilde{\mathfrak{H}}(\varpi, \alpha, \xi)$;
(x) $\tilde{\mathfrak{H}}(\alpha, \breve{o}, \xi + v) \leq \tilde{\mathfrak{H}}(\alpha, \varpi, \xi)\hat{\otimes}\tilde{\mathfrak{G}}(\varpi, \breve{o}, v)$;
(xi) $\tilde{\mathfrak{H}}(\alpha, \varpi,) : (0, \infty) \rightarrow [0, 1]$ *is continuous;*

for all $\alpha, \varpi, \breve{o} \in \Lambda$ *and* $v, \xi > 0$.

In an $\mathfrak{IFMS}$ $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$, a sequence $\{\alpha_n\}$ is said to converge to $\alpha \in \Lambda$ if, for all $\xi > 0$, the limits as n approaches infinity satisfy: $\lim_{n\to\infty} \tilde{\mathfrak{G}}(\alpha_n, \alpha, \xi) = 1$
$\lim_{n\to\infty} \tilde{\mathfrak{H}}(\alpha_n, \alpha, \xi) = 0$ In such a scenario, we designate α as the limit of $\{\alpha_n\}$. The space $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$ is termed compact if every sequence in Λ possesses a convergent subsequence within Λ. A mapping $\eta : \Lambda \rightarrow \Lambda$ is termed continuous at $\alpha \in \Lambda$ if, for any convergent sequence $\{\alpha_n\}$ in Λ with a limit of α, the sequence $\{\tilde{\varrho}(\alpha_n)\}$ converges to $\tilde{\eta}(\alpha)$. Moreover, η is considered continuous on $\Omega \subseteq \Lambda$ if it displays continuity at every point within Ω. If $\eta : \Lambda \rightarrow \Lambda$ is a continuous function and Ω is a compact set, then the image of Ω under η, denoted as $\eta(\Omega)$, is also a compact set.

Definition 2.6. *In the context of an* $\mathfrak{IFMS}(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$, *where* $\Omega \subseteq \Lambda$ *and* $\eta : \Omega \rightarrow \Omega$ *is a mapping,* η *is termed Edelstein contractive on* Ω *if:*

$$\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) < \tilde{\mathfrak{G}}(\eta(\alpha), \eta(\varpi), \xi)$$
$$\tilde{\mathfrak{H}}(\alpha, \varpi, \xi) > \tilde{\mathfrak{H}}(\eta(\alpha), \eta(\varpi), \xi)$$

for all $\alpha, \varpi \in \Omega, \alpha \neq \varpi, \xi > 0$.

3. Main Results

In this section, we begin by introducing the concept of Edelstein ϱ-contractive mappings within the context of $\mathfrak{IFMS}$s. Following that, we aim to present two noteworthy findings: a theorem regarding coincidence points and another concerning common fixed points..

Definition 3.1. *Consider an* $\mathfrak{IFMS}$ $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$, *where* $\Omega \subseteq \Lambda$ *is a subset, and* $\eta, \varrho : \Omega \rightarrow \Omega$ *are two mappings. We define* η *to be an Edelstein* ϱ*-contractive mapping on* Ω *if:*

$$\left.\begin{array}{l} \tilde{\mathfrak{G}}(\varrho(\alpha), \varrho(\varpi), \xi) < \tilde{\mathfrak{G}}(\eta(\alpha), \eta(\varpi), \xi) \\ \tilde{\mathfrak{H}}(\varrho(\alpha), \varrho(\varpi), \xi) > \tilde{\mathfrak{H}}(\eta(\alpha), \eta(\varpi), \xi) \end{array}\right\} \tag{3.1}$$

for all $\alpha, \varpi \in \Omega, \varrho(\alpha) \neq \varrho(\varpi), \xi > 0$. *It is clear that every Edelstein contractive mapping* η *on* Ω *also satisfies the criteria for being Edelstein* ϱ*-contractive on* Ω, *with* ϱ *being the identity mapping* $\mathfrak{I}_\Omega$ *on* Ω.

Example 3.1. *Let $\Lambda = (0, \infty)$, $\lambda \hat{\otimes} \alpha = \max\{\lambda, \alpha\}$ for all $\lambda, \alpha \in [0,1]$ and the fuzzy sets $\tilde{\mathfrak{G}}, \tilde{\mathfrak{H}} : \Lambda \times \Lambda \times (0, \infty) \to [0,1]$ be defined by:*

$$\tilde{\mathfrak{G}}(\alpha, \varpi, \xi) = \frac{\min\{\alpha, \varpi\}}{\max\{\alpha, \varpi\}} \quad \tilde{\mathfrak{H}}(\alpha, \varpi, \xi) = 1 - \frac{\min\{\alpha, \varpi\}}{\max\{\alpha, \varpi\}}$$

for all $\alpha, \varpi \in \Lambda$. Then $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$ is an $\mathfrak{IFMS}$. Consider the mappings $\eta, \varrho : \Lambda \to \Lambda$ defined by $\eta(\alpha) = \alpha^2 + 2$ and $\varrho(\alpha) = \alpha^2 + 1$ for all $\alpha, \varpi \in \Lambda$. If $\alpha < \varpi$, then $\frac{\alpha^2+1}{\varpi^2+1} < \frac{\alpha^2+2}{\varpi^2+2}$ and $1 - \frac{\alpha^2+1}{\varpi^2+1} > 1 - \frac{\alpha^2+2}{\varpi^2+2}$, i.e., $\tilde{\mathfrak{G}}(\varrho(\alpha), \varrho(\varpi), \xi) < \tilde{\mathfrak{G}}(\eta(\alpha), \eta(\varpi), \xi)$ and $\tilde{\mathfrak{H}}(\varrho(\alpha), \varrho(\varpi), \xi) > \tilde{\mathfrak{H}}(\eta(\alpha), \eta(\varpi), \xi)$. The inequality remains valid when $\varpi < \alpha$, thus indicating that η maintains Edelstein ϱ-contractivity on Λ. However, η is deemed Edelstein contractive over Λ since equation (3.1) fails to satisfy conditions for $\alpha = 1$ and $\varpi = 2$.

From the example above, it's evident that $\eta(\Lambda) \subset \varrho(\Lambda)$. Thus, it's always feasible to formulate a Jungck($\mathcal{JK}$) sequence for the ordered pair (η, ϱ) with some initial value $\alpha_0 \in \Lambda$ (e.g., one can choose $\alpha_0 = 1$), while η and ϱ lack coincidence points and periodic points (as for any $\{\varpi_n\} \in J_\Lambda(\eta, \varrho)$, $\varpi_{n+p} \neq \varpi_n$ for all $p \in \mathbb{N}$). In the following proposition, we establish that if η and ϱ lack coincidence points, they also lack periodic points. Furthermore, we'll determine the conditions under which η and ϱ will have coincidence point.

