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An Algorithm for Construction of Optimal Integration Formulas in Hilbert Spaces and Its Realization

Kh. M. Shadimetov[1,2], **A. R. Hayotov**[2,3,4], **G. N. Akhmadaliev**[1,*]

[1]*Tashkent State Transport University, Odilkhodjaev street, 1, Tashkent, 100167, Uzbekistan*

[2]*V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, University Street, 9, Tashkent, 100174, Uzbekistan*

[3]*Central Asian University, Milliy Bog Str., 264, Barkamol MFY, M. Ulugbek District, Tashkent, 111221, Uzbekistan*

[4]*National University of Uzbekistan named after M. Ulugbek, University Str., 4, Tashkent, 100174, Uzbekistan*

[*]*Corresponding author:* ahmadaliyev78@mail.ru

Abstract. It is known that many problems in science and technology are reduced to the calculation of singular or regular integrals. Basically, these integrals are calculated approximately using quadrature and cubature formulas. In the present paper we develop an algorithm for construction of optimal quadrature formulas in some Hilbert spaces based on discrete analogues of the linear differential operators. Then we apply the algorithm to construct optimal quadrature formulas which are exact for hyperbolic functions and polynomials. We get explicit expressions for the coefficients of the optimal quadrature formulas. The obtained optimal quadrature formulas have $m-$th order of convergence.

1. Introduction

Many problems of science and technology lead to integral and differential equations or their systems. Solutions to such equations are often expressed in terms of definite integrals. In most cases, these integrals cannot be calculated exactly. Therefore, it is necessary to calculate the approximate value of such integrals with the highest possible accuracy and at a low cost.

Based on the known geometric meaning, the calculation of the numerical value for integrals is often called quadrature and cubature formulas, respectively. Various quadrature and cubature

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methods allow to calculate the integral using a finite number of values of the integrated function. These methods are universal and can be used where other calculation methods fail.

Many researchers have constructed various quadrature formulas based on certain ideas and taking into account the properties of the integrand. Thus, the well-known quadrature formulas of Gregory, Newton-Cotes, Simpson, Euler, Gauss, Chebyshev, Markov and others appeared, still used in practice.

Currently, in the theory of constructing quadrature and cubature formulas, there are the following main approaches: *algebraic, probabilistic, number-theoretic and functional*.

- In the algebraic approach, it is necessary to choose the nodes and coefficients of quadrature and cubature formulas so that these formulas are exact for all functions of a given set F. Taking into account the properties of the integrand, usually, the set F is taken to be algebraic or trigonometric polynomials whose degrees do not exceed a certain number of m or bounded rational functions.
- The probabilistic approach to constructing cubature formulas is based on the Monte-Carlo method.
- Number-theoretic approach of constructing cubature formulas is based on methods of number theory.
- The approach which uses the methods of functional analysis to constructing quadrature and cubature formulas in the functional spaces. It is believed that the integrands belong to some Banach space, and the difference between an integral and a quadrature sum that approximates it is considered some linear continuous functional. This functional is called the *error functional* of the formula, and the error of the formula is estimated by the norms of the error functional. By minimizing the norms of the error functional according to the parameters of quadrature and cubature formulas, optimal formulas for numerical integration of various meanings are obtained.

We note that in the present work we use the methods of functional analysis to get a new numerical integration formulas of high accuracy.

Due to this we consider a weighted quadrature formula

$$\int_a^b p(x)\varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = p(x)\varepsilon_{[a,b]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta), \tag{1.2}$$

where $p(x)$ is an integrable weight function, φ is an element of a Banach space B, C_β are coefficients and x_β are nodes of formula (1.1), $\varepsilon_{[a,b]}(x)$ is the characteristic function of the interval $[a, b]$, δ is the Dirac delta-function.

We suppose that the Banach space B is embedded into the space of continuous functions [27], i.e., $B \to C$. This is sufficient for the linear functional (1.2) to be defined on all functions from the space B.

The difference between the integral and the quadrature sum

$$(\ell, \varphi) = \int_a^b p(x)\varphi(x)dx - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) \tag{1.3}$$

is called *the error* of the quadrature formula (1.1), where $\ell(\varphi) = (\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx$ is the value of the error functional ℓ at the function φ.

The quadrature formula (1.1) is called exact for the function φ if the difference (1.3) is equal to zero. A set of all functions on which the quadrature formula (1.1) is exact is the kernel of the error functional ℓ.

The error of the quadrature formula (1.1) on functions of the space B is estimated from above by the norm of the error functional (1.2) in the dual space B^* as follows

$$|(\ell, \varphi)| \leq \|\ell\|_{B^*} \cdot \left\|\varphi\right\|_B ,$$

where the norm of the error functional (1.2) is defined as follows

$$\|\ell\|_{B^*} = \sup_{\|\varphi\|_B = 1} |(\ell, \varphi)| \tag{1.4}$$

It is clear that the norm of the error functional (1.2) depends on the coefficients and nodes. The problem of finding the minimum of the norm of the error functional with respect to the coefficients and nodes is called *the Nikolskii problem*, and the resulting formula is called *the optimal formula in the sense of Nikolskii* (or *the best formula*). Finding the minimum of the norm for the error functional at fixed nodes is called *the Sard problem*. The obtained formula is called *the optimal quadrature formula in the sense of Sard*.

Since this work is devoted to construction of optimal quadrature formulas in the sense of Sard, we will discuss only such formulas below.

Thus we consider the following.

Problem A. For the quadrature formulas of the form (1.1) find such coefficients $C_\beta = \mathring{C}_\beta$ that give a minimum to the norm of the error functional (1.2) and calculate the quantity

$$\left\|\mathring{\ell}\right\|_{B^*} = \inf_{C_\beta} \|\ell\|_{B^*} . \tag{1.5}$$

Here, $\mathring{C}_\beta$ are called *the optimal coefficients* and $\mathring{\ell}$ is the error functional corresponding to the optimal quadrature formulas of the form (1.1).

Problem A for quadrature formulas of the form

$$\int_0^N \varphi(x)dx \cong \sum_{k=0}^{N} p_k \varphi(k) \tag{1.6}$$

in the space $L_2^{(m)}$ was first studied by A. Sard [19], where $L_2^{(m)}$ is the Sobolev space of functions which are square integrable with m-th generalazed derivative. A.Sard and L.F.Meyers [18] obtained solutions for this problem when $m = 2$ for $N \leq 20$, when $m = 3$ for $N \leq 12$ and when $m = 4$ for $N \leq 19$. In the works of G.Coman [6,7] Sard's problem was studied in the case $m = 2$ for arbitrary N. In these works, G.Coman obtained a recursive formula for finding the optimal coefficients of quadrature formulas of the form (1.6) in the case $m = 2$.

