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Qualitative Behavior of Sixth Order of Rational Difference Equation

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Abstract. The field of difference equations (DEs)has gained considerable prominence in applied analysis. The primary aim of this research is to conduct a comprehensive analysis on the periodicity of solutions, local asymptotic stability, and global behavior of DEs

$$U_{n+1} = sU_{n-2} + tU_{n-5} + \frac{hU_{n-2} + rU_{n-5}}{cU_{n-5} - e},\ n = 0, 1, 2, \ldots.$$

1. Introduction

Lately, there has been a growing interest among academics in nonlinear difference equations (NLDE). Over the past decade, we have witnessed a notable surge in attention toward these equations, possibly fueled by their diverse applications beyond mathematics. Fields such as biology, engineering, ecology, discrete temporal systems, economics, physics, and other math-related disciplines have found utility in these equations. Anticipating that this research area will continue to attract more scholars, we expect the allure of intriguing outcomes reported in studies to contribute to its sustained appeal. One persistent challenge in this field is the difficulty of obtaining closed-form solutions for NLDEs. Despite common assumptions, scholars actively engage in attempts to solve NLDEs through various means, as exemplified in references [9,16,34]. It is evident that determining a general solution form for such equations can be exceedingly complex. Nevertheless, recent efforts have introduced several strategies to transform challenging NLDEs into linear forms, thereby revealing recognized solution forms. Notably, this approach has successfully led to the closed-form solutions of a significant class of NLDEs, as demonstrated in references [5–8]. Many scholars have studied the behavior of equations of difference equations

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(DEs) and investigated the behavior of systems of solved DEs.,for instance : Khaliq and Elsayed [25] examined the DEs periodic solutions existence and dynamics:

$$T_{n+1} = \zeta_1 T_{n-2} + \frac{\zeta_2 T_{n-2}^2}{\beta_1 T_{n-2} + \beta_2 T_{n-5}}.$$

El-Metwally [11] examined the solutions form for the following systems of DEs:

$$A_{n+1} = \frac{A_{n-1}E_n}{\pm A_{n-1} \pm E_{n-2}},\ E_{n+1} = \frac{E_{n-1}A_n}{\pm E_{n-1} \pm A_{n-2}}.$$

R.P. Agarwal et al. [1] examined solutions for qualitative behavior of the DE:

$$Q_{n+1} = \alpha + \frac{aQ_{n-l}Q_{n-p}}{c - eQ_{n-d}},$$

where α, a, c, e are positive real constants.
A. Gelisken [24] examined behaviors of clearly stated solutions for the system that follows

$$S_{n+1} = \frac{E_1 X_{n-(3p-1)}}{W_1 + C_1 X_{n-(3p-1)} S_{n-(2p-1)} X_{n-(p-1)}},\ X_{n+1} = \frac{E_2 S_{n-(3p-1)}}{W_2 + C_2 S_{n-(3p-1)} X_{n-(2p-1)} S_{n-(p-1)}},$$

where $n, p \in \mathbb{N}_0$, the coefficients $E_1, E_2, W_1, W_2, C_1, C_2$ and the initial conditions are arbitrary numbers.
In [14,15], E.M. Elsayed found the solutions of the following DEs

$$T_{n+1} = \frac{T_{n-7}}{\pm 1 \pm T_{n-1}T_{n-3}T_{n-5}T_{n-7}},\ T_{n+1} = \frac{T_{n-9}}{\pm 1 \pm T_{n-4}T_{n-9}}.$$

E. Tasdemir [36] investigatedstudied the global asymptotic stability of the following system of DEs

$$X_{n+1} = u + t\frac{Y_n}{Y_{n-1}^2},\ Y_{n+1} = u + t\frac{X_n}{X_{n-1}^2},$$

Kostrov et al. [31] examined the following second order recursive equation to determine if it is bounded and whether it is stable locally and globally.

$$w_{n+1} = \frac{\eta + \kappa w_{n-1}}{\gamma w_n + \alpha w_n w_{n-1} + w_{n-1}}.$$

In [13], Zayed et al. examined some of the solutions qualitative characteristics for the NLDE

$$X_{n+1} = EX_n + CX_{n-p} + RX_{n-k} + FX_{n-j} + \frac{\alpha X_{n-p} + \beta X_{n-k}}{\gamma X_{n-p} + \lambda X_{n-k}},\ n = 0, 1, \ldots$$

References [1]- [39] contain further results on systems and rational difference equations that are related.
Differential equations and discrete difference equations serve as means to depict the dynamic characteristics of population systems, where the former is employed for species with overlapping generations, and the latter for those with non-overlapping generations.