Proposition 3.1. *Consider $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$ within the framework of $\mathfrak{IFMS}$, where $\Omega \subseteq \Lambda$, and two mappings $\eta, \varrho : \Omega \to \Omega$ are specified such that $J_\Lambda(\eta, \varrho) \neq \emptyset$. For η to be Edelstein ϱ-contractive on Ω, it is necessary to have either a coincidence point between ϱ and η or the lack of a periodic point between them.*

Proof. Let $\{\varpi_n\} \in J_\Lambda(\eta, \varrho)$ with an initial value of α_0. It's worth noting that any periodic point of order 1 serves as a coincidence point of η and ϱ. Therefore, we assume the absence of a periodic point of order 1. We assert that there exists no $n \in \mathbb{N}$ such that for any $p \geq 2$, $\varpi_{n+p} = \varpi_n$. Suppose to the contrary that there exist $n \in \mathbb{N}$ and $p \geq 2$ such that $\varpi_{n+p} = \varpi_n$. Then, as there is no periodic point of order 1, by condition (3.1) we have: for all $\xi > 0$.

$$\begin{aligned}
\tilde{\mathfrak{G}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right) &= \tilde{\mathfrak{G}}\left(\eta\left(\alpha_{n+p}\right), \eta\left(\alpha_{n+p-1}\right), \xi\right) \\
&> \tilde{\mathfrak{G}}\left(\varrho\left(\alpha_{n+p}\right), \varrho\left(\alpha_{n+p-1}\right), \xi\right) \\
&= \tilde{\mathfrak{G}}\left(\varpi_{n+p-1}, \varpi_{n+p-2}, \xi\right). \\
\tilde{\mathfrak{H}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right) &= \tilde{\mathfrak{H}}\left(\eta\left(\alpha_{n+p}\right), \eta\left(\alpha_{n+p-1}\right), \xi\right) \\
&< \tilde{\mathfrak{H}}\left(\varrho\left(\alpha_{n+p}\right), \varrho\left(\alpha_{n+p-1}\right), \xi\right) \\
&= \tilde{\mathfrak{H}}\left(\varpi_{n+p-1}, \varpi_{n+p-2}, \xi\right).
\end{aligned}$$

Proceeding in similar way, we obtain

$$\left.\begin{array}{l}
\tilde{\mathfrak{G}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right) > \tilde{\mathfrak{G}}\left(\varpi_{n+1}, \varpi_n, \xi\right) \\
\tilde{\mathfrak{H}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right) < \tilde{\mathfrak{H}}\left(\varpi_{n+1}, \varpi_n, \xi\right)
\end{array}\right\} \tag{3.2}$$

for all $\xi > 0$.
Again,

$$\begin{aligned}
\tilde{\mathfrak{G}}\left(\varpi_{n+1}, \varpi_n, \xi\right) &= \tilde{\mathfrak{G}}\left(\varpi_{n+1}, \varpi_{n+p}, \xi\right) \\
&= \tilde{\mathfrak{G}}\left(\eta\left(\alpha_{n+1}\right), \eta\left(\alpha_{n+p}\right), \xi\right) \\
&> \tilde{\mathfrak{G}}\left(\varrho\left(\alpha_{n+1}\right), \varrho\left(\alpha_{n+p}\right), \xi\right) \\
&= \tilde{\mathfrak{G}}\left(\varpi_{n}, \varpi_{n+p-1}, \xi\right) \\
&= \tilde{\mathfrak{G}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right). \\
\tilde{\mathfrak{H}}\left(\varpi_{n+1}, \varpi_n, \xi\right) &= \tilde{\mathfrak{H}}\left(\varpi_{n+1}, \varpi_{n+p}, \xi\right) \\
&= \tilde{\mathfrak{H}}\left(\eta\left(\alpha_{n+1}\right), \eta\left(\alpha_{n+p}\right), \xi\right) \\
&< \tilde{\mathfrak{H}}\left(\varrho\left(\alpha_{n+1}\right), \varrho\left(\alpha_{n+p}\right), \xi\right) \\
&= \tilde{\mathfrak{H}}\left(\varpi_{n}, \varpi_{n+p-1}, \xi\right) \\
&= \tilde{\mathfrak{H}}\left(\varpi_{n+p}, \varpi_{n+p-1}, \xi\right).
\end{aligned}$$

The stated inequality, when combined with equation (3.2), leads to a contradiction, therefore validating the claim. □

Lemma 3.1. *In the context of $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$ being an $\mathfrak{IFMS}$, with $\Omega \subseteq \Lambda$, and $\eta, \varrho : \Omega \to \Omega$ being two mappings, if η satisfies the Edelstein ϱ-contractive condition on Ω and ϱ is continuous on Ω, then η is also continuous on Ω.*

Proof. If $\Omega = \emptyset$, the result is trivial. Suppose $\alpha \in \Omega$ and $\{\alpha_n\}$ is a sequence in Ω converging to α. Due to the continuity of ϱ, the sequence $\{\varrho\left(\alpha_n\right)\}$ converges to $\varrho(\alpha)$, implying $\lim_{n\to\infty} \tilde{\mathfrak{G}}\left(\varrho\left(\alpha_n\right), \varrho(\alpha), \xi\right) = 1$ and $\lim_{n\to\infty} \tilde{\mathfrak{H}}\left(\varrho\left(\alpha_n\right), \varrho(\alpha), \xi\right) = 0$ for all $\xi > 0$. Since η is Edelstein ϱ-contractive on Ω by (3.1), we have:

$$\lim_{n\to\infty} \tilde{\mathfrak{G}}\left(\eta\left(\alpha_n\right), \eta(\alpha), \xi\right) \geq \lim_{n\to\infty} \tilde{\mathfrak{G}}\left(\varrho\left(\alpha_n\right), \varrho(\alpha), \xi\right) = 1$$

$$\lim_{n\to\infty} \tilde{\mathfrak{H}}\left(\eta\left(\alpha_n\right), \eta(\alpha), \xi\right) \leq \lim_{n\to\infty} \tilde{\mathfrak{H}}\left(\varrho\left(\alpha_n\right), \varrho(\alpha), \xi\right) = 0$$

for all $\xi > 0$. Thus, $\{\eta\left(\alpha_n\right)\}$ converges to $\eta(\alpha)$, implying that η is also continuous at $\alpha \in \Omega$ for all $\alpha \in \Omega$. □

Theorem 3.1. *Let $(\Lambda, \tilde{\mathfrak{G}}, \tilde{\mathfrak{H}}, \hat{\otimes}, \hat{\oplus})$ denote an $\mathfrak{IFMS}$, where Ω is a nonempty compact subset of Λ, and $\eta, \varrho : \Omega \to \Omega$ are two mappings satisfying $\eta(\Omega) \subseteq \varrho(\Omega)$. Assuming η is an Edelstein ϱ-contractive mapping and ϱ is continuous, then $P(\eta, \varrho) \cap \varrho(\Omega)$ contains only one element. Furthermore, for each initial value $\alpha_0 \in \Omega$, there exists $\{\varpi_n\} \in J_\Omega(\eta, \varrho)$ such that $\varpi_n = \eta\left(\alpha_{n-1}\right) = \varrho\left(\alpha_n\right)$ for $n \in \mathbb{N}$, converging to the unique element of $P(\eta, \varrho) \cap \varrho(\Omega)$. In this context, $P(\eta, \varrho)$ refers to the set of coincidence points between the mappings η and ϱ, defined as follows:*

$$P(\eta, \varrho) = \{x \in \Omega : \eta(x) = \varrho(x)\}$$

Proof. Let $\xi > 0$ be given. We introduce two mappings φ and χ defined on Ω into the interval $(0, 1]$:

$$\varphi(c)(\xi) = \tilde{\mathfrak{G}}(\varrho(c), \eta(c), \xi)$$
$$\chi(c)(\xi) = \tilde{\mathfrak{H}}(\varrho(c), \eta(c), \xi), \text{ for all } c \in \Omega.$$

As ϱ is continuous on Ω, according to Lemma (3.1), η is also continuous on Ω. Additionally, since $\tilde{\mathfrak{G}}$ and $\tilde{\mathfrak{H}}$ are continuous on $\Lambda \times \Lambda \times (0, \infty)$ (as stated in [31]), and ϱ and η are continuous, the functions φ and χ are continuous as well. Therefore, they attain their maximum and minimum values in Ω respectively.