In the work [20] by I. Schoenberg and S. Silliman, Sard's problem was studied for $N \to \infty$, i.e., for formulas of the form

$$\int_0^\infty \varphi(x)dx \cong \sum_{k=0}^{\infty} B_k^{(m)} \varphi(k). \tag{1.7}$$

In [20], an algorithm was given for finding the coefficients $B_k^{(m)}$ using splines of degree $2m-1$, and the coefficients $B_k^{(m)}$ were numerically computed on a computer in the cases $m = 1, 2, ..., 7$.

In the work of A.A. Maljukov and I.I. Orlov [17], A. Sard's problem was solved in the case $m = 2$, where the optimal coefficients of quadrature formulas of the form (1.6) were obtained using a cubic spline.

S. L. Sobolev [27] reduced the problem of constructing optimal quadrature formulas in the space $L_2^{(m)}(\mathbb{R})$ to solving a discrete problem of the Wiener-Hopf type. It should be noted that S.L.Sobolev studied more general case, i.e. constructing optimal lattice cubature formulas in the space $L_2^{(m)}(\mathbb{R}^n)$. He gave an algorithm for finding the optimal coefficients of cubature formulas. For quadrature formulas, he reduced finding the optimal coefficients of quadrature formulas of the form (1.1) in the space $L_2^{(m)}(\mathbb{R})$ to solving a system of $(2m-2)$ linear equations.

In [31], in the space $L_2^{(m)}$, F.Ya.Zagirova obtained numerical values of optimal coefficients for $m \leq 30$.

In the works [32] and [21], using the discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}}$ optimal quadrature formulas are constructed in the spaces $L_2^{(m)}(\Omega)$ and $L_2^{(m)}(\mathbb{R})$. It should be noted that the same result was independently obtained by P. Köhler in [15] using the spline method.

Further, in [23], the weighted optimal quadrature formulas of the form (1.1) were constructed in the spaces $L_2^{(m)}(0, 1)$ and $L_2^{(m)}(0, N)$ for all natural m.

F. Lanzara [16] gave a procedure to construct quadrature formulae which are exact for solutions of linear differential equations and are optimal in the sense of Sard. She presented the necessary and sufficient conditions under which such formulae do exist. At the end of the work, the author obtained several quadrature formulas by applying this method and compared them with well known formulas.

In [5], a review of papers on $\phi-$function method for constructing optimal quadrature formulas in the sense of Sard was given and this method was described. In addition, using this method, the authors constructed optimal quadrature formulas in the sense of Sard in the spaces $L_2^{(m)}$ for $m = 1, 2$.

Here we consider Problem A in more general space.

Let L be a linear differential operator defined as

$$L \equiv a_m \frac{d^m}{dx^m} + a_{m-1}\frac{d^{m-1}}{dx^{m-1}} + ... + a_1 \frac{d}{dx} + a_0,$$

where a_j are real numbers and $a_m \neq 0$. As L^* we denote the adjoint operator of L and it is defied as follows

$$L^* \equiv (-1)^m \frac{d^m}{dx^m}\{a_m \cdot\} + (-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\{a_{m-1}\cdot\} + ... + (-1)\frac{d}{dx}\{a_1 \cdot\} + a_0.$$

For convenience, a set of basis functions of the kernel for the operator L of order m we denote as

$$\{\phi_1, \phi_2, ..., \phi_m\}, \tag{1.8}$$

then a set of basis functions for the kernel of the differential operator L^*L of order $2m$ can be taken as follows

$$\{\phi_1, \phi_2, ..., \phi_m, \phi_{m+1}, ..., \phi_{2m}\}. \tag{1.9}$$

Now we consider Problem A on optimal quadrature formulas in the space $K^{(m)}(a,b)$, where $K^{(m)}(a,b)$ is the class of functions which have absolutly continuous $(m-1)$-st derivative and m-th derivatives are square integrable on the interval $[a,b]$ [1]. The space $K^{(m)}(a,b)$ is a generalization of the Sobolev space $L_2^{(m)}(a,b)$. After factorization with respect to the kernel of the operator L the space $K^{(m)}(a,b)$ becomes a Hilbert space with the inner product

$$\langle f, g\rangle_{K^{(m)}} = \int_a^b (Lf(x))(Lg(x))dx \tag{1.10}$$

and the corresponding norm is defined by the formula

$$\|f\|_{K^{(m)}} = \sqrt{\int_a^b (Lf(x))^2 dx}. \tag{1.11}$$

The main aim of this work is to give an algorithm for construction of optimal quadrature formulas of the form (1.1) in the sense of Sard in the Hilbert space $K^{(m)}(a,b)$. Then, based on this algorithm, for the operator $L \equiv \frac{d^m}{dx^m} - \omega^2 \frac{d^{m-2}}{dx^{m-2}}$, $m \geq 3$, to get analytic expressions of the coefficients of optimal quadrature formulas (1.1) when $p(x) = 1$.

The rest of the paper is organized as follows. In Section 2, using the Riesz representation theorem, the upper bound for the error of the quadrature formulas of the form (1.1) is obtained. This upper bound is expressed by the norm of the error functional (1.2). In Section 3 the problem of finding the conditional minimum of the norm of the error functional (1.2) is discussed. For the coefficients of the optimal quadrature formula (1.1) the system of linear equations is obtained. In Section 4 a new algorithm for analytic solution of the system of linear equations for optimal coefficients is developed. Finally, in Section 5, using this algorithm, the analytic expressions for the coefficients of the optimal quadrature formulas (1.1) are obtained. The obtained optimal quadrature formulas are exact for hyperbolic functions $\sinh(\omega x)$, $\cosh(\omega x)$ and for algebraic polynomials of degree $m-3$. It is easy to show that the order of convergence of the optimal quadrature formulas is $O(h^m)$, where

h is the mesh step. In addition, optimal and effective quadrature formulas are considered in the following works [33–39].

2. A Riesz element for the error functional (1.2)

In order to solve Problem A in the space $K^{(m)}(a,b)$, first, we need to calculate the norm of the error functional (1.2) which gives the sharp upper bound for the error of the quadrature formula (1.1).

Since $K^{(m)}(a,b)$ is the Hilbert space then by the Riesz representation theorem for the linear continuous functional ℓ defined as (1.2) on this space there exists a function ψ_ℓ from $K^{(m)}(a,b)$ such that

$$(\ell, f) = \langle \psi_\ell, f\rangle_{K^{(m)}} \text{ for all } f \in K^{(m)}(a,b) \tag{2.1}$$

and $\|\ell\|_{K^{(m)*}} = \|\psi_\ell\|_{K^{(m)}}$. Here $\langle \psi_\ell, f\rangle_{K^{(m)}}$ is the inner product of the functions ψ_ℓ and f.