In practical scenarios, tests and observations can be directly utilized to formulate a discrete model. When confronted with the numerical solution of a differential equation, particularly when an explicit solution is unattainable, it becomes beneficial to propose a finite-difference scheme. The most suitable difference equation approximation is the one where the solution coincides with the differential equation at discrete points [6]. However, meeting these criteria proves challenging unless both equations can be explicitly solved.

When a differential equation originates from a difference equation, it is generally preferred to retain the dynamical properties of the associated continuous-time model, includes maintaining equilibrium points, as well as analyzing their stability both locally and globally, along with studying bifurcation phenomena. El-Metwally et al. [12] examined the asymptotic tendencies of the population model:

$$X_{n+1} = \gamma + \alpha X_{n-1} e^{-X_n},$$

The generalized Beverton–Holt stock recruitment model has been examined in [4]:

$$X_{n+1} = \alpha X_n + \frac{r X_{n-1}}{1 + d X_{n-1} + e X_n}.$$

Elettreby and El-Metwally [10] examined certain qualitative characteristics of the following discrete model system in the field of economy

$$W_{n+1} = (1-E)W_n + PW_n(1-W_n)e^{-(W_n+H_n)},\ H_{n+1} = (1-E)H_n + PH_n(1-H_n)e^{-(W_n+H_n)}.$$

Khaliq et al. [26] examined dynamical analysis of the discrete-time Lotka-Volterra model system with two predators and one prey was conducted.

$$X_{n+1} = \frac{aX_n - bX_nY_n - cX_nZ_n}{1+dX_n},\ Y_{n+1} = \frac{eY_n + rX_nY_n - sY_nZ_n}{1+mY_n},\ Z_{n+1} = \frac{\alpha Z_n + hX_nZ_n - gY_nZ_n}{1+wZ_n}.$$

The authors in [27] studied local dynamics in a discrete-time COVID-19 epidemic model using topological classifications, bifurcation analysis, and chaos management.See also [33]- [38]. This research work aims to study the following new rational difference equation (RDE).

$$U_{n+1} = sU_{n-2} + tU_{n-5} + \frac{hU_{n-2} + rU_{n-5}}{cU_{n-5} - e},\ n = 0, 1, 2, \ldots, \tag{1.1}$$

constants s, t, h, r, c and e are positive real numbers and $U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}$ and U_0 are arbitrary real numbers.

2. Linearized Stability of Eq.(1)

This section shows that, under specific conditions, Eq. (1) has two equilibrium points (EQPs)and is asymptotically stable (AS). Eq. (1) fixed points is provided by

$$\bar{U}(1-s-t) = \frac{h\bar{U} + r\bar{U}}{c\bar{U} - e},$$

then

$$c\bar{U}^2(1-s-t) - \bar{U}(e(1-s-t) + h + r) = 0,$$

from which we can obtain the following two EQPs :

$$\bar{U}_1 = 0,\ \bar{U}_2 = \frac{h+r}{c(1-s-t)} + \frac{e}{c}.$$

Theorem 2.1. *If $\bar{U}_1 = 0$ is the first EQP of Eq.(1), it is locally asymptotically stable (LAS) if*

$$|-h-r| < e(1-s-t).$$

Proof. Let $\psi : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuous function defined by:

$$\psi(f,g) = sf + tg + \frac{hf+rg}{cg-e} \tag{2.1}$$

The following step is to locate the partial derivatives

$$\frac{\partial\psi(f,g)}{\partial f} = s + \frac{h}{cg-e}, \quad \frac{\partial\psi(f,g)}{\partial g} = t - \frac{(chf+re)}{(cg-e)^2}, \tag{2.2}$$