Suppose $\zeta \in \Omega$ and for $\xi > 0$ the following holds:

$$\varphi(\zeta)(\xi) = \max_{c \in \Omega} \varphi(c)(\xi)$$
$$\chi(\zeta)(\xi) = \min_{c \in \Omega} \chi(c)(\xi)$$

We claim that $\zeta \in \mathcal{CP}(\eta, \varrho)$. Suppose the contrary, that is, $\eta(\zeta) \neq \varrho(\zeta)$. Since $\eta(\Omega) \subseteq \varrho(\Omega)$, there exists $\beta \in \Omega$ such that $\eta(\zeta) = \varrho(\beta) \neq \varrho(\zeta)$. Given the Edelstein ϱ-contractive property of η, for each $\xi > 0$, we obtain:

$$\begin{aligned}
\varphi(\beta)(\xi) &= \tilde{\mathfrak{G}}(\varrho(\beta), \eta(\beta), \xi) = \tilde{\mathfrak{G}}(\eta(\zeta), \eta(\beta), \xi) \\
&> \tilde{\mathfrak{G}}(\varrho(\zeta), \varrho(\beta), \xi) = \tilde{\mathfrak{G}}(\varrho(\zeta), \eta(\zeta), \xi) \\
&= \varphi(\zeta)(\xi). \\
\chi(\beta)(\xi) &= \tilde{\mathfrak{H}}(\varrho(\beta), \eta(\beta), \xi) = \tilde{\mathfrak{H}}(\eta(\zeta), \eta(\beta), \xi) \\
&< \tilde{\mathfrak{H}}(\varrho(\zeta), \varrho(\beta), \xi) = \tilde{\mathfrak{H}}(\varrho(\zeta), \eta(\zeta), \xi) \\
&= (\zeta)(\xi).
\end{aligned}$$

This contradicts the definition of ι. Therefore, we must have $\eta(\zeta) = \varrho(\zeta)$, implying $\zeta \in \mathcal{CP}(\eta, \varrho)$ and $\varrho(\zeta) \in \mathcal{P}(\varrho, \varrho)$. The uniqueness of the point of coincidence $\varrho(\zeta)$ follows from the contractive condition (3.1). Thus, $\mathcal{P}(\eta, \varrho) \cap \varrho(\Omega) = \{\varrho(\zeta)\}$.

Let $\alpha_0 \in \Omega$. We construct a sequence $\{\varpi_n\} \in J_\Omega(\eta, \varrho)$ with initial value α_0 as follows:

Since $\eta(\alpha_0) \in \eta(\Omega) \subseteq \varrho(\Omega)$, there exists $\alpha_1 \in \Omega$ such that $\eta(\alpha_0) = \varrho(\alpha_1) = \varpi_0$. Similarly, since $\eta(\alpha_1) \in \eta(\Omega) \subseteq \varrho(\Omega)$, there exists $\alpha_2 \in \Omega$ such that $\eta(\alpha_1) = \varrho(\alpha_2) = \varpi_1$. Continuing this process, we obtain $\varpi_n = \eta(\alpha_{n-1}) = \varrho(\alpha_n)$ for all $n \in \mathbb{N}$, i.e., $\{\varpi_n\} \in J_\Omega(\eta, \varrho)$ with an initial value α_0. We consider the following cases I and II:

Case I: If during the sequence construction, we encounter a term equal to $\varrho(\zeta)$ at any stage, denoted by some $n_0 \in \mathbb{N}$, we can set $\alpha_{n_0} = \zeta$ since $\zeta \in \Omega$. Consequently, we have $\varpi_{n_0-1} = \eta(\alpha_{n_0-1}) = \varrho(\zeta)$. Additionally, as $\eta(\zeta) \in \eta(\Omega) \subseteq \varrho(\Omega)$ and $\zeta \in \mathcal{CP}(\eta, \varrho)$, implying $\varrho(\zeta) = \eta(\zeta)$, we can then choose $\alpha_{n_0+1} = \zeta$. Consequently, $\varpi_{n_0} = \eta(\alpha_{n_0}) = \varrho(\alpha_{n_0+1}) = \eta(\zeta) = \varrho(\zeta)$. Continuing in a similar manner, we find that the sequence $\{\varpi_n\}$ eventually becomes constant (i.e., $\varpi_{n_0+i} = \varrho(\zeta)$ for all $i \in \mathbb{N}$), converging to $\varrho(\zeta)$.

Case II: Let's assume that $\varpi_{n-1} = \eta(\alpha_{n-1}) = \varrho(\alpha_n) \neq \varrho(\zeta)$ for all $n \in \mathbb{N}$.
We define $\tilde{\mathfrak{G}}_n(\xi) = \tilde{\mathfrak{G}}(\varpi_n, \varrho(\zeta), \xi)$ and $\tilde{\mathfrak{H}}_n(\xi) = \tilde{\mathfrak{H}}(\varpi_n, \varrho(\zeta), \xi)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi > 0$. Then, for each $\xi > 0$, we have:

$$\begin{aligned}\tilde{\mathfrak{G}}_n(\xi) &= \tilde{\mathfrak{G}}(\varpi_n, \varrho(\zeta), \xi) = \tilde{\mathfrak{G}}(\eta(\alpha_n), \eta(\zeta), \xi)\\ &> \tilde{\mathfrak{G}}(\varrho(\alpha_n), \varrho(\zeta), \xi)\\ &= \tilde{\mathfrak{G}}(\varpi_{n-1}, \varrho(\zeta), \xi)\\ &= \tilde{\mathfrak{G}}_{n-1}(\xi).\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak{H}}_n(\xi) &= \tilde{\mathfrak{H}}(\varpi_n, \varrho(\zeta), \xi) = \tilde{\mathfrak{H}}(\eta(\alpha_n), \eta(\zeta), \xi)\\ &< \tilde{\mathfrak{H}}(\varrho(\alpha_n), \varrho(\zeta), \xi)\\ &= \tilde{\mathfrak{H}}(\varpi_{n-1}, \varrho(\zeta), \xi)\\ &= \tilde{\mathfrak{H}}_{n-1}(\xi).\end{aligned}$$

Therefore, $\left\{\tilde{\mathfrak{G}}_n(t)\right\}$ forms an increasing sequence and $\left\{\tilde{\mathfrak{H}}_n(t)\right\}$ is a decreasing function in $(0,1]$, with both being convergent. Let's assume:

$$\lim_{n\to\infty} \tilde{\mathfrak{G}}_n(\xi) = \mathcal{M}(\xi) \in (0,1] \quad \text{and} \lim_{n\to\infty} \tilde{\mathfrak{H}}_n(\xi) = \mathcal{M}'(\xi) \in (0,1]$$

for each $\xi > 0$

Since $\varrho(\alpha) \neq \varrho(\zeta)$, by (3.1) we obtain

$$\begin{aligned}\tilde{\mathfrak{G}}(\varrho(\alpha), \varrho(\zeta), \xi_0) &= \mathcal{M}(\xi_0)\\ &= \lim_{i\to\infty} \tilde{\mathfrak{G}}(\varpi_{n_i}, \varrho(\zeta), \xi_0) = \tilde{\mathfrak{G}}(\eta(\alpha), \eta(\zeta), \xi_0)\\ &> \tilde{\mathfrak{G}}(\varrho(\alpha), \varrho(\zeta), \xi_0).\\ \tilde{\mathfrak{H}}(\varrho(\alpha), \varrho(\zeta), \xi_0) &= \mathcal{M}'(\xi_0)\\ &= \lim_{i\to\infty} \tilde{\mathfrak{H}}(\varpi_{n_i}, \varrho(\zeta), \xi_0) = \tilde{\mathfrak{H}}(\eta(\alpha), \eta(\zeta), \xi_0)\\ &< \tilde{\mathfrak{H}}(\varrho(\alpha), \varrho(\zeta), \xi_0).\end{aligned}$$

This contradiction shows that
$\mathcal{M}(\xi) = \lim_{n\to\infty} \tilde{\mathfrak{G}}_n(\xi) = \lim_{n\to\infty} \tilde{\mathfrak{G}}(\varpi_n, \varrho(\zeta), \xi) = 1$ and
$\mathcal{M}'(\xi) = \lim_{n\to\infty} \tilde{\mathfrak{H}}_n(\xi) = \lim_{n\to\infty} \tilde{\mathfrak{H}}(\varpi_n, \varrho(\zeta), \xi) = 0$ for all $\xi > 0$. Consequently, $\{\varpi_n\}$ converges to $\varrho(\zeta)$. □

Remark 3.1. *If the mappings η and ϱ are weakly compatible in the above theorem, then according to Proposition (2.1), they possess a unique common fixed point.*

4. Break-even Analysis

The break-even analysis model is a structured method for determining the point at which total costs equal revenue, known as the break-even point. This calculation involves finding the contribution per unit sold, which is the selling price minus the variable cost per unit. By dividing fixed costs by the unit contribution, one can identify the number of units needed to cover all fixed costs. It's important to note that the break-even point isn't fixed; it fluctuates as costs and prices change over time.