In particular, from (2.1), based on the Reisz representation theorem, we get the following relation

$$(\ell, \psi_\ell) = \langle \psi_\ell, \psi_\ell\rangle_{K^{(m)}} = \|\ell\|^2_{K^{(m)*}}. \tag{2.2}$$

Thus, to calculate $\|\ell\|^2_{K^{(m)*}}$ we need ψ_ℓ from equation (2.1) which is a Riesz element for the error functional ℓ.

The following holds.

Theorem 1. *A solution ψ_ℓ of equation* (2.1) *has the form*

$$\psi_\ell(x) = (-1)^m \ell(x) * G_L(x) + \sum_{k=1}^{m} \lambda_k \phi_k(x), \tag{2.3}$$

where $G_L(x)$ is a fundamental solution of the operator L^*L and $\phi_1, \phi_2, ..., \phi_m$ are basis functions (1.8) *for the kernel of the operator L.*

Proof. First, we consider equation (2.1) for functions $f(x)$ from the space $\mathring{C}^{(\infty)}(a,b)$, where $\mathring{C}^{(\infty)}(a,b)$ is the space of infinitely differentiable and finite functions on $[a,b]$. Then from equation (2.1), integrating by parts its right-hand side we come to the equation for ψ_ℓ:

$$L^*L\psi_\ell(x) = (-1)^m \ell(x). \tag{2.4}$$

For the solution ψ_ℓ of equation (2.1) when f from the space $K^{(m)}(a,b)$, taking into account that the space $\mathring{C}^{(\infty)}(a,b)$ is dense in the space $K^{(m)}(a,b)$, together with equation (2.4) we get the following boundary conditions

$$\{L\psi_\ell(x)\}^{(k)}|_{x=a}^{x=b} = 0,\ k = 0, 1, ..., m-1. \tag{2.5}$$

Solving equation (2.4) with the boundary conditions (2.5) we get (2.3).

Theorem 1 is proved. □

We note that $G_L(x)$ is a fundamental solution of the operator L^*L, i.e. it satisfies the equation

$$L^*LG_L(x) = \delta(x) \tag{2.6}$$

and has the form (see [30, p. 92])

$$G_L(x) = \frac{\text{sign}(x)}{2} Z(x), \tag{2.7}$$

where $Z(x)$ is a solution of the homogeneous equation

$$L^* L Z(x) = 0, \tag{2.8}$$

with the initial conditions

$$Z(0) = Z'(0) = ... = Z^{(2m-2)}(0) = 0, \text{ and } Z^{(2m-1)}(0) = 1.$$

From here we conclude that $Z(x)$ is a linear combination of the basis functions (1.9).

It should be noted that since the error functional ℓ defined on the space $K^{(m)}(a,b)$ the following conditions should be imposed

$$(\ell, \phi_k) = 0, \ k = 1, 2, ..., m, \tag{2.9}$$

where ϕ_k, $k = 1, 2, ..., m$ are basis functions (1.8) of the kernel for the operator L.

We note that equalities (2.9) mean that our quadrature formula of the form (1.1) will be exact for the functions (1.8). These equalities can be written as follows

$$\int_a^b p(x)\phi_k(x)dx = \sum_{\beta=0}^{N} C_\beta \phi_k(x_\beta), \ k = 1, 2, ..., m.$$

Hence we conclude that for the existence of quadrature formulas of the form (1) exact on the functions ϕ_k, $k = 1, 2, ..., m$, it is sufficient to fulfillment the condition $N \geq m - 1$.

Now we can calculate the norm of the error functional (1.2). For this from (2.2), using (2.3), taking into account (2.9), we have the following

$$\|\ell\|^2_{K^{(m)*}} = (-1)^m \Bigg(\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta C_\gamma G_L(x_\beta - x_\gamma)$$

$$-2 \sum_{\beta=0}^{N} C_\beta \int_a^b p(x) G_L(x - x_\beta) dx + \int_a^b \int_a^b p(x)p(y) G_L(x-y) dx dy \Bigg). \tag{2.10}$$

Thus, we have obtained the upper bound for the error of quadrature formulas of the form (1.1) in the space $K^{(m)}(a,b)$.

Further, to solve Problem A, we find the minimum of the multi variable function (2.10) with respect to coefficients C_β, $\beta = 0, 1, ..., N$ under the conditions (2.9). Next sections are devoted to this problem.

3. The conditional minimum value of the expression (2.10)

For finding an extremum of the function (2.10) under the conditions (2.9) we consider the Lagrange function

$$\Lambda(C_0, C_1, ..., C_N, \lambda_1, \lambda_2, ..., \lambda_m) = \|\ell\|^2_{K^{(m)}} - 2(-1)^m \sum_{k=1}^{m} (\ell, \phi_k). \tag{3.1}$$

The function Λ depends on $N+m+1$ variables C_β, $\beta = 0, 1, ..., N$ and λ_k, $k = 1, 2, ..., m$. For stationary points of the function Λ we get the following system of $N+m+1$ linear equations

$$\sum_{\gamma=0}^{N} \mathring{C}_\gamma G_L(x_\beta - x_\gamma) + \sum_{k=1}^{m} \mathring{\lambda}_k \phi_k(x_\beta) = \int_a^b p(x) G_L(x - x_\beta) dx, \ \ \beta = 0, 1, ..., N, \tag{3.2}$$

$$\sum_{\beta=0}^{N} \mathring{C}_\beta \phi_k(x_\beta) = \int_a^b p(x) \phi_k(x) dx, \ \ k = 1, 2, ..., m, \tag{3.3}$$

where $\mathring{C}_\beta$, $\beta = 0, 1, ..., N$ are optimal coefficients.

The system (3.2)-(3.3) has a unique solution when $N \geq m-1$ and this solution gives the minimum to $\|\ell\|^2_{K^{(m)}}$ under the conditions (3.3). The uniqueness of the solution of such type of systems was discussed, for instance, in [26–28].

In the next section, we present an algorithm for finding the analytic solution of the system (3.2)-(3.3).

4. A new algorithm for analytic solution of the system (3.2)-(3.3)

Here we first present the main concept of the functions of discrete argument which will be used and operations on them. The theory of discrete argument functions is given, for example, in the works [27, 28]. For the purposes of completeness we give some definitions about functions of discrete argument.

Assume that the nodes x_β are equally spaced, i.e. $x_\beta = h\beta$, $h = \frac{b-a}{N}$, $N = 1, 2, ...$ Suppose that $f(x)$ and $g(x)$ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

Definition 1 A function $f(h\beta)$ is called *a function of discrete argument* if it is defined on some set of integer values of β.

