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Almost (τ_1, τ_2)-Continuity and $\tau_1\tau_2$-δ-Open Sets

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Abstract. This paper deals with the concepts of upper and lower almost (τ_1, τ_2)-continuous multifunctions. Moreover, we investigate some characterizations of upper and lower almost (τ_1, τ_2)-continuous multifunctions by utilizing $\tau_1\tau_2$-δ-open sets.

1. Introduction

δ-open sets, α-open sets and β-open sets, semi-open sets and preopen sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of weak forms of continuity for functions and multifunctions. Singal and Singal [53] introduced the concept of almost continuous functions as a generalization of continuity. Popa [50] defined almost quasi-continuous functions as a generalization of almost continuity [53] and quasi-continuity [36]. Munshi and Bassan [38] studied the notion of almost semi-continuous functions. Maheshwari et al. [35] introduced the concept of almost feebly continuous functions as a generalization of almost continuity [53]. Noiri [43] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [39] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions by utilizing the notions of preopen sets and β-open sets due to Mashhour et al. [37] and Abd El-Monsef et al. [1], respectively. The class of almost precontinuity is a generalization of each of almost feeble continuity and almost α-continuity. The class of almost β-continuity is a

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generalization of almost quasi-continuity and almost semi-continuity. Keskin and Noiri [31] introduced the concept of almost b-continuous functions by utilizing the notion of b-open sets due to Andrijević [2]. The class of almost b-continuity is a generalization of almost precontinuity and almost semi-continuity. The class of almost β-continuity is a generalization of almost b-continuity. Duangphui et al. [29] introduced and studied the notion of almost $(\mu, \mu')^{(m,n)}$-continuous functions. Viriyapong and Boonpok [61] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [17]. Furthermore, some characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were investigated in [55], [57], [4], [52], [10], [16], [9], [21], [28], [5], [6] and [7], respectively.

In 1982, Popa [49] introduced and studied the notion of almost continuous multifunctions. Popa and Noiri [47] introduced the notion of almost quasi-continuous multifunctions. Moreover, several characterizations of almost quasi-continuous multifunctions were investigated in [42]. Popa et al. [45] introduced the concept of almost precontinuous multifunctions. Additionally, some characterizations of almost precontinuous multifunctions were studied in [48]. Popa and Noiri [46] introduced and investigated the notion of almost α-continuous multifunctions. Noiri and Popa [41] introduced the concept of almost β-continuous multifunctions. The further characterizations of almost β-continuous multifunctions were investigated in [44]. Ekici and Park [30] introduced and studied the notion of almost γ-continuous multifunctions. Noiri and Popa [40] introduced and investigated the notion of almost m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. In [27], the present author introduced and studied the notion of pairwise almost M-continuous functions in biminimal structure spaces. Pue-on et al. [51] introduced and investigated the notions of upper and lower (τ_1, τ_2)-continuous multifunctions. Srisarakham et al. [54] introduced and studied the concept of weakly (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [33] introduced and investigated the notions of upper and lower almost (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of almost $\star$-continuous multifunctions, almost $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, almost $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\iota^{\star}$-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions and almost $\beta(\Lambda, sp)$-continuous multifunctions were established in [25], [22], [8], [11], [12], [14], [13], [19], [24], [60], [15], [32], [18] and [58], respectively. Laprom et al. [34] introduced and studied the notion of almost $\beta(\tau_1,\tau_2)$-continuous multifunctions. Viriyapong and Boonpok [62] introduced and investigated the concept of almost $(\tau_1,\tau_2)\alpha$-continuous multifunctions. Moreover, some characterizations of almost $(\tau_1,\tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1,τ_2)-continuous multifunctions were presented in [23] and [20], respectively. In this paper, we investigate some characterizations of upper and lower almost (τ_1,τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X,τ_1,τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1,2$. A subset A of a bitopological space (X,τ_1,τ_2) is called *$\tau_1\tau_2$-closed* [26] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. Let A be a subset of a bitopological space (X,τ_1,τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [26] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [26] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [26] *Let A and B be subsets of a bitopological space (X,τ_1,τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-$Cl(A)$ and $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Cl(A)) = \tau_1\tau_2$-$Cl(A)$.*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-$Cl(A) \subseteq \tau_1\tau_2$-$Cl(B)$.*
(3) *$\tau_1\tau_2$-$Cl(A)$ is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-$Cl(A)$.*
(5) *$\tau_1\tau_2$-$Cl(X - A) = X - \tau_1\tau_2$-$Int(A)$.*