Break-even analysis is a vital tool for business owners as it pinpoints the point where total costs match revenue, termed the break-even point. This figure indicates the minimum level of sales required to cover operating expenses. At this juncture, no profit is made, yet no losses are incurred either. This metric serves as a crucial indicator for both budding enterprises, gauging the feasibility of their ventures, and established businesses, helping to pinpoint operational shortcomings.

Components of Break-even Analysis:

The three components of Break-even Analysis are as follows:

Fixed Costs: Fixed costs, also referred to as overhead costs, are the expenses that a company must cover regardless of its level of production. These costs remain stable and do not fluctuate with changes in production volume. Examples of fixed costs include rent or mortgage payments, equipment expenses, salaries, taxes, insurance premiums, and other ongoing operational expenses that remain consistent over time.

Variable Costs: Variable costs are expenses that fluctuate in direct proportion to changes in production output. As production increases, variable costs also rise, and conversely, they decrease when production decreases. Examples of variable costs include expenses such as packaging costs, wages for production workers, raw material costs, and other expenses directly tied to the production volume.

Total cost = Fixed cost + Variable cost

Selling Price: The selling price represents the amount that a seller or company charges customers in exchange for their products or services. This price is determined based on various factors such as the cost of raw materials used in production, labor wages, fixed expenses, and other relevant costs associated with bringing the product or service to market. The selling price plays a crucial role in determining profitability and competitiveness in the market.

Limitations:

(1) The challenges in determining the break-even point in many cases are due to the potential fluctuations in market conditions over the projected capacity range.
(2) The total cost line, which combines variable costs and fixed costs, is ideally represented as a straight line, but in reality, actual costs often do not vary in direct proportion.
(3) Additionally, the break-even analysis chart becomes more complex when a company produces a variety of products.

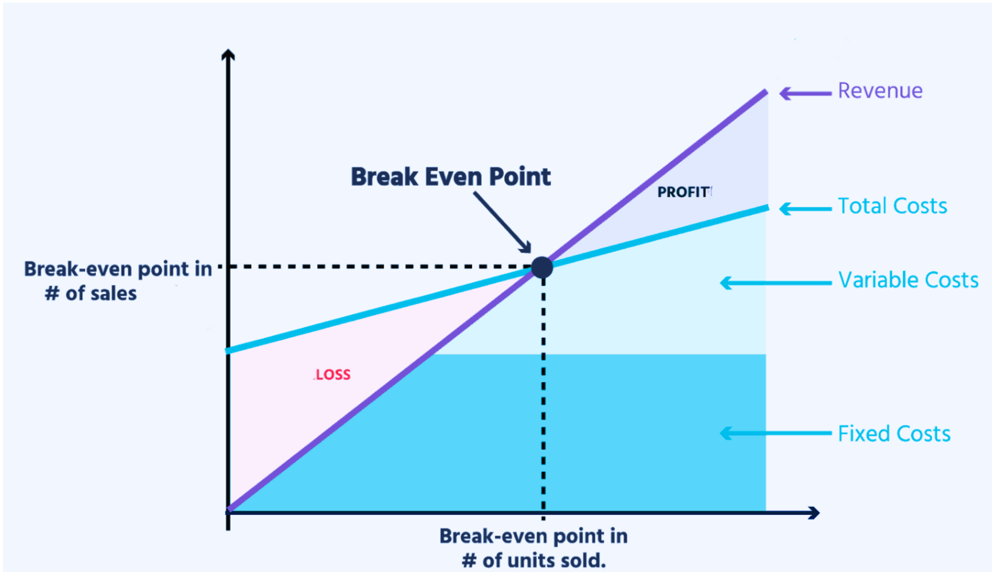

Figure 1. Break-even chart

5. Application to a Profitable Business

In this segment, we leverage Theorem (3.1) to establish that, given certain favorable conditions, a business currently operating at a loss can be transitioned into a profitable enterprise.

A company's total cost and revenue cost structure should be considered. On the $\mathcal{P}$ (vertical) axis, we show the several phases of a company's development, while on the $\mathcal{Q}$ (horizontal) axis, we show the product's sales volume and its price (total cost and revenue cost). Both the revenue cost function ($\mathcal{R}_C$) and the total cost function ($\mathcal{T}_C$) represent the sales volume (q) in terms of the price (p), respectively, and these functions entirely characterise these types of organisations. F_C stands for the constant function that represents fixed cost. Both the total cost function $\mathcal{T}_C$ and the revenue cost function $\mathcal{R}_C$ are growing given typical business circumstances (refer to Figure 2).

In practical scenarios, we operate under the assumption that the sales volume, total cost, and revenue cost prices of a product are non-negative finite values. These aspects are modeled as continuous functions of the quantity q. We stipulate that the quantity ($\mathcal{R}_C$ or $\mathcal{T}_C$) always falls within the range $[0, \lambda]$, while the corresponding prices vary within $[0, \alpha]$, where both λ and α are non-negative constants. Additionally, we assert that for each price p within $[0, \alpha]$, there exists a specific quantity corresponding to the revenue q within $[0, \lambda]$, and the same holds true for total cost. For practical applicability, with appropriate selection of functions, we can assume that both p and q lie within the interval $[0, \lambda]$ (see Remark 2 below). Throughout the discussion, we denote this interval as $\mathfrak{I} = [0, \lambda]$.

Definition 5.1. *The sensitive index of a business within an interval* $[\mathfrak{u}, \mathfrak{v}] \subseteq \mathfrak{I}$ *is denoted by* $\imath(\mathfrak{u}, \mathfrak{v})$ *and is defined as follows:*

$$\imath(\mathfrak{u}, \mathfrak{v}) = \frac{|\mathcal{T}_C(\mathfrak{u}) - \mathcal{T}_C(\mathfrak{v})|}{|\mathcal{R}_C(\mathfrak{u}) - \mathcal{R}_C(\mathfrak{v})|}, \mathfrak{u} \neq \mathfrak{v}.$$