Definition 2 We define *the inner product* of two discrete argument functions $f(h\beta)$ and $g(h\beta)$ as the following number

$$[f(h\beta), g(h\beta)] = \sum_{\beta=-\infty}^{\infty} f(h\beta) \cdot g(h\beta),$$

if the series on the right-hand side of the last equality converges absolutely.

Definition 3 We define *the convolution* of two discrete argument functions $f(h\beta)$ and $g(h\beta)$ as the inner product

$$f(h\beta) * g(h\beta) = \sum_{\gamma=-\infty}^{\infty} f(h\gamma) \cdot g(h\beta - h\gamma).$$

Further, when we use a discrete function we mean the function of a discrete argument.

In the algorithm for finding the coefficients of the optimal quadrature formulas of the form (1.1) the discrete analogue $D_L(h\beta)$ of the differential operator L^*L plays the main role.

The discrete argument function $D_L(h\beta)$ satisfying the following equation is the discrete analogue of the differential operator L^*L:

$$D_L(h\beta) * G_L(h\beta) = \delta_d(h\beta), \tag{4.1}$$

where $G_L(h\beta)$ is the discrete function corresponding to the function $G_L(x)$ defined by equality (2.7), $\delta_d(h\beta)$ is the discrete delta function and $\delta_d(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0. \end{cases}$

We note that equation (4.1) is the discrete variant of equation (2.6). The discrete analogue $D_L(h\beta)$ has similar properties as the differential operator L^*L. The convolution of the discrete function $D_L(h\beta)$ with the discrete argument functions $\phi_k(h\beta)$, $k = 1, 2, ..., 2m$, corresponding to the basis functions (1.9) of the kernel of the operator L^*L, is zero. That is

$$D_L(h\beta) * \phi_k(h\beta) = 0, \; k = 1, 2, ..., 2m. \tag{4.2}$$

The discrete analogue $D_L(h\beta)$ of the operator L^*L can be constructed by solution of equation (4.1). Equation (4.1) is solved using the theory of harrow shaped functions and the Fourier transforms [27,28]. In particular, for the differential operators $\frac{d^{2m}}{dx^{2m}}$, $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ and $\frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$ the corresponding discrete analogues were constructed in the works [11,22,25]-.

Suppose, we have the discrete analogue $D_L(h\beta)$ of the differential operator L^*L, satisfying equality (4.1) with properties (4.2).

Further, we consider the optimal coefficients $\mathring{C}_\beta$, $\beta = 0, 1, ..., N$ as a discrete function. For this we determine this function for other integer values of β as zero. That is we assume that $\mathring{C}_\beta = 0$ for $\beta = -1, -2, ...$ and $\beta = N+1, N+2, ...$ Then we can rewrite equality (3.2) in the following convolution form

$$G_L(h\beta) * \mathring{C}_\beta + \sum_{k=1}^{m} \mathring{\lambda}_k \phi_k(h\beta) = \int_a^b p(x) G_L(x - h\beta) dx, \;\; \beta = 0, 1, ..., N, \tag{4.3}$$

The left-hand side of (4.3) we denote as

$$u_L(h\beta) = G_L(h\beta) * \mathring{C}_\beta + \sum_{k=1}^{m} \mathring{\lambda}_k \phi_k(h\beta). \tag{4.4}$$

Then, it is easy to see that

$$\mathring{C}_\beta = D_L(h\beta) * u_L(h\beta). \tag{4.5}$$

Indeed, applying $D_L(h\beta)$ to the function $u_L(h\beta)$ defined by equality (4.4) we get

$$\begin{aligned} D_L(h\beta) * u_L(h\beta) &= D_L(h\beta) * \left(G_L(h\beta) * \mathring{C}_\beta + \sum_{k=1}^{m} \mathring{\lambda}_k \phi_k(h\beta) \right) \\ &= D_L(h\beta) * \left(G_L(h\beta) * \mathring{C}_\beta \right) + \sum_{k=1}^{m} \mathring{\lambda}_k \left(D_L(h\beta) * \phi_k(h\beta) \right). \end{aligned}$$

Hence, taking (4.1) and (4.2) into account, we come

$$D_L(h\beta) * u_L(h\beta) = \mathring{C}_\beta * \delta_d(h\beta) = \mathring{C}_\beta.$$

Thus, in order to find the coefficients $\mathring{C}_\beta$ we need to find the discrete function $u_L(h\beta)$ for all integer values of β. From equality (4.3), it is clear that for $\beta = 0, 1, ..., N$ the function $u_L(h\beta)$ is known and $u_L(h\beta) = \int_a^b p(x) G_L(x - h\beta) dx$. Now we define the function $u_L(h\beta)$ for $\beta = -1, -2, ...$ and $\beta = N+1, N+2, ...$ For this we use the function $G_L(x)$ defined by (2.7), where $Z(x)$ is the linear combination of the basis functions $\phi_1, ...\phi_{2m}$. Then, taking into account (4.1), we get

$$u_L(h\beta) = -Q_{2m}(h\beta) - R_m(h\beta) + P_m(h\beta) \text{ for } \beta = -1, -2, ...,$$

and

$$u_L(h\beta) = Q_{2m}(h\beta) + R_m(h\beta) + P_m(h\beta) \text{ for } \beta = N+1, N+2, ...,$$

where $Q_{2m}(h\beta) = \sum_{j=1}^{2m} \mu_j \phi_j(h\beta)$ is known, $R_m(h\beta) = \sum_{j=1}^{m} \nu_j \phi_j(h\beta)$ and $P_m(h\beta) = \sum_{k=1}^{m} \mathring{\lambda}_k \phi_k(h\beta)$ are unknown functions.

We denote

$$P_m^-(h\beta) = P_m(h\beta) - R_m(h\beta) \text{ and } P_m^+(h\beta) = P_m(h\beta) + R_m(h\beta), \tag{4.6}$$

where $P_m^-(h\beta) = \sum_{k=1}^{m} p_k^- \phi_k(h\beta)$ and $P_m^+(h\beta) = \sum_{k=1}^{m} p_k^+ \phi_k(h\beta)$. Then we get the following problem.