After that, calculating these partial derivatives at the EQP results in

$$\frac{\partial\psi(\bar{U}_1,\bar{U}_1)}{\partial f} = s - \frac{h}{e} = -p_1, \quad \frac{\partial\psi(\bar{U}_1,\bar{U}_1)}{\partial g} = t - \frac{r}{e} = -p_2$$

About the EQP $\bar{U}_1$, the relevant linearized DE of Eq. (1) is given by

$$S_{n+1} + p_1 S_n + p_2 S_{n-1} = 0.$$

The fixed point of Eq. (1) is AS , if

$$|p_1| + |p_2| < 1.$$

This could be expressed as

$$|s - \frac{h}{e}| + |t - \frac{r}{e}| < 1,$$

this implies,

$$|se - h + te - r| < e,$$

$$|-h-r| < e(1-s-t).$$

The proof is complete.

Theorem 2.2. *If*

$$|h\gamma - (h+e\gamma)\gamma| < h + r - s - t.$$

Where $\gamma = (1-s-t)$, then the second EQP $\bar{U}_2$ of Eq. (1) locally asymptotically stable .

Proof. Substituting $\bar{U}_2 = \frac{h+r}{c\gamma} + \frac{e}{c}$ into Eq. (6). We get

$$\frac{\partial\psi(\bar{U}_2,\bar{U}_2)}{\partial f} = s + \frac{h\gamma}{h+r} = -K_1, \quad \frac{\partial\psi(\bar{U}_2,\bar{U}_2)}{\partial g} = t - \frac{(h+e\gamma)\gamma}{h+r} = -K_2$$

Where $\gamma = (1-s-t)$. So, the linearized Eq. (1) about $\bar{U}_2$ is

$$S_{n+1} + K_1 S_n + K_2 S_{n-1} = 0.$$

$\bar{U}_2$ of Eq.(1) is asymptotically stable if

$$|K_1| + |K_2| < 1.$$

Thus,

$$|s+\frac{h\gamma}{h+r}|+|t-\frac{(h+e\gamma)\gamma}{h+r}|<1,$$

thus,

$$|s+h\gamma+t-(h+e\gamma)\gamma|<h+r.$$

Therefore,

$$|h\gamma-(h+e\gamma)\gamma|<h+r-s-t.$$

The proof is completed.

3. Global Attractivity Results

We will examine the global stability of the equilibrium points in this section.

Theorem 3.1. *The EQPs $\bar{U}$ of Eq. (1) is globally asymptotically stable (GAS) if*
(i) $se+tc+r>h+te+e$
(ii) $e+h+r>c$

Proof. Suppose that a and b be real numbers and assume $\psi(a,b)^2 \longrightarrow (a,b)$ is a function that defined by

$$\psi(f,g)=sf+tg+\frac{hf+rg}{cg-e}.$$

Now, we consider two cases

Case (i). Suppose that $\psi(f,g)$ is increasing in f and g . Then, assume (Q,q) is a solution of the following system

$$Q=\psi(Q,Q),$$
$$q=\psi(q,q),$$

So,

$$Q=sQ+tQ+\frac{hQ+rQ}{cQ-e},$$
$$q=sq+tq+\frac{hq+rq}{cq-e},$$

this gives,

$$cQ^2\gamma-eQ\gamma=Q(h+r), \tag{3.1}$$

$$cq^2\gamma-eq\gamma=q(h+r), \tag{3.2}$$

where $\gamma=(1-s-t)$ after subtracting (8) from (7). We get

$$(Q^2-q^2)c\gamma-(Q-q)e\gamma-(Q-q)(h+r)=0,$$

this implies,

$$(Q-q)\{(Q+q)c\gamma-e\gamma-(h+r)\}=0.$$

Thus, when $e+h+r>c$,

$$Q=q.$$

The EQPs $\bar{U}$ of Eq.(1) is a global attractor. The proof is completed.