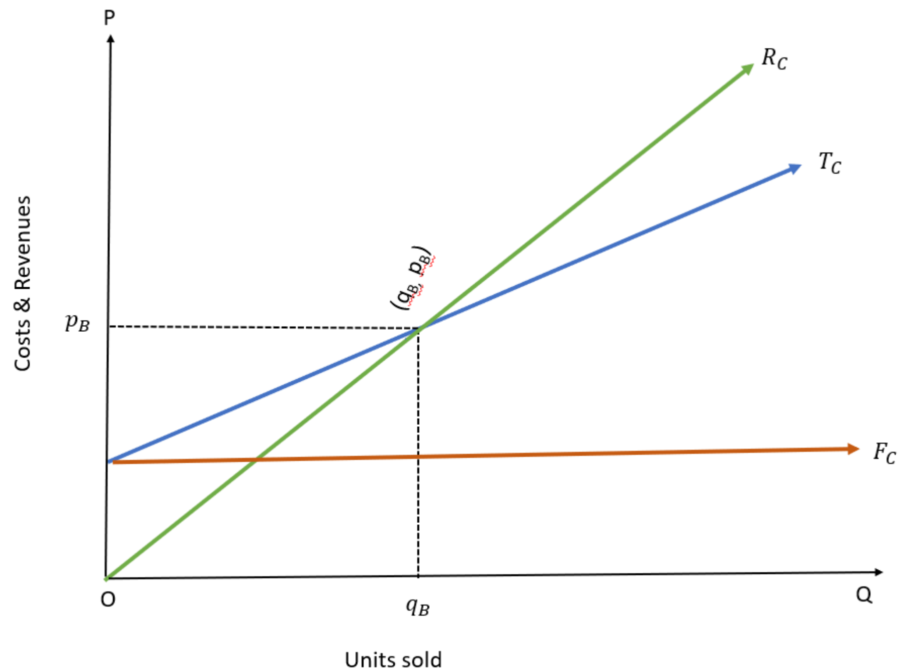

Figure 2. $\mathcal{R}_C$: Revenue line, $\mathcal{T}_C$: Total cost line

A business is considered profit-sensitive if $\imath(\mathfrak{u}, \mathfrak{v}) < 1$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{J}$, *indicating that:*

$$|\mathcal{T}_C(\mathfrak{u}) - \mathcal{T}_C(\mathfrak{v})| < |\mathcal{R}_C(\mathfrak{u}) - \mathcal{R}_C(\mathfrak{v})| \text{ for all } \mathfrak{u}, \mathfrak{v} \in \mathfrak{J} \text{ with } \mathfrak{u} \neq \mathfrak{v}. \tag{5.1}$$

According to the previous criteria, a profit-sensitive company is one in which the total revenue changes more quickly than the total cost of the product in response to a specific change in the quantity.

On the contrary, a company is deemed to be loss-sensitive if, when there's a change in quantity, the total product cost fluctuates more rapidly than the revenue of the product. If the ratio $t(\mathfrak{u}, \mathfrak{s})$, *where* $\mathfrak{u}, \mathfrak{s} \in \mathfrak{J}$, *exceeds 1 for all instances, then the company is mathematically categorized as loss-sensitive.*

$$|\mathcal{R}_C(\mathfrak{u}) - \mathcal{R}_C(\mathfrak{s})| < |\mathcal{T}_C(\mathfrak{u}) - \mathcal{T}_C(\mathfrak{s})| \text{ for all } \mathfrak{u}, \mathfrak{s} \in \mathfrak{J} \text{ with } \mathfrak{u} \neq \mathfrak{s}. \tag{5.2}$$

Theorem 5.1. *In the context of a profit-sensitive business, stability is maintained consistently, ensuring continuity in its operations. Additionally, within this framework, there exists a sequence of prices denoted as* $\{p_n\} \in \mathcal{JK}_{\mathfrak{J}}(\mathcal{T}_C, \mathcal{R}_C)$, *which converges towards the break-even price* p_B.

Proof. Let q denote a quantity within the range $\mathfrak{J}$. Considering the price corresponding to this quantity, associated with the total cost as $\mathcal{T}_C(q)$, and since each price has a specific quantity linked with the revenue cost, there exists another quantity $q' \in \mathfrak{J}$ such that $p = \mathcal{R}_C(q')$. This implies that the set of prices determined by total cost, $\mathcal{T}_C(\mathfrak{J})$, is a subset of those determined by revenue cost, $\mathcal{R}_C(\mathfrak{J})$.

Given that the business is profit-sensitive, we have:

$$|\mathcal{T}_C(\alpha)-\mathcal{T}_C(\varpi)|<|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)| \text{ for all } \alpha,\varpi\in\mathfrak{J} \text{ with } \alpha\neq\varpi.$$

The inequality described above remains valid for any α and ϖ belonging to $\mathfrak{J}$, given that $\mathcal{R}_C(\alpha)\neq\mathcal{R}_C(\varpi)$.

Let's define fuzzy sets $\tilde{\mathfrak{G}}$ and $\tilde{\mathfrak{H}}$ on $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ as follows:

$$\tilde{\mathfrak{G}}(\alpha,\varpi,\xi)=\frac{\xi}{\xi+|\alpha-\varpi|}, \text{ and } \tilde{\mathfrak{H}}(\alpha,\varpi,\xi)=\frac{|\alpha-\varpi|}{\xi+|\alpha-\varpi|}, \text{ for all } \alpha,\varpi\in\mathfrak{J}.$$

Then, it is evident that $(\mathbb{R},\tilde{\mathfrak{G}},\tilde{\mathfrak{H}},\hat{\otimes},\hat{\oplus})$ forms an $\mathfrak{IFMS}$, where $\xi_1\hat{\otimes}\xi_2=\max\{\xi_1,\xi_2\}$ and $\xi_1\hat{\oplus}\xi_2=\min\{\xi_1,\xi_2\}$ for all $\xi_1,\xi_2\in(0,\infty)$. Moreover, given $\mathcal{R}_C(\alpha)\neq\mathcal{R}_C(\varpi)$, as $|\mathcal{T}_C(\alpha)-\mathcal{T}_C(\varpi)|<|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|$, we deduce:

$$\tilde{\mathfrak{G}}(\mathcal{R}_C(\alpha),\mathcal{R}_C(\varpi),\xi)=\frac{\xi}{\xi+|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|}<\frac{\xi}{\xi+|\mathcal{T}_C(\alpha)-\mathcal{T}_C(\varpi)|}=\tilde{\mathfrak{G}}(\mathcal{T}_C(\alpha),\mathcal{T}_C(\varpi),\xi).$$

$$\tilde{\mathfrak{H}}(\mathcal{R}_C(\alpha),\mathcal{R}_C(\varpi),\xi)=\frac{|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|}{\xi+|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|}>\frac{|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|}{\xi+|\mathcal{T}_C(\alpha)-\mathcal{T}_C(\varpi)|}=\tilde{\mathfrak{H}}(\mathcal{T}_C(\alpha),\mathcal{T}_C(\varpi),\xi).$$

In the standard metric space $(\mathbb{R},|\cdot|)$, $\mathfrak{J}$ is compact. For any sequence $\{q_n\}$ in $\mathfrak{J}$ such that $\lim_{n\to\infty}\tilde{\mathfrak{G}}(q_n,q,\xi)=1$ and the limit of the set $\{\mathfrak{H}(q_n,q,\xi)\}$ as $n\to\infty$ is zero, given that $\lim_{n\to\infty}\left|q_n-q\right|=0$, holds for all $\{q_n\}$ sequences in $\mathbb{R}$ and q not in it. Thus, $\mathfrak{J}$ is a small subset of the $\mathfrak{IFMS}$ collection $(\mathbb{R},\tilde{\mathfrak{G}},\tilde{\mathfrak{H}},\hat{\otimes},\hat{\oplus})$.