Problem B. Find the discrete argument function $u_L(h\beta)$ satisfying the equation

$$D_L(h\beta) * u_L(h\beta) = 0 \text{ for } \beta = -1, -2, ... \text{ and } \beta = N+1, N+2, ... \tag{4.7}$$

and having the form

$$u_L(h\beta) = \begin{cases} -Q_{2m}(h\beta) + P_m^-(h\beta), & \beta < 0, \\ \int_a^b p(x) G_L(x - h\beta), & 0 \le \beta \le N, \\ Q_{2m}(h\beta) + P_m^+(h\beta), & \beta > N, \end{cases} \tag{4.8}$$

In (4.8) there are $2m$ unknowns p_k^- and p_k^+, $k = 1, 2, ..., m$. For these unknowns from equation (4.7) we get the following system of $2m$ linear equations

$$\begin{aligned} D_L(h\beta) * u_L(h\beta) &= 0, \quad \text{for } \beta = -1, -2, ..., -m, \\ D_L(h\beta) * u_L(h\beta) &= 0, \quad \text{for } \beta = N+1, N+2, ..., N+m. \end{aligned}$$

The unknowns p_k^- and p_k^+, $k = 1, 2, ..., m$ are found from the last system. Then, we have $P_m^-(h\beta)$ and $P_m^+(h\beta)$.

Thus, Problem B is solved.

Since we have got $P_m^-(h\beta)$ and $P_m^+(h\beta)$, then from (4.6) we obtain

$$P_m(h\beta) = \frac{1}{2}(P_m^+(h\beta) + P_m^-(h\beta)) \text{ and } R_m(h\beta) = \frac{1}{2}(P_m^+(h\beta) - P_m^-(h\beta)).$$

Finally, from (4.5), using the discrete function $D_L(h\beta)$ and the function $u_L(h\beta)$ defined by equality (4.8) we get the optimal coefficients

$$\mathring{C}_\beta = D_L(h\beta) * u_L(h\beta) \text{ for } \beta = 0, 1, ..., N.$$

Therefore, we have got the algorithm for construction of the weighted optimal quadrature formulas of the form (1.1) in the sense of Sard in the space $K^{(m)}(a, b)$.

Thus, Problem A is also solved.

In the next section we apply this algorithm for construction of optimal quadrature formulas which are exact for hyperbolic functions $\sinh(\omega x)$, $\cosh(\omega x)$ and algebraic polynomials of certain degree.

It should be noted that the similar algorithm was applied for construction of the optimal numerical integration formulas for the Fourier coefficients, Fourier integrals and they used for reconstruction of Computed Tomography images in the works [3, 4, 12–14].

5. Optimal quadrature formulas exact for hyperbolic functions and algebraic polynomials of degree $m-3$

In the present section, based on the above given algorithm, we construct optimal quadrature formulas of the form (1.1) for the case $p(x) = 1$ and $[a, b] = [0, 1]$. The obtained optimal formulas will be exact for hyperbolic functions $\sinh(\omega x)$, $\cosh(\omega x)$ and for algebraic polynomials of degree $\leq (m-3)$ when $m \geq 3$.

5.1. **Statement of the problem.** We consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta), \tag{5.1}$$

where C_β are coefficients, $h = \frac{1}{N}$, $N = 1, 2, ...$, functions $\varphi(x)$ from the Hilbert space $K^{(m)}(0, 1)$ with the operator $L = \frac{d^m}{dx^m} - \omega^2 \frac{d^{m-2}}{dx^{m-2}}$, where $m \geq 3$ and $\omega \in \mathbb{R}\backslash\{0\}$.

For clarity, in the rest part of the paper, the space $K^{(m)}(0, 1)$ we denote as $K_{2,\omega}^{(m)}(0, 1)$.

Thus, $K_{2,\omega}^{(m)}(0, 1)$ is the Hilbert space defined as follows

$$K_{2,\omega}^{(m)}(0, 1) = \{\varphi : [0, 1] \to \mathbb{R} \mid \varphi^{(m-1)} \text{ is absulute cont. and } \varphi^{(m)} \in L_2(0, 1)\}$$

and equipped with the norm

$$\|\varphi\|_{K_{2,\omega}^{(m)}} = \sqrt{\int_0^1 \left(\varphi^{(m)}(x) - \omega^2 \varphi^{(m-2)}(x)\right)^2 dx}. \tag{5.2}$$

Every element of this space is a class of functions φ which are differ from each other to the solution of the equation $\varphi^{(m)}(x) - \omega^2\varphi^{(m-2)}(x) = 0$.

Here, the error

$$(\ell_1, \varphi) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N} C_\beta \varphi(h\beta)$$

of the formula (5.1) is estimated by the norm of the error functional

$$\ell_1(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - h\beta) \tag{5.3}$$

as follows

$$|(\ell_1, \varphi)| \leq \|\ell_1\|_{K_{2,\omega}^{(m)*}} \|\varphi\|_{K_{2,\omega}^{(m)}}. \tag{5.4}$$

From inequality (5.4), taking into account (5.2), we conclude that quadrature formulas of the form (5.1) are exact for hyperbolic functions $\sinh(\omega x)$, $\cosh(\omega x)$ and algebraic polynomials $P_{m-3}(x)$ of degree $(m-3)$. Then, in order for the functional ℓ_1 to be defined on the space $K_{2,\omega}^{(m)}$, the following conditions should be imposed

$$\begin{aligned} &(\ell_1, \sinh(\omega x)) = 0, \\ &(\ell_1, \cosh(\omega x)) = 0, \\ &(\ell_1, x^\alpha) = 0,\ \alpha = 0, 1, ..., m-3. \end{aligned} \tag{5.5}$$

Thus, to construct optimal quadrature formulas of the form (5.1) which are exact for hyperbolic functions $\sinh(\omega x)$, $\cosh(\omega x)$ and algebraic polynomials $P_{m-3}(x)$ of degree $(m-3)$ we solve the following problem.

Problem $\mathbf{A}_1$. For the quadrature formulas of the form (5.1) find such coefficients $C_\beta = \mathring{C}_\beta$ that give a minimum to the norm of the error functional (5.3).

Here, for the operator $L = \frac{d^m}{dx^m} - \omega^2 \frac{d^{m-2}}{dx^{m-2}}$ doing calculations given in section 2, for the coefficients of the optimal quadrature formulas (5.1) we get the following system of linear equations

$$\sum_{\gamma=0}^{N} \mathring{C}_\gamma G_m(h\beta - h\gamma) + \mathring{d}_1 \sinh(\omega h\beta) + \mathring{d}_2 \cosh(\omega h\beta) + \sum_{k=0}^{m-3} \mathring{p}_k (h\beta)^k = f_m(h\beta), \tag{5.6}$$

$$\beta = 0, 1, ..., N,$$

$$\sum_{\gamma=0}^{N} \mathring{C}_\gamma \sinh(h\gamma) = \frac{\cosh\omega - 1}{\omega}, \tag{5.7}$$

$$\sum_{\gamma=0}^{N} \mathring{C}_\gamma \cosh(h\gamma) = \frac{\sinh\omega}{\omega}, \tag{5.8}$$

$$\sum_{\gamma=0}^{N} \mathring{C}_{\gamma}(h\gamma)^{\alpha} = \frac{1}{\alpha+1},\ \alpha = 0,1,...,m-3, \tag{5.9}$$

where

$$G_m(x) = \frac{\operatorname{sign}x}{4\omega^{2m-1}}\left(\omega x\cosh\omega x - (2m-3)\sinh\omega x + 2\sum_{k=1}^{m-2}\frac{(m-k-1)(\omega x)^{2k-1}}{(2k-1)!}\right), \tag{5.10}$$

$$f_m(h\beta) = \int_0^1 G_m(x-h\beta)dx, \tag{5.11}$$

$m \geq 3$.