A subset A of a bitopological space (X,τ_1,τ_2) is said to be *$(\tau_1,\tau_2)r$-open* [62] (resp. *$(\tau_1,\tau_2)s$-open* [23], *$(\tau_1,\tau_2)p$-open* [23], *$(\tau_1,\tau_2)\beta$-open* [23]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). The complement of a $(\tau_1,\tau_2)r$-open (resp. $(\tau_1,\tau_2)s$-open, $(\tau_1,\tau_2)p$-open, $(\tau_1,\tau_2)\beta$-open) set is called *$(\tau_1,\tau_2)r$-closed* (resp. *$(\tau_1,\tau_2)s$-closed*, *$(\tau_1,\tau_2)p$-closed*, *$(\tau_1,\tau_2)\beta$-closed*). A subset A of a bitopological space (X,τ_1,τ_2) is said to be *$\alpha(\tau_1,\tau_2)$-open* [59] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1,\tau_2)$-open set is called *$\alpha(\tau_1,\tau_2)$-closed* [6]. A subset A of a bitopological space (X,τ_1,τ_2) is said to be *$\tau_1\tau_2$-δ-open* [6] if A is the union of $(\tau_1,\tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called *$\tau_1\tau_2$-δ-closed*. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the *$\tau_1\tau_2$-δ-interior* [6] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the *$\tau_1\tau_2$-δ-closure* [6] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [3] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Characterizations of almost (τ_1, τ_2)-continuous multifunctions

In this section, we investigate some characterizations of upper and lower almost (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. [33] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost (τ_1, τ_2)-continuous if F has this property at each point of X.*

Lemma 3.1. [33] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost (τ_1, τ_2)-continuous;*
(2) *$F^+(V) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;*
(5) *$F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ for every subset B of Y;*
(6) *$F^+(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(7) *$F^-(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Lemma 3.2. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, the following properties hold:*

(1) *If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$.*
(2) *$\tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.*

Theorem 3.1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost (τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;*
(3) *$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 3.2, $\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by Lemma 3.1, $\tau_1\tau_2\text{-}\mathrm{Cl}(F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-}\mathrm{Cl}(B) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B)$.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (3), we have

$$\begin{aligned}\tau_1\tau_2\text{-}\mathrm{Cl}(F^-(K)) &= \tau_1\tau_2\text{-}\mathrm{Cl}(F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))))\\ &= \tau_1\tau_2\text{-}\mathrm{Cl}(F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(K)))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(K))\\ &= F^-(K)\end{aligned}$$

and hence $F^-(K)$ is $\tau_1\tau_2$-closed in X. By Lemma 3.1, F is upper almost (τ_1, τ_2)-continuous. □

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-δ-$Cl(B))))) \subseteq F^+(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ for every subset B of Y;*
(3) *$\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(B))))) \subseteq F^+(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ for every subset B of Y.*

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.2. [6] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V))$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$\delta$-$Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ for every subset B of Y;*
(3) *$\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ for every subset B of Y.*

Definition 3.3. [33] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)) \cap F(z) \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost (τ_1, τ_2)-continuous if F has this property at each point of X.*

Lemma 3.3. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost (τ_1, τ_2)-continuous, then for each $x \in X$ and each subset B of Y with $\sigma_1\sigma_2$-δ-$Int(B) \cap F(x) \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^-(B)$.*

Proof. Let $x \in X$ and B be a subset of Y with $\sigma_1\sigma_2$-δ-Int$(B) \cap F(x) \neq \emptyset$. Since $\sigma_1\sigma_2$-δ-Int$(B) \cap F(x) \neq \emptyset$, there exists a nonempty $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower almost (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$; hence $U \subseteq F^-(B)$. □