Everything stated in Theorem (3.1) holds because $\Lambda=\mathbb{R}$, $\Omega=\mathfrak{J}$, $\eta\equiv\mathcal{T}_C$, and $\varrho\equiv\mathcal{R}_C$. According to Theorem (3.5), the $\mathcal{JK}$ sequence $\{p_n\}\in\mathcal{JK}_{\mathfrak{J}}(\mathcal{T}_C,\mathcal{R}_C)$ holds for all starting values $q_0\in\mathfrak{J}$. And the sequence converges to the unique point where $\mathcal{T}_C$ and $\mathcal{R}_C$ coincide, satisfying the condition $\mathcal{T}_C(q_n)=\mathcal{R}_C(q_{n-1})$ for all $n\in\mathbb{N}$. This unique point of coincidence is $p_{\mathrm{B}}=\mathcal{T}_C(q_{\mathrm{B}})=\mathcal{R}_C(q_{\mathrm{B}})$. Thus, the break-even price p_{B} of the company is constant. □

Theorem 5.2. *An organisation that is loss-sensitive maintains consistent stability, and it generates a sequence of prices denoted as the $\mathcal{JK}$ sequence, represented by $\{p_n\}$., within the framework of $\mathcal{R}_C$ and $\mathcal{T}_C$, that converges to the break-even price p_{B} in a decreasing order of value.*

Proof. Let's consider the relationship between price, p, and quantity, q, defined by the revenue cost function $\mathcal{R}_C$ for q within the feasible set $\mathfrak{J}$. Given that each price corresponds to a specific quantity of total cost, denoted by $q'\in\mathfrak{J}$, where $p=\mathcal{T}_C(q')$, it becomes evident that the set of prices generated by $\mathcal{R}_C$ is a subset of those generated by the total cost function $\mathcal{T}_C$. Considering the business's sensitivity to losses, this implies,

$$|\mathcal{R}_C(\alpha)-\mathcal{R}_C(\varpi)|<|\mathcal{T}_C(\alpha)-\mathcal{T}_C(\varpi)|, \text{ for all } \alpha,\varpi\in\mathfrak{J} \text{ with } \alpha\neq\varpi.$$

The inequality mentioned above applies to all pairs of quantities $\alpha,\varpi\in\mathfrak{J}$ where $\mathcal{T}_C(\alpha)\neq\mathcal{T}_C(\varpi)$. We then examine the fuzzy sets $\tilde{\mathfrak{G}}$ and $\tilde{\mathfrak{H}}$ defined on $\mathbb{R}\times\mathbb{R}\times(0,\infty)$, along with the CTN $\hat{\otimes}$ and

CTCN $\hat{\oplus}$ as outlined in the proof of Theorem 2.

Given that $|\mathcal{R}_C(\alpha) - \mathcal{R}_C(\varpi)| < |\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)|$ whenever $\mathcal{T}_C(\alpha) \neq \mathcal{T}_C(\varpi)$, we can conclude that,

$$\tilde{\mathfrak{G}}(\mathcal{T}_C(\alpha), \mathcal{T}_C(\varpi), \xi) = \frac{\xi}{\xi + |\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)|} < \frac{\xi}{\xi + |\mathcal{R}_C(\alpha) - \mathcal{R}_C(\varpi)|} = \tilde{\mathfrak{G}}(\mathcal{R}_C(\alpha), \mathcal{R}_C(\varpi), \xi).$$

$$\tilde{\mathfrak{H}}(\mathcal{T}_C(\alpha), \mathcal{T}_C(\varpi), \xi) = \frac{|\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)|}{\xi + |\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)|} > \frac{|\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)|}{\xi + |\mathcal{R}_C(\alpha) - \mathcal{R}_C(\varpi)|} = \tilde{\mathfrak{H}}(\mathcal{R}_C(\alpha), \mathcal{R}_C(\varpi), \xi).$$

Therefore, with $\Lambda = \mathbb{R}, \Omega = \mathfrak{J}, \eta \equiv \mathcal{R}_C$, and $\varrho \equiv \mathcal{T}_C$, all the conditions outlined in Theorem (3.1) are fulfilled. Consequently, according to Theorem (3.5), for any initial value $q_0 \in \mathfrak{J}$, the $\mathcal{JK}$ sequence $\{p_n\} \in \mathcal{JK}_{\mathfrak{J}}(\mathcal{R}_C, \mathcal{T}_C)$, where $p_n = \mathcal{R}_C(q_n) = \mathcal{T}_C(q_{n-1})$ for $n \in \mathbb{N}$, converges to the unique point where $\mathcal{T}_C$ and $\mathcal{R}_C$ coincide. Denoting this unique point as $p_{\mathrm{B}} = \mathcal{R}_C(q_{\mathrm{B}}) = \mathcal{T}_C(q_{\mathrm{B}})$, we identify p_{B} as the break-even price of the business, signifying its stability. □

Convergence for profit and loss-sensitive businesss: Let's begin by examining a firm that is focused on maximising profits. In reality, the price of a good or service does not correspond to the cost of equilibrium. At first, the price p_0 is seen as a constant expense. Assume that the price p_1 is not equal to p_B, specifically p_1 is more than p_B (if p_1 is less than p_B, the same reasoning will apply). Currently, p_1 is equal to the function $T_C(q_1)$. By utilizing the provided price p_1, we can determine a specific revenue cost, represented by q_0, by drawing a line parallel to the quantity axis and passing through the point $(0, p_1)$, as illustrated in Figure 2. In essence, this implies that $p_1 = T_C(q_1) = R_C(q_0)$.

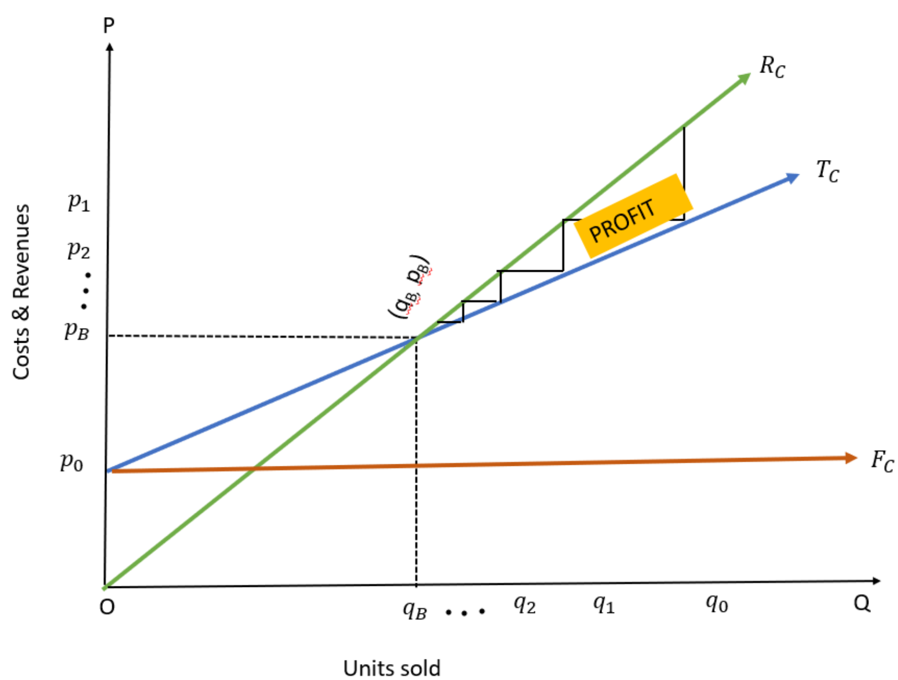

FIGURE 3. A profit-sensitive business

Referring to Figure 3, it becomes apparent that $q_0 < q_1$, indicating that the revenue generated is less than the total cost incurred for the product. Consequently, there exists a surplus of the commodity amounting to $q_1 - q_0$. Consequently, business owners respond by increasing the price to a level denoted as p_2. This price p_2 is directly related to the total cost accumulated from selling q_2 units, represented as $p_2 = T_C(q_2)$. This series of actions propels the company towards greater success. The initial price p_2 surpasses the break-even point, represented as p_B. At this stage, the total cost is q_2 and the revenue cost is q_1, with q_1 being less than q_2. In other words, p_2 equals both the total cost $T_C(q_2)$ and the revenue cost $R_C(q_1)$. Consequently, the difference between q_2 and q_1 signifies the quantity of excess goods sold. These consecutive outcomes drive the company to higher levels of achievement.