Further, solving the system (5.6)-(5.9), we find analytic forms of the optimal coefficients $\mathring{C}_{\beta}$.

5.2. **Auxiliary results.** First, we present well-known formulas that are used in the proof of the main result. The following formula from [10] is valid:

$$\sum_{\gamma=0}^{n-1} q^{\gamma}\gamma^k = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i\gamma^k|_{\gamma=n}, \tag{5.12}$$

where $\Delta^i\gamma^k$ is the finite difference of order i from γ^k, $\Delta^i 0^k = \Delta^i\gamma^k|_{\gamma=0}$ and $\Delta^i 0^k = \sum\limits_{l=1}^{i}(-1)^{i-l}\binom{i}{j}l^k$.

For $|q| < 1$, from (5.12) we have

$$\sum_{\gamma=0}^{\infty} q^{\gamma}\gamma^k = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k. \tag{5.13}$$

We give the following formulas from [9]:

$$\sum_{\gamma=0}^{\beta-1}\gamma^k = \sum_{j=1}^{k+1}\frac{k!\,B_{k+1-j}}{j!\,(k+1-j)!}\beta^j, \tag{5.14}$$

where B_{k+1-j} are Bernoulli numbers,

$$\Delta^{\alpha}x^{\beta} = \sum_{p=0}^{\beta}\begin{pmatrix}\beta\\ p\end{pmatrix}\Delta^{\alpha}0^p x^{\beta-p}. \tag{5.15}$$

It is known in [8,29] that the *Euler-Frobenius polynomials* $E_k(x)$ are given by the formula

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k\frac{x}{(1-x)^2},\quad k = 1,2,..., \tag{5.16}$$

$E_0(x) = 1$. In addition, all roots $x_j^{(k)}$ of the polynomial $E_k(x)$ are real, negative and simple, i.e.

$$x_1^{(k)} < x_2^{(k)} < ... < x_k^{(k)} < 0. \tag{5.17}$$

Also, the roots equidistant from the ends of the chain (5.17) are mutually inverse, i.e.

$$x_j^{(k)}\cdot x_{k+1-j}^{(k)} = 1,\quad j = 1,2,..,k.$$

Euler obtained the following formula for the coefficients $a_s^{(k)}$, $s = 0, 1, .., k$ of the polynomial $E_k(x) = \sum_{s=0}^{k} a_s^{(k)} x^s$:

$$a_s^{(k)} = \sum_{j=0}^{s} (-1)^j \begin{pmatrix} k+2 \\ j \end{pmatrix} (s+1-j)^{k+1}. \tag{5.18}$$

Polynomial $E_k(x)$ satisfies the following identity

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right), \tag{5.19}$$

i.e. $a_s^{(k)} = a_{k-s}^{(k)}$, $s = 0, 1, .., k$. The following theorem in [24] is true.

Theorem 2. *Polynomials*

$$P_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i}, \tag{5.20}$$

$$P_k\left(\frac{1}{x}\right) = \left(\frac{1}{x} - 1\right)^{k+1} \sum_{i=0}^{k+1} \left(\frac{x}{1-x}\right)^i \Delta^i 0^{k+1}, \tag{5.21}$$

are the Euler-Frobenius polynomials $E_k(x)$ and $E_k\left(\frac{1}{x}\right)$, respectively.

To solve the system (5.6)-(5.9) we use the discrete argument function $D_m(h\beta)$ which satisfies the equation

$$D_m(h\beta) * G_m(h\beta) = \delta_d(h\beta), \tag{5.22}$$

where $G_m(h\beta)$ is the discrete argument function corresponding to the function $G_m(x)$ defined by equality (5.10). It should be noted that equation (5.22) is the discrete analogue of the differential equation

$$\left(\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}\right) G_m(x) = \delta(x), \tag{5.23}$$

Furthermore, the discrete operator $D_m(h\beta)*$ has similar properties as the differential operator $\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$. The zeros of the discrete operator $D_m(h\beta)*$ are the discrete argument functions corresponding to the zeros of the operator $\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$.

The discrete argument function $D_m(h\beta)$ was constructed in the work [2] and the following was proved.

Theorem 3. *The discrete analogue $D_m(h\beta)$, to the differential operator $\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$, satisfying equation* (5.22) *has the form*

$$D_m(h\beta) = \frac{2\omega^{2m-1}}{p_{2m-2}^{(2m-2)}} \begin{cases} \sum_{k=1}^{m-1} A_k \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} A_k, & |\beta| = 1, \\ C + \sum_{k=1}^{m-1} \frac{A_k}{\lambda_k}, & \beta = 0, \end{cases} \tag{5.24}$$

where

$$A_k = \frac{(1-\lambda_k)^{2m-4}(\lambda_k^2 - 2\lambda_k \cosh h\omega + 1)^2 p_{2m-2}^{(2m-2)}}{\lambda_k P'_{2m-2}(\lambda_k)}, \tag{5.25}$$

$$C = 4 - 4\cosh h\omega - 2m - \frac{p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}}, \tag{5.26}$$

$$p_{2m-2}^{(2m-2)} = h\omega \cosh h\omega - (2m-3)\sinh h\omega + 2\sum_{k=1}^{m-2} \frac{(m-k-1)(h\omega)^{2k-1}}{(2k-1)!}, \tag{5.27}$$

$p_{2m-2}^{(2m-2)}$, $p_{2m-3}^{(2m-2)}$ *are leading and second coefficients,* λ_k $(|\lambda_k| < 1)$ *are roots of* $(2m-2)$ *degree polynomial*

$$P_{2m-2}(x) = \sum_{s=0}^{2m-2} p_s^{(2m-2)} x^s = (1-x)^{2m-4}\Big[[h\omega \cosh h\omega - (2m-3)\sinh h\omega]x^2 + [(2m-3)\sinh(2h\omega) - 2h\omega]x + [h\omega\cosh h\omega - (2m-3)\sinh h\omega]\Big]$$

$$+2(x^2 - 2x\cosh h\omega + 1)^2 \sum_{k=1}^{m-2} \frac{(m-k-1)(h\omega)^{2k-1}(1-x)^{2m-2k-4}E_{2k-2}(x)}{(2k-1)!}, \tag{5.28}$$

here $E_{2k-2}(x)$ *is the Euler-Frobenius polynomial of degree* $(2k-2)$.