Case (ii) Let $\psi(f,g)$ be increasing in f and decreasing in g. Then, assume (Q,q) is a solution of the following system

$$Q = \psi(Q,q),$$
$$q = \psi(q,Q),$$

So,

$$Q = sQ + tq + \frac{hQ + rq}{cq - e},$$
$$q = sq + tQ + \frac{hq + rQ}{cQ - e},$$

this implies,

$$Q(1-s)(cq-e) - tq(cq-e) - hQ - rq = 0, \tag{3.3}$$

$$q(1-s)(cQ-e) - tQ(cQ-e) - hq - rQ = 0. \tag{3.4}$$

Now, subtracting (10) from (9). We get

$$(Q-q)\{se - e + tc(Q+q) - te - h + r\} = 0.$$

Therefore, when $se + tc + r > h + te + e$

$$Q = q.$$

The EQPs $\bar{U}$ of Eq. (1) is a global attractor. The proof is completed.

4. Existence of Periodic Solutions

We shall discuss a principal theorem in this section that establishes the existence of periodic two solutions to Eq. (1).

Theorem 4.1. *Eq.(1) has solution of period two if and only if*

$$e(1-s-t) + h + r \neq 0 \tag{4.1}$$

Proof. . Assume that Eq. (1) has a solution of period two

$$\ldots \eta, \zeta, \eta, \zeta, \ldots,$$

with $\eta \neq \zeta$

$$\eta = s\eta + t\eta + \frac{h\eta + r\eta}{c\eta - e},$$
$$\zeta = s\zeta + t\zeta + \frac{h\zeta + r\zeta}{c\zeta - e}.$$

So,

$$c\eta^2(1-s-t)-e\eta(1-s-t)=\eta(h+r), \tag{4.2}$$

$$c\zeta^2(1-s-t)-e\zeta(1-s-t)=\zeta(h+r), \tag{4.3}$$

Subtracting (9) from (10) gives

$$c(1-s-t)(\eta^2-\zeta^2)-e(1-s-t)(\eta-\zeta)=(h+r)(\eta-\zeta),$$

this implies,

$$c(1-s-t)(\eta+\zeta)-e\eta(1-s-t)=(h+r),$$

$$\eta+\zeta=\frac{e(1-s-t)+h+r}{c(1-s-t)}. \tag{4.4}$$

Again, adding (9) and (10). We get

$$c(1-s-t)(\eta^2+\zeta^2)=\{e(1-s-t)+(h+r)\}(\eta+\zeta). \tag{4.5}$$

By using (11),(12), and the relation $(\eta+\zeta)^2=\eta^2+2\eta\zeta+\zeta^2$, we obtain

$$c(1-s-t)\{(\eta+\zeta)^2-2\eta\zeta\}=\{e(1-s-t)+(h+r)\}(\eta+\zeta).$$

then,

$$2c(1-s-t)\eta\zeta=c(1-s-t)(\eta+\zeta)^2-\{e(1-s-t)+(h+r)\}(\eta+\zeta),$$

$$2c(1-s-t)\eta\zeta=\frac{(e(1-s-t)+h+r)^2}{c(1-s-t)}-\{e(1-s-t)+h+r\}(\frac{e(1-s-t)+h+r)}{c(1-s-t)}.$$

Thus,

$$\eta\zeta=0. \tag{4.6}$$

Hence, based on equations (11) and (13), it can be deduced that η and ζ represent the two distinct roots of the quadratic equation

$$\lambda^2-(\eta+\zeta)\lambda+\eta\zeta=0. \tag{4.7}$$

This is,

$$\lambda^2-(\frac{e(1-s-t)+h+r}{c(1-s-t)})\lambda=0.$$

then,

$$c(1-s-t)\lambda^2-(e(1-s-t)+h+r)\lambda=0.$$

so,

$$(e(1-s-t)+h+r)^2>0.$$

If $(e(1-s-t)+h+r)\neq 0$, the condition (8) is satisfied. Conversely, assuming that condition (8) holds true, we will illustrate that Eq.(1) possesses a periodic solution with a prime period (PP) of two. Set