For every stage of the company, we establish a sequence of pricing $\{p_n\}$, where
$p_n = T_C(q_n) = R_C(q_{n-1})$, for all $n \geq 0$.

The progression of prices adheres to a specific sequence known as the $\mathcal{JK}$ sequence of the ordered pair (T_C, R_C). In simpler terms, $\{p_n\} \in \mathcal{JK}_1(T_C, R_C)$. The eventual recovery of the total cost of production as profit, even with a fifty percent selling price, underscores its susceptibility to changes in profitability. It's important to note that, due to the business being profit-sensitive, Theorem (5.1) necessitates the convergence of the $\mathcal{JK}$ sequence. The limit of this sequence signifies the break-even point for the functions T_C and R_C, essentially representing the break-even price for the business. Presently, the business is not undergoing significant alterations. Anything sold beyond that point is considered profit.

Alternatively, for a loss-sensitive company, if the initial price is $p_1 \neq p_{\text{B}}$ and $p_1 < p_{\text{B}}$ (similar explanation applies if $p_1 > p_{\text{B}}$). In this scenario, let's assume that p_1 corresponds to the result of applying the function $\mathcal{R}_C$ to q_1. Given a price p_1, there exists a specific quantity, denoted as q_0, which can be determined by drawing a line parallel to the P axis intersecting with the total cost curve and a line parallel to the Q axis passing through the point $(0, p_1)$ (see Figure 3). In essence, p_1 equals the total cost function $\mathcal{R}_C$ evaluated at q_1, which is also equivalent to the total cost function $\mathcal{T}_C$ evaluated at q_0.

Based on Figure 4, it becomes apparent that the quantity q_1 exceeds q_0, indicating an excess of the product amounting to $q_1 - q_0$. In response, producers increase the price of the product, reaching p_2. At this price, the corresponding revenue cost amount is q_2, represented as $p_2 = \mathcal{R}_C(q_2)$. This sequence of events propels the company to the subsequent level.

At the initial stage of the business, the price p_2 is lower than the break-even price, i.e., $p_2 < p_{\text{B}}$. At this point, the quantity of revenue cost is q_2, while the quantity of total cost is q_1, with $q_1 < q_2$. Thus, there is a shortage of product amounting to $q_2 - q_1$. Subsequently, producers raise the price to a value p_3, such that the corresponding quantity of revenue cost is q_3, i.e., $p_3 = \mathcal{R}_C(q_3)$. Given that there is a specific quantity of total cost corresponding to this price p_3, denoted as q_2, i.e., $p_3 = \mathcal{R}_C(q_3) = \mathcal{T}_C(q_2)$, this process drives the business to the next stage.

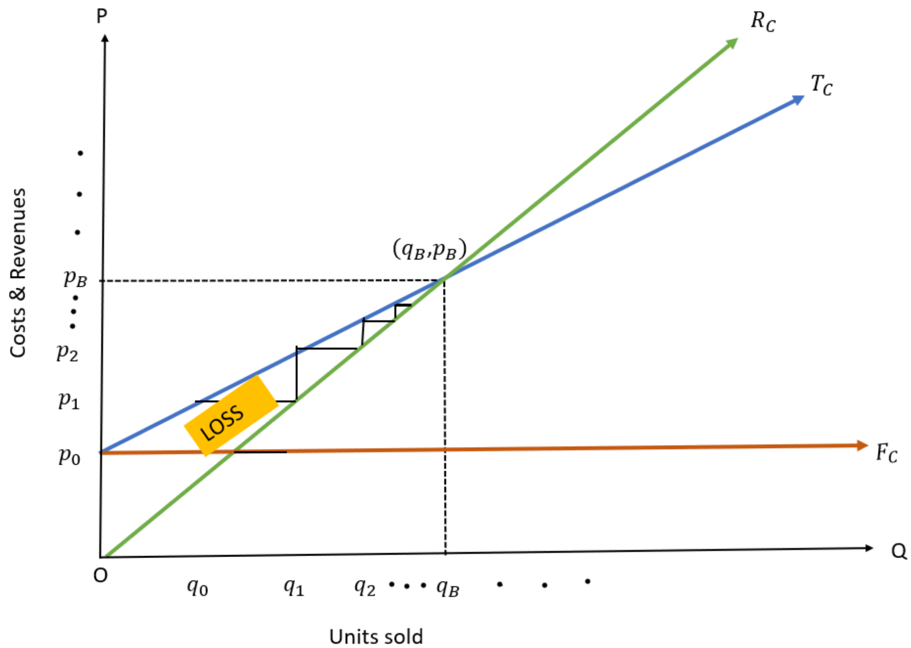

FIGURE 4. A loss-sensitive business

Similarly, for each stage of the business, we derive a sequence of prices $\{p_n\}$, where

$$p_n = \mathcal{R}_C\,(q_n) = \mathcal{T}_C\,(q_{n-1}) \text{ for all } n \geq 0.$$

The sequence of prices clearly forms a $\mathcal{JK}$ sequence of the ordered pair $(\mathcal{R}_C, \mathcal{T}_C)$, meaning that $\{p_n\}$ belongs to the set $\mathcal{JK}_{\mathfrak{J}}(\mathcal{R}_C, \mathcal{T}_C)$." However, if the costs used in making a product are only recovered by selling more than the intended amount, it leads to a financial loss, suggesting a high sensitivity to losses. Furthermore, given the business's vulnerability to losses, Theorem (5.2) requires the $\mathcal{JK}$ sequence to converge. The limit of this series corresponds to the point at which the functions $\mathcal{T}_C$ and $\mathcal{R}_C$ intersect, indicating the price at which the market reaches a break-even point. Based on this study, it is evident that the firm is maintaining stability at the break-even threshold. Nevertheless, any sales made below this threshold lead to financial losses.

Remark 5.1. *In practical situations, there may be cases where the quantities of total cost (or revenue cost) are within the range $[0, \lambda]$, while the corresponding product prices fall within $[0, \alpha]$, where $\alpha \neq \lambda$. To handle such scenarios, a technique called "scaling" is employed. We illustrate this method using the example of a profit-sensitive business, although the same approach can be applied to a loss-sensitive business:*

Let $\mathcal{T}_C, \mathcal{R}_C : [0, \lambda] \to [0, \alpha]$ be functions satisfying $|\mathcal{T}_C(\alpha) - \mathcal{T}_C(\varpi)| < |\mathcal{R}_C(\alpha) - \mathcal{R}_C(\varpi)|$ for all $\alpha, \varpi \in [0, \lambda]$ with $\alpha \neq \varpi$. Assume $\mathcal{T}_C([0, \lambda]) \subseteq \mathcal{R}_C([0, \lambda])$ and $\mathcal{R}_C$ is continuous.

