Int. J. Anal. Appl. (2024), 22:125

International Journal of Analysis and Applications

Weakly Quasi (τ_1, τ_2)-Continuous Functions

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Abstract. This paper deals with the notion of weakly quasi (τ_1, τ_2)-continuous functions. Furthermore, some characterizations of weakly quasi (τ_1, τ_2)-continuous functions are discussed.

1. Introduction

The notion of quasi continuous functions was introduced by Marcus [24]. Popa [31] introduced and investigated the notion of almost quasi continuous functions. Neubrunnovaá [25] showed that quasi continuity is equivalent to semi-continuity due to Levine [22]. Popa and Stan [32] introduced and studied the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [23] which are independent of each other. It is shown in [28] that weak quasi continuity is equivalent to weak semi-continuity due to Arya and Bhamini [1] and Kar and Bhattacharyya [20]. Duangphui et al. [19] introduced and investigated the notion of weakly $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, some characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, $\delta p(\Lambda, s)$-continuous functions and pairwise weakly M-continuous functions were presented in [35], [36], [6], [33], [7], [8], [9], [13], [17], [38], [34] and [18], respectively. Bânzaru and Crivăţ [2] introduced and studied the concept of quasi continuous multifunctions.

Received: May 2, 2024.

2020 *Mathematics Subject Classification.* 54C08, 54E55.

Key words and phrases. $\tau_1\tau_2$-open set; $\tau_1\tau_2$-closed set; weakly quasi (τ_1, τ_2)-continuous function.

https://doi.org/10.28924/2291-8639-22-2024-125
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ISSN: 2291-8639

Popa and Noiri [29] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. The notion of weakly quasi continuous multifunctions was introduced and investigated by the present authors [27]. Several characterizations of weakly quasi continuous multifunctions have been obtained in [29]. Popa and Noiri [30] introduced and studied the notion of θ-quasi continuous multifunctions. Moreover, some characterizations of θ-quasi continuous multifunctions were studied in [26]. In [14], the present author introduced and studied the notions of almost quasi $\star$-continuous multifunctions and weakly quasi $\star$-continuous multifunctions. Additionally, some characterizations of weakly $\star$-continuous multifunctions and almost $\beta(\star)$-continuous multifunctions were established in [15] and [11], respectively. Laprom et al. [21] introduced and investigated the concept of $\beta(\tau_1, \tau_2)$-continuous multifunctions. In [39], the authors introduced and studied the notion of weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. In particular, some characterizations of weakly $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were investigated in [12], [10], [3], [4] and [5], respectively. In this paper, we introduce the notion of weakly quasi (τ_1, τ_2)-continuous functions. Moreover, several characterizations of weakly quasi (τ_1, τ_2)-continuous functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [16] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [16] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [16] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [16] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-$Cl(A)$ *and* $\tau_1\tau_2$-$Cl(\tau_1\tau_2$- $Cl(A)) = \tau_1\tau_2$-$Cl(A)$.
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$- $Cl(A) \subseteq \tau_1\tau_2$-$Cl(B)$.
(3) $\tau_1\tau_2$-$Cl(A)$ *is* $\tau_1\tau_2$-*closed.*
(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-$Cl(A)$.
(5) $\tau_1\tau_2$-$Cl(X - A) = X - \tau_1\tau_2$-$Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [39] (resp. *$(\tau_1, \tau_2)s$-open* [12], *$(\tau_1, \tau_2)p$-open* [12], *$(\tau_1, \tau_2)\beta$-open* [12]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq$

$\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)), $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)), $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)))). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be *$(\tau_1, \tau_2)r$-closed, $(\tau_1, \tau_2)s$-closed, $(\tau_1, \tau_2)p$-closed, $(\tau_1, \tau_2)\beta$-closed*. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\alpha(\tau_1, \tau_2)$-open* [37] if $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))). The complement of an $\alpha(\tau_1, \tau_2)$-open set is called *$\alpha(\tau_1, \tau_2)$-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the *$(\tau_1, \tau_2)s$-closure* [12] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the *$(\tau_1, \tau_2)s$-interior* [12] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 2.2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) (τ_1, τ_2)-*sCl*$(A) = \tau_1\tau_2$-*Int*$(\tau_1\tau_2$-*Cl*$(A)) \cup A$ [10];
(2) (τ_1, τ_2)-*sInt*$(A) = \tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Int*$(A)) \cap A$.

Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a *$(\tau_1, \tau_2)\theta$-cluster point* [39] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the *$(\tau_1, \tau_2)\theta$-closure* [39] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)\theta$-closed* [39] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be *$(\tau_1, \tau_2)\theta$-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the *$(\tau_1, \tau_2)\theta$-interior* [39] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2.3. [39] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.*
(2) *$(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.*

3. Weakly quasi (τ_1, τ_2)-continuous functions

In this section, we introduce the notion of weakly quasi (τ_1, τ_2)-continuous functions. Furthermore, several characterizations of weakly quasi (τ_1, τ_2)-continuous functions are discussed.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$, $f(G) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly quasi (τ_1, τ_2)-continuous if f is weakly quasi (τ_1, τ_2)-continuous at each point of X.*

Theorem 3.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly quasi (τ_1, τ_2)-continuous;*
(2) *for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)s$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$;*
(3) *$\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$f^{-1}(V) \subseteq (\tau_1, \tau_2)$-sInt$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(5) *(τ_1, τ_2)-sCl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open sets V of Y.*

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{U}(x)$ the family of all $\tau_1\tau_2$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\tau_1\tau_2$-open set G_H such that $G_H \subseteq H$ and $f(G_H) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Put $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then, W is $\tau_1\tau_2$-open in X and $x \in \tau_1\tau_2\text{-Cl}(W)$. Let $U = W \cup \{x\}$. Then, U is a $(\tau_1, \tau_2)s$-open set of X containing x and $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(2) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $x \in U \subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (4), we have

$$\begin{aligned}
X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) &= f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\
&\subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\
&= (\tau_1, \tau_2)\text{-sInt}(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\
&\subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(Y - V)) \\
&= (\tau_1, \tau_2)\text{-sInt}(X - f^{-1}(V)) \\
&= X - (\tau_1, \tau_2)\text{-sCl}(f^{-1}(V))
\end{aligned}$$

and hence $(\tau_1, \tau_2)\text{-sCl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(5) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. By (5) and Lemma 2.2,

$$\begin{aligned}
\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K)))) &\subseteq (\tau_1, \tau_2)\text{-sCl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(K)) \\
&= f^{-1}(K).
\end{aligned}$$

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. By (3) and Lemma 2.2,

$$\begin{aligned}
X - (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= (\tau_1, \tau_2)\text{-sCl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\
&\subseteq f^{-1}(Y - V) \\
&= X - f^{-1}(V)
\end{aligned}$$

and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (4), we have

$$f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Put $U = (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, U is a $(\tau_1, \tau_2)s$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that f is weakly quasi (τ_1, τ_2)-continuous. □

Theorem 3.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly quasi (τ_1, τ_2)-continuous;*
(2) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;*
(3) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;*
(4) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(5) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;*
(6) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Since $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y, by Theorem 3.1

$$\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$$

and by Lemma 2.2, we have (τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq (\sigma_1, \sigma_2)\theta$-Cl$(B)$ for every subset B of Y.

(3) ⇒ (4): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have $V \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ and $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Now, put $G = \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$, then G is $\sigma_1\sigma_2$-open in Y and $\sigma_1\sigma_2$-Cl$(G) = \sigma_1\sigma_2$-Cl(V). Thus by (4), (τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$.

(5) ⇒ (6): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Since $\sigma_1\sigma_2$-Int(K) is $(\sigma_1, \sigma_2)p$-open in Y and by (5), we have

$$\begin{aligned}(\tau_1, \tau_2)\text{-sCl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= (\tau_1, \tau_2)\text{-sCl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))\\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))\\ &= f^{-1}(K).\end{aligned}$$

(6) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y and by (6),

$$\begin{aligned}(\tau_1, \tau_2)\text{-sCl}(f^{-1}(V)) &\subseteq (\tau_1, \tau_2)\text{-sCl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

It follows from Theorem 3.1 that f is weakly quasi (τ_1, τ_2)-continuous □

Theorem 3.3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly quasi (τ_1, τ_2)-continuous;*
(2) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;*
(3) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;*
(4) *(τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have $V \subseteq \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ and hence $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Since

$\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y and by Theorem 3.2, (τ_1, τ_2)-sCl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) ⇒ (3): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) ⇒ (4): For any $(\sigma_1, \sigma_2)p$-open set V of Y, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed and $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)s$-open in Y.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y. By (4), we have

$$(\tau_1, \tau_2)\text{-sCl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

It follows from Theorem 3.2 that f is weakly quasi (τ_1, τ_2)-continuous. □

Theorem 3.4. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly quasi (τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(f^{-1}(V))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;*
(3) *(τ_1, τ_2)-sCl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;*
(4) *$f^{-1}(V) \subseteq (\tau_1, \tau_2)$-sInt$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since f is weakly quasi (τ_1, τ_2)-continuous, by Theorem 3.2 and Lemma 2.2,

$$\begin{aligned}\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(V))) &\subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2) and Lemma 2.2, we have

$$\begin{aligned}(\tau_1, \tau_2)\text{-sCl}(f^{-1}(V)) &= f^{-1}(V) \cup \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (3), we have

$$\begin{aligned}X - (\tau_1, \tau_2)\text{-sInt}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= (\tau_1, \tau_2)\text{-sCl}(X - (f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= (\tau_1, \tau_2)\text{-sCl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= (\tau_1, \tau_2)\text{-sCl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - f^{-1}(V)\end{aligned}$$

and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)$-sInt$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

(4) ⇒ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)p$-open, this follows from Theorem 3.1. □

Acknowledgements: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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