$$U_{-5}=U_{-3}=U_{-1}=P=\frac{e(1-s-t)+h+r}{c(1-s-t)}$$

$$U_{-4} = U_{-2} = U_0 = Q = 0$$

Now, we want to show that

$$U_1 = P, \; U_2 = 0.$$

It follows Eq. (1) that

$$U_1 = sP + tP + \frac{hP + rP}{cP - e},$$

so,

$$\begin{aligned} U_1 &= (s+t)(\frac{e(1-s-t)+h+r}{c(1-s-t)}) + \frac{(h+r)(\frac{e(1-s-t)+h+r)}{c(1-s-t)})}{c(\frac{e(1-s-t)+h+r)}{c(1-s-t)}) - e}, \\ &= (s+t)(\frac{e(1-s-t)+h+r}{c(1-s-t)}) + \frac{(h+r)(e(1-s-t)+h+r)}{c(h+r)} \\ &= (s+t)(\frac{e(1-s-t)+h+r}{c(1-s-t)}) + \frac{(e(1-s-t)+h+r)}{c} \\ &= (\frac{e(1-s-t)+h+r}{c})(1 + \frac{(s-t)}{(1-s-t)}) = \frac{e(1-s-t)+h+r}{c(1-s-t)} = P, \end{aligned}$$

$$U_2 = sQ + tQ + \frac{hQ + rQ}{cQ - e} = 0 = Q.$$

So, by induction we get:

$$U_{2n} = Q, \; U_{2n+1} = P$$

for all $n \leq -5$. Hence, Eq.(1) has two solutions P and Q.where P and Q represent the different quadratic roots Eq.(14).

5. Numerical Examples

The purpose of this section is to verify the theoretical work we did in the previous sections.

Example 1. This example demonstrates behavior of Eq.(1) tends to $\bar{U}_1 = 0$ when we assume that $s = 0.2,\ t = 0.1,\ h = 0.4,\ r = 0.4,\ c = 1,\ e = 3,\ U_{-5} = 5,\ U_{-4} = 0.4,\ U_{-3} = 0.4,\ U_{-2} = 5,\ U_{-1} = 5,$ and $U_0 = 0.4$. See Figure 1.

Example 2. Figure 2 shows how Eq. (1) behaves as it approaches the second equilibrium point. $\bar{U}_2$ is seen in figure 2 when we take the supposition that $s = 0.1,\ t = 0.2,\ h = 0.3,\ r = 2,\ c = 1,\ e = 1,\ U_{-5} = 7,\ U_{-4} = 3,\ U_{-3} = 3,\ U_{-2} = 2,\ U_{-1} = 2,$ and $U_0 = 1$.

Example 3. Figure 3 illustrates Eq. (1) unstable behavior. We presuppose that $s = 0.1,\ t = 0.13,\ h = 0.5,\ r = 2,\ c = 3,\ e = 2,\ U_{-5} = 12,\ U_{-4} = -5,\ U_{-3} = 2,\ U_{-2} = 5,\ U_{-1} = 5,$ and $U_0 = -3$.

Example 4. Figure 4 illustrates how Eq. (1) behaves globally in terms of stability. The behavior of Eq. (1) clearly tends to the fixed point $\bar{U}_1$ when we assume $s = 0.6,\ t = 0.2,\ h = 4,\ r = 2,\ c = 3,\ e = 1,\ U_{-5} = -8,\ U_{-4} = 2,\ U_{-3} = 5,\ U_{-2} = 3,\ U_{-1} = 6,$ and $U_0 = -1$.

Example 5. Figure 5 illustrates the fixed point $\bar{U}_2$ behavior in terms of global stability when $s = 0.1,\ t = 0.2,\ h = 0.3,\ r = 2,\ c = 1,\ e = 1,\ U_{-5} = 7,\ U_{-4} = 3,\ U_{-3} = 3,\ U_{-2} = 2,\ U_{-1} = 2,$ and $U_0 = 1$.