If $\alpha \leq \lambda$ (i.e., $[0, \alpha] \subseteq [0, \lambda]$), then $\mathcal{T}_C, \mathcal{R}_C$ are self-mappings of $[0, \lambda]$. Consequently, following a similar process to the proof of Theorem (5.2), one can ensure that $\mathcal{T}_C, \mathcal{R}_C$ have a point of coincidence, denoted by $\zeta\,(= q_E) \in [0, \lambda]$, such that $\mathcal{T}_C(\zeta) = \mathcal{R}_C(\zeta)$. However, if $\lambda < \alpha$ (i.e., $[0, \alpha] \supset [0, \lambda]$), then $\mathcal{T}_C, \mathcal{R}_C$ are not *self-mappings of* $[0,\lambda]$*, rendering the same procedure inapplicable, and the existence of a point of coincidence cannot be determined.*

To create self-mappings on $[0,\lambda]$ *with the specified properties, we utilize the "scaling" technique on* $\mathcal{T}_C, \mathcal{R}_C$ *to enable the determination of a coincidence point in a manner akin to the proof of Theorem (5.1). Suppose* $\lambda < \alpha$*, and consider the functions:*

$$\mathcal{T}_{C1}(q) = \frac{\lambda}{\alpha}\mathcal{T}_C(q), \mathcal{R}_{C1}(q) = \frac{\lambda}{\alpha}\mathcal{R}_C(q) \text{ for all } q \in [0,\lambda].$$

Then, we notice that $\mathcal{T}_{C1}, \mathcal{R}_{C1} : [0,\lambda] \to [0,\lambda]$*. Moreover, for* $\alpha, \varpi \in [0,\lambda]$*, where* $\alpha \neq \varpi$*, we have:*

$$\begin{aligned}\left|\mathcal{T}_{C1}(\alpha) - \mathcal{T}_{C1}(\varpi)\right| &= \left|\frac{\lambda}{\alpha}\mathcal{T}_C(\alpha) - \frac{\lambda}{\alpha}\mathcal{T}_C(\varpi)\right| < \frac{\lambda}{\alpha}|\mathcal{R}_C(\alpha) - \mathcal{R}_C(\varpi)| \\ &= \frac{\lambda}{\alpha}\left|\frac{\alpha}{\lambda}\mathcal{R}_{C1}(\alpha) - \frac{\alpha}{\lambda}\mathcal{R}_{C1}(\varpi)\right| \\ &= \left|\mathcal{R}_{C1}(\alpha) - \mathcal{R}_{C1}(\varpi)\right|.\end{aligned}$$

Since $\mathcal{T}_C([0,\lambda]) \subseteq \mathcal{R}_C([0,\lambda])$*, we have*

$$\mathcal{T}_{C1}([0,\lambda]) = \left\{\frac{\lambda}{\alpha}\mathcal{T}_C(q) : q \in [0,\lambda]\right\} \subseteq \left\{\frac{\lambda}{\alpha}\mathcal{R}_C(q) : q \in [0,\lambda]\right\} = \mathcal{R}_{C1}([0,\lambda]).$$

Thus, $\mathcal{T}_{C1}$ *and* $\mathcal{R}_{C1}$ *both map from* $[0,\lambda]$ *to* $[0,\lambda]$*. By employing a procedure similar to the proof of Theorem (5.1), one can establish the existence of the break-even point of* $\mathcal{R}_{C1}$ *and* $\mathcal{T}_{C1}$*. Let* $\zeta\,(= q_E) \in \mathrm{BE}\,(\mathcal{T}_{C1}, \mathcal{R}_{C1})$*. It's worth noting that* $\zeta \in \mathrm{BE}\,(\mathcal{T}_{C1}, \mathcal{R}_{C1})$ *if and only if* $\zeta \in \mathrm{BE}(\mathcal{T}_C, \mathcal{R}_C)$*; thus, we have established the desired result.*

6. Differentiating Profit-Sensitive and Loss-Sensitive Market Strategies: Concluding Perspectives

In this section, we examine two strategies devised for profit-sensitive and loss-sensitive businesses, drawing comparisons with the approach outlined in [13].

The preceding discussion highlights that to evaluate the stability of a business, it suffices to focus on the convergence behavior of the $\mathcal{JK}$ sequence p_n of prices. If this sequence converges, it does so to the break-even price, indicating business stability. Furthermore, in profit-sensitive businesses, when the product price deviates from the break-even point, producers adjust the price until the total cost quantity aligns with the revenue cost quantity from the previous stage, termed as a "total cost-based" strategy. Consequently, for effective control of profit-sensitive businesses, producers should adopt a total cost-based approach. Conversely, in loss-sensitive businesses, price adjustments aim to match the supplied quantity with the demanded quantity from the prior stage, known as a "profit-based" strategy. Thus, to effectively manage loss-sensitive businesses, producers should employ a profit-based strategy.

The implications of these strategies are illustrated in Figures 2 and 3. Representing profit-sensitive businesses in Figure 2, the converging price sequence p_n forms a descending staircase leading towards the break-even price p_{B}. Conversely, in Figure 3, illustrating loss-sensitive businesses, the sequence forms an ascending staircase converging towards p_{B}. Failure to adhere to the prescribed strategies, such as employing a “loss-based" strategy for a loss-sensitive business (or a “profit-based" strategy for a profit-sensitive business), results in a divergent staircase, preventing the attainment of the business's break-even state.

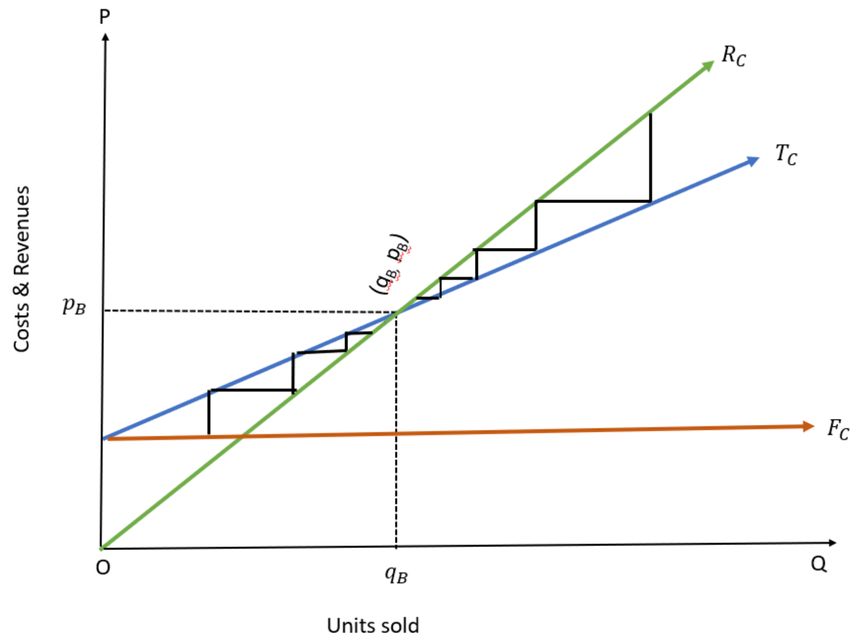

FIGURE 5. Failure of convergence

If the business doesn't satisfy either condition (5.1) or (5.2)—in other words, if it's neither profit-sensitive nor loss-sensitive—then it might lack stability. In such scenarios, the discussed procedures, especially regarding the sequences of prices, may fail to converge, making the break-even point unreachable. For example, if at any given stage, $|\mathcal{T}_C(q_{n-1}) - \mathcal{T}_C(q_n)| = |\mathcal{R}_C(q_{n-1}) - \mathcal{R}_C(q_n)|$ for some $q_{n-1}, q_n \in \mathfrak{I}$ (where the sensitivity index $\imath(q_{n-1}, q_n) = 1$) with $q_{n-1} \neq q_n$, then both discussed procedures fail to converge, as depicted in Figure 4.

In such cases, both profit-based and loss-based strategies yield a staircase-like sequence of prices where the price repeats after each consecutive stage. Mathematically, these cases display periodic points (prices) in the $\mathcal{JK}$ sequence, denoted by $p_0 = p_2 = \cdots$, thus hindering convergence for business operations. Conversely, if either condition (5.1) or (5.2) is satisfied, Proposition (3.1) guarantees that such obstacles are overcome, maintaining stability in the business.

7. CONCLUSION

This paper introduced a new theorem for finding break-even points $\mathfrak{IFMS}$. It explored how businesses analyze costs and revenue, focusing on dynamic environments. By using this theorem, the study shown how businesses naturally move toward their break-even points, whether they're profit-sensitive or loss-sensitive. This research highlights the importance of understanding financial equilibrium in dynamic business settings, providing useful insights for both practitioners and researchers.

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