For the properties of the discrete argument function $D_m(h\beta)$ the following are valid [2].

Theorem 4. *The discrete analogue* $D_m(h\beta)$ *to the differential operator* $\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$ *satisfies the following equalities*

1) $D_m(h\beta) * \sinh(h\omega\beta) = 0$,
2) $D_m(h\beta) * \cosh(h\omega\beta) = 0$,
3) $D_m(h\beta) * (h\omega\beta)\sinh(h\omega\beta) = 0$,
4) $D_m(h\beta) * (h\omega\beta)\cosh(h\omega\beta) = 0$,
5) $D_m(h\beta) * (h\beta)^\alpha = 0$, $\alpha = 0, 1, ..., 2m-5$.

In the next subsection we get analytic formulas for coefficients of the optimal quadrature formulas of the form (5.1).

5.3. **The coefficients of the optimal quadrature formulas of the form** (5.1)**.** In this subsection we find analytic expressions for the optimal coefficients $\mathring{C}_\beta$, $\beta = 0, 1, ..., N$, which are solution of the system (5.6) - (5.9). For this we consider $\mathring{C}_\beta$ as a function of discrete argument β. Here we assume that $\mathring{C}_\beta = 0$ for $\beta = -1, -2, ...$ and $\beta = N+1, N+2, ...$ Then, using Definition 3, we rewrite equation (5.6) in the following form

$$\mathring{C}_\beta * G_m(h\beta) + \mathring{d}_1 \sinh(\omega h\beta) + \mathring{d}_2 \cosh(\omega h\beta) + \sum_{k=0}^{m-3} \mathring{p}_k (h\beta)^k = f_m(h\beta), \tag{5.29}$$

$$\beta = 0, 1, ..., N.$$

We denote the left-hand side of equation (5.29) as

$$u_m(h\beta) = \mathring{C}_\beta * G_m(h\beta) + \mathring{d}_1 \sinh(\omega h\beta) + \mathring{d}_2 \cosh(\omega h\beta) + \sum_{k=0}^{m-3} \mathring{p}_k (h\beta)^k.$$

Then, taking into account (5.22), using Theorem 4, we get

$$\mathring{C}_\beta = D_m(h\beta) * u_m(h\beta), \tag{5.30}$$

where $D_m(h\beta)$ defined by equality (5.24) in Theorem 3.

We note that in (5.30) the function $u_m(h\beta)$ is only defined for $\beta = 0, 1, ..., N$ and the following holds

$$u_m(h\beta) = f_m(h\beta),\ \beta = 0, 1, ..., N. \tag{5.31}$$

Now, in order to find the function $u_m(h\beta)$ for $\beta = -1, -2, ...$ and $\beta = N+1, N+2, ...$, we use the assumption that $\mathring{C}_\beta = 0$ for $\beta = -1, -2, ...$ and $\beta = N+1, N+2, ...$ Then, after some calculations we com to the following problem.

Problem B$_1$. Find the function $u_m(h\beta)$ that satisfies the equation

$$D_m(h\beta) * u_m(h\beta) = 0 \text{ for } \beta = -1, -2, ... \text{ and } \beta = N+1, N+2, ... \tag{5.32}$$

and has the form

$$u_m(h\beta) = \begin{cases} \frac{-1}{4\omega^{2m-1}}\Big((h\omega\beta\cosh(h\omega\beta) - (2m-3)\sinh(h\omega\beta))\frac{\sinh\omega}{\omega} \\ \quad +((2m-3)\cosh(h\omega\beta) - h\omega\beta\sinh(h\omega\beta))\frac{\cosh\omega - 1}{\omega}\Big) \\ \quad + d_1^- \sinh(h\omega\beta) + d_2^- \cosh(h\omega\beta) - Q_{2m-5}(h\beta) + P_{m-3}^-(h\beta), & \beta < 0, \\ f_m(h\beta), & 0 \le \beta \le N, \\ \frac{1}{4\omega^{2m-1}}\Big((h\omega\beta\cosh(h\omega\beta) - (2m-3)\sinh(h\omega\beta))\frac{\sinh\omega}{\omega} \\ \quad +((2m-3)\cosh(h\omega\beta) - h\omega\beta\sinh(h\omega\beta))\frac{\cosh\omega - 1}{\omega}\Big) \\ \quad + d_1^+ \sinh(h\omega\beta) + d_2^+ \cosh(h\omega\beta) + Q_{2m-5}(h\beta) + P_{m-3}^+(h\beta), & \beta > N, \end{cases} \tag{5.33}$$

where

$$\begin{aligned} Q_{2m-5}(h\beta) \quad = \quad & \frac{1}{2\omega^{2m-1}}\Bigg(\sum_{k=1}^{[\frac{m}{2}]-1} \sum_{\alpha=0}^{2k-1} \frac{(-1)^\alpha (m-k-1)\omega^{2k-1}}{(2k-1-\alpha)!\ (\alpha+1)!}(h\beta)^{2k-1-\alpha} \\ & + \sum_{k=[\frac{m}{2}]}^{m-2} \sum_{\alpha=0}^{m-3} \frac{(-1)^\alpha (m-k-1)\omega^{2k-1}}{(2k-1-\alpha)!\ (\alpha+1)!}(h\beta)^{2k-1-\alpha}\Bigg), \end{aligned} \tag{5.34}$$

$d_1^-, d_1^+, d_2^-, d_2^+$ are unknown constants,

$$P_{m-3}^-(h\beta) = \sum_{k=0}^{m-3} p_k^-(h\beta)^k \text{ and } P_{m-3}^+(h\beta) = \sum_{k=0}^{m-3} p_k^+(h\beta)^k \tag{5.35}$$

are unknown polynomials of degree $(m-3)$. Here

$$\mathring{d}_1 = \frac{1}{2}(d_1^- + d_1^+),\ \mathring{d}_2 = \frac{1}{2}(d_2^- + d_2^+),\ \text{and } \mathring{p}_k = \frac{1}{2}(p_k^- + p_k^+),\ k = 0, 1, ..., m-3. \tag{5.36}$$

The function $u_m(h\beta)$ can be found from equation (5.32). To find this function we need to find $d_1^-, d_1^+, d_2^-, d_2^+, P_{m-3}^-(h\beta)$ and $P_{m-3}^+(h\beta)$. Then from equation (5.30) for $\beta = 0, 1, ..., N$ we can find the optimal coefficients $\mathring{C}_\beta$.