Example 6. Eq. (1) has a prime period two solution, as shown in Figure 6, when $s = 0.1$, $t = 0.2$, $h = 0.3$, $r = 2$, $c = 1$, $e = 1$, $U_{-5} = P$, $U_{-4} = Q$, $U_{-3} = P$, $U_{-2} = Q$, $U_{-1} = P$, and $U_0 = Q$ where P and Q satisfied Theorem 4.

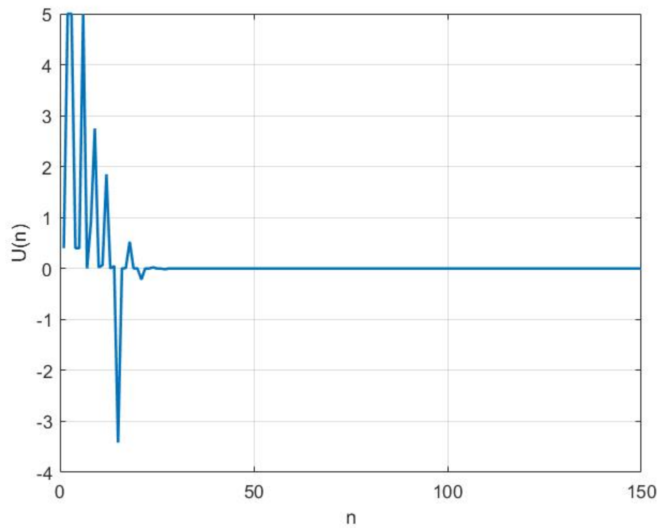

Figure 1.

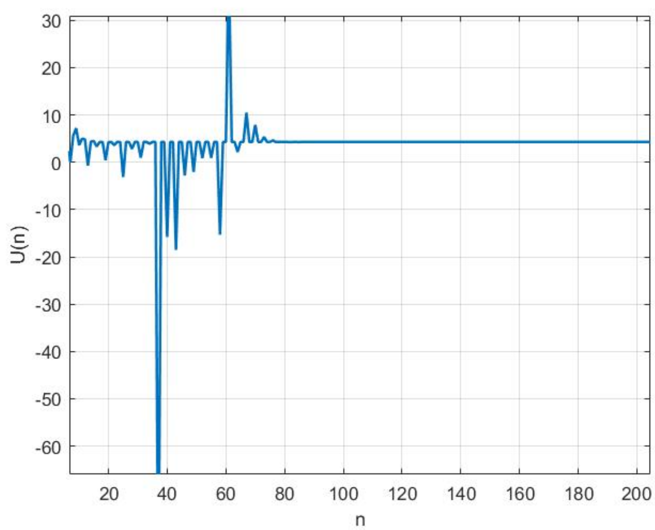

Figure 2.

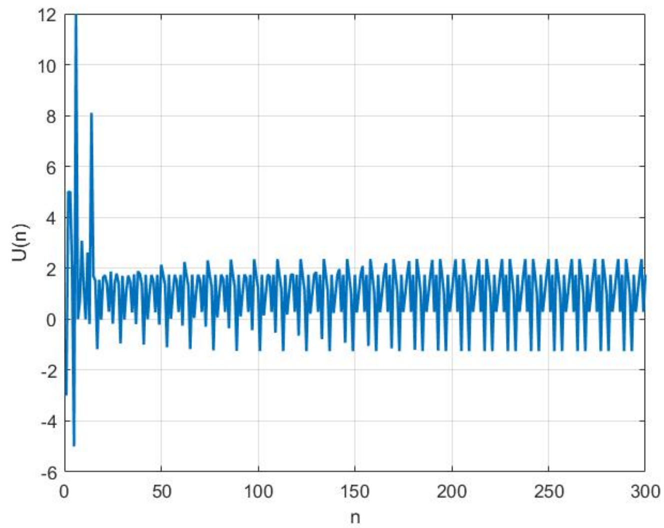

Figure 3.