Lemma 3.4. [33] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-continuous;*
(2) *$F^-(V) \subseteq \tau_1\tau_2$-$Int(F^-(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2$-$Cl(F^+(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(B))))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y;*
(5) *$F^-(\sigma_1\sigma_2$-$Int(B)) \subseteq \tau_1\tau_2$-$Int(F^-(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(\sigma_1\sigma_2$-$Int(B)))))$ for every subset B of Y;*
(6) *$F^-(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(7) *$F^+(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Theorem 3.3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-continuous;*
(2) $\tau_1\tau_2$-$Cl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ *for every subset B of Y;*
(3) $F(\tau_1\tau_2$-$Cl(A)) \subseteq \sigma_1\sigma_2$-$\delta$-$Cl(F(A))$ *for every subset A of X;*
(4) $F^+(K)$ *is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(5) $F^-(V)$ *is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;*
(6) $F^-(\sigma_1\sigma_2$-δ-$Int(B)) \subseteq \tau_1\tau_2$-$Int(F^-(B))$ *for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$. Then, we have $x \in F^-(Y - \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)) = F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(Y - B))$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^-(Y - B) = X - F^+(B)$. Thus, $U \cap F^+(B) = \emptyset$ and hence $x \in X - \tau_1\tau_2\text{-Cl}(F^+(B))$. This shows that $\tau_1\tau_2\text{-Cl}(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2), we have

$$\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(F(A)))$$

and hence $F(\tau_1\tau_2\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(F(A))$.

(3) $\Rightarrow$ (1): Let B be any subset of Y. Then, by the hypothesis and Lemma 3.2,

$$\begin{aligned} F(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))) &\subseteq \tau_1\tau_2\text{-}\delta\text{-Cl}(F(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(B) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$. By Lemma 3.4, F is lower almost (τ_1, τ_2)-continuous.

(2) $\Rightarrow$ (4): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K) = K$. By (2), we have

$$\tau_1\tau_2\text{-Cl}(F^+(K)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) = F^+(K)$$

and hence $F^+(K)$ is $\tau_1\tau_2$-closed in X.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (6): Let B be any subset of Y. By (5), we have

$$F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(B)) = \tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(B))) \subseteq \tau_1\tau_2\text{-Int}(F^-(B)).$$

(6) $\Rightarrow$ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, we have V is $\sigma_1\sigma_2$-δ-open and $\sigma_1\sigma_2\text{-}\delta\text{-Int}(V) = V$. Thus, by (6) $F^-(V) \subseteq \tau_1\tau_2\text{-Int}(F^-(V))$ and hence $F^-(V)$ is $\tau_1\tau_2$-open in X. By Lemma 3.4, F is lower almost (τ_1, τ_2)-continuous. □

Definition 3.4. [51] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(i) *upper (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$;*
(ii) *lower (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

Definition 3.5. [56] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)s-regular if for each (τ_1, τ_2)s-closed set F and each $x \notin F$, there exist disjoint (τ_1, τ_2)s-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 3.5. [56] *Let (X, τ_1, τ_2) be a (τ_1, τ_2)s-regular space. Then, the following properties hold:*

(1) *$\tau_1\tau_2$-$Cl(A) = \tau_1\tau_2$-δ-$Cl(A)$ for every subset A of X.*
(2) *Every $\tau_1\tau_2$-open set is $\tau_1\tau_2$-δ-open.*

Lemma 3.6. [56] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is a (σ_1, σ_2)s-regular space, the following properties are equivalent:*

(1) *F is lower (τ_1, τ_2)-continuous;*
(2) *$F^+(\sigma_1\sigma_2$-δ-$Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;*
(3) *$F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(4) *$F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.*

Theorem 3.4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is a (σ_1, σ_2)s-regular space, the following properties are equivalent:*

(1) *F is lower (τ_1, τ_2)-continuous;*
(2) *$F^+(\sigma_1\sigma_2$-δ-$Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;*
(3) *$F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(4) *$F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;*
(5) *F is lower almost (τ_1, τ_2)-continuous.*

Proof. The proofs of the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are similar as in Lemma 3.6.

$(4) \Rightarrow (5)$: Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $\sigma_1\sigma_2$-open in Y and by Lemma 3.5, V is $\sigma_1\sigma_2$-δ-open in Y. By (4), we have $F^-(V)$ is $\tau_1\tau_2$-open in X. Thus by Lemma 3.4, F is lower almost (τ_1, τ_2)-continuous.

$(5) \Rightarrow (1)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)s-regular, there exists a $(\sigma_1, \sigma_2)r$-open set W such that $F(x) \cap W \neq \emptyset$ and $W \subseteq V$. Since F is lower almost (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap W \neq \emptyset$ for every $z \in U$. Thus, $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower (τ_1, τ_2)-continuous. □

Definition 3.6. [5] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is a (σ_1, σ_2)s-regular space, the following properties are equivalent:*

(1) *f is (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(\sigma_1\sigma_2$-δ-$Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;*

(3) *$f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(4) *$f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;*
(5) *f is almost (τ_1, τ_2)-continuous.*

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