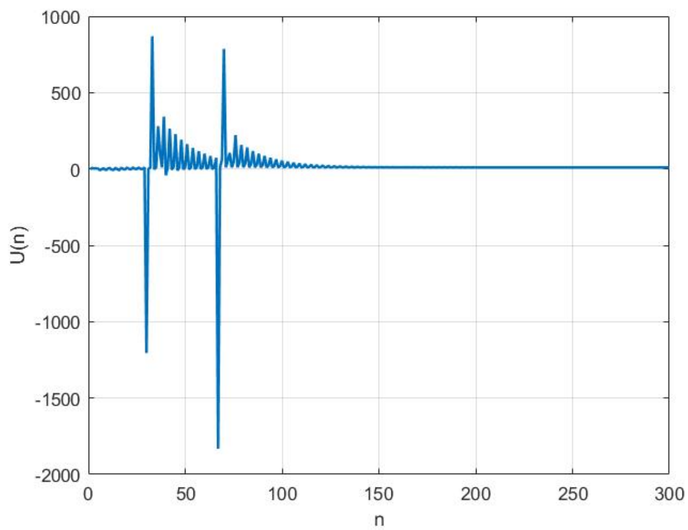

Figure 4.

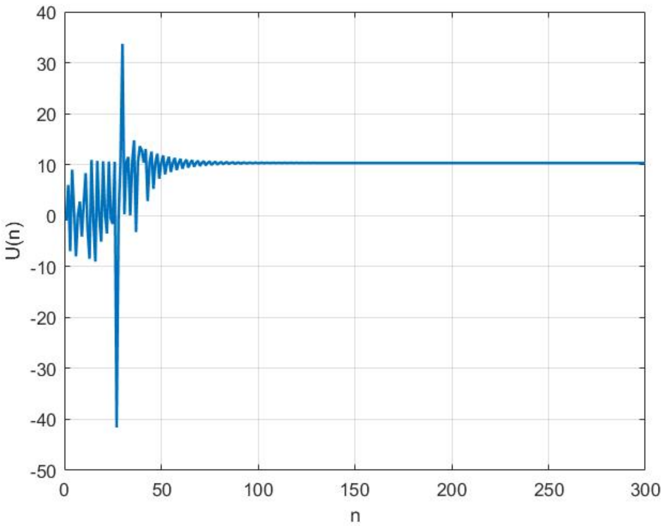

Figure 5.

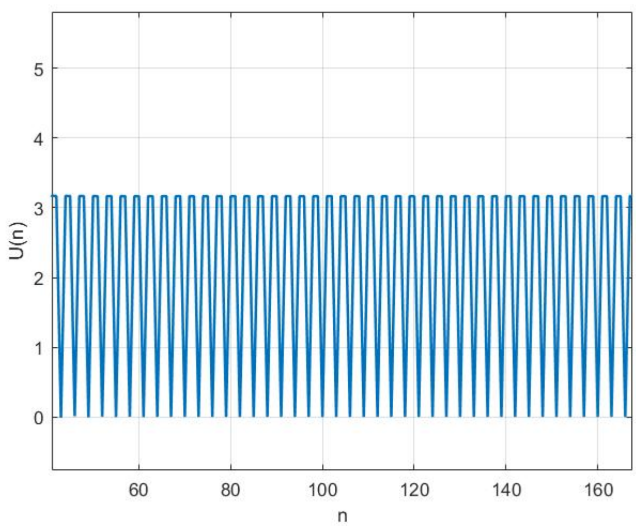

Figure 6.

6. Conclusion

This research explores the dynamics of NLDE (1). In Section 2, we demonstrate that when the LAS condition described in Theorems 1 and 2, $|-h-r| < e(1-s-t)$ is satisfied, the behavior converges towards the stability state of the EQP $\bar{U}_1 = 0$. While, the EQP $\bar{U}_2$ achieves LAS if $|h\gamma - (h+e\gamma)\gamma| < h+r-s-t$. The global solution of the EQPs is presented in Section 3. Section 4 examines the necessary and sufficient conditions for obtaining periodic solutions of Eq.(1). To validate our theoretical analysis, numerical examples are provided in Section 5, with Figures 1-6 confirming the results.

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