But, in this subsection, we find the optimal coefficients $\mathring{C}_\beta$ by other way.

Here, using the discrete argument functions $D_m(h\beta)$ and $u_m(h\beta)$ we get expressions for the optimal coefficients $\mathring{C}_\beta$ with $(2m-2)$ unknowns. Putting these expressions to equation (5.29) we get the identity with respect to $(h\beta)$. From this identity the optimal coefficients $\mathring{C}_\beta$ will be completely found.

For the optimal coefficients we have.

Theorem 5. *In the Hilbert space $K_{2,\omega}^{(m)}$ for $m \geq 3$ the coefficients of the optimal quadrature formulas of the form* (5.1) *have the following form*

$$\mathring{C}_\beta = T + \sum_{k=1}^{m-1}\left(m_k\lambda_k^\beta + n_k\lambda_k^{N-\beta}\right),\ \beta = 1, 2, ..., N-1, \tag{5.37}$$

where

$$T = D_m(h\beta) * f_m(h\beta), \tag{5.38}$$

$$\begin{aligned} m_k = \frac{A_k p}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^\gamma\Big\{ & \frac{-1}{4\omega^{2m-1}}\Big[[(2m-3)\sinh(h\omega\gamma) - h\omega\gamma\cosh(h\omega\gamma)]\frac{\sinh\omega}{\omega} \\ & + [(2m-3)\cosh(h\omega\gamma) - h\omega\gamma\sin(h\omega\gamma)]\frac{\cosh\omega - 1}{\omega}\Big] + d_1^-\sinh(-h\omega\gamma) \\ & + d_2^-\cosh(h\omega\gamma) - Q_{2m-5}(-h\gamma) + P_{m-3}^-(-h\gamma) - f_m(-h\gamma)\Big\}, \end{aligned} \tag{5.39}$$

$$\begin{aligned} n_k = \frac{A_k p}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^\gamma\Big\{ & \frac{1}{4\omega^{2m-1}}\Big[[h\omega(N+\gamma)\cosh(h\omega(N+\gamma)) \\ & -(2m-3)\sinh(h\omega(N+\gamma))]\frac{\sinh\omega}{\omega} \\ & + [(2m-3)\cosh(h\omega(N+\gamma)) - h\omega(N+\gamma)\sinh(h\omega(N+\gamma))]\frac{\cosh\omega - 1}{\omega}\Big] \\ & + d_1^+\sinh(h\omega(N+\gamma)) + d_2^+\cosh(h\omega(N+\gamma)) + Q_{2m-5}(1+h\gamma) \end{aligned}$$

$$+P^{+}_{m-3}(1+h\gamma) - f_m(1+h\gamma)\Big\}, \tag{5.40}$$

$$k = 1, 2, ..., m-1.$$

The series in the last equalities are convergent due to $|\lambda_k| < 1$.

Proof. Let $\beta = 1, 2, ..., N-1$. Then, taking into account equalities (5.24) and (5.33), using denotations (5.39) and (5.40), from equality (5.30) we get (5.37). Theorem 5 is proved.

Then, using Theorem 5, from equations (5.7) and (5.8), after some simplifications, we get the following result.

Theorem 6. *In the Hilbert space* $K^{(m)}_{2,\omega}$ *for* $m \geq 3$ *the first and last coefficients of the optimal quadrature formulas of the form* (5.1) *have the following form*

$$\begin{aligned}\mathring{C}_0 \;=\;& \frac{2\cdot(\cosh(\omega)-1)(\cosh(\omega h)-1) - h\omega\left[\sinh(\omega)-\sinh(\omega-\omega h)-\sinh(\omega h)\right]}{2\omega\sinh(\omega)(\cosh(\omega h)-1)}\\ & -\sum_{k=1}^{m-1}\Bigg[m_k\cdot\frac{\lambda_k^{N+1}\sinh(\omega h)-\lambda_k^2\sinh(\omega)+\lambda_k\sinh(\omega-\omega h)}{(\lambda_k^2+1-2\lambda_k\cosh(\omega h))\sinh(\omega)}\\ & +n_k\cdot\frac{\lambda_k^{N+1}\sinh(\omega-\omega h)-\lambda_k^N\sinh(\omega)+\lambda_k\sinh(\omega h)}{(\lambda_k^2+1-2\lambda_k\cosh(\omega h))\sinh(\omega)}\Bigg],\end{aligned} \tag{5.41}$$

$$\begin{aligned}\mathring{C}_N \;=\;& \frac{2(\cosh(\omega)-1)(\cosh(\omega h)-1) - \omega h\left[\sinh(\omega)-\sinh(\omega-\omega h)-\sinh(\omega h)\right]}{2\omega\sinh(\omega)(\cosh(\omega h)-1)}\\ & -\sum_{k=1}^{m-1}\Bigg[m_k\frac{\lambda_k^{N+1}\sinh(\omega-\omega h)-\lambda_k^N\sinh(\omega)+\lambda_k\sinh(\omega h)}{(\lambda_k^2-2\lambda_k\cosh(\omega h)+1)\sinh(\omega)}\\ & +n_k\frac{\lambda_k^{N+1}\sinh(\omega h)-\lambda_k^2\sinh(\omega)+\lambda_k\sinh(\omega-\omega h)}{(\lambda_k^2-2\lambda_k\cosh(\omega h)+1)\sinh(\omega)}\Bigg].\end{aligned} \tag{5.42}$$

Finally, the main result of this subsection is as follows.

Theorem 7. *The coefficients of optimal quadrature formulas of the form (5.1) with equal spaced nodes, in the Hilbert space* $K^{(m)}_{2,\omega}$ *for* $m \geq 3$*, are expressed as follows*

$$\begin{aligned}\mathring{C}_0 &= \frac{h}{2}+\sum_{k=1}^{m-1} m_k\left(\frac{\lambda_k-\lambda_k^N}{\lambda_k-1}\right),\\ \mathring{C}_\beta &= h+\sum_{k=1}^{m-1} m_k\left(\lambda_k^{\beta}+\lambda_k^{N-\beta}\right),\ \ \beta = 1, 2, ..., N-1,\\ \mathring{C}_N &= \frac{h}{2}+\sum_{k=1}^{m-1} m_k\left(\frac{\lambda_k-\lambda_k^N}{\lambda_k-1}\right),\end{aligned}$$

where m_k $(k=\overline{1,m-1})$ are defined by the system of $(m-1)$ linear equations

$$\sum_{k=1}^{m-1} m_k\left[\frac{\lambda_k+(-1)^{\alpha}\lambda_k^{N+1}}{(\lambda_k-1)^{\alpha}}E_{\alpha-2}(\lambda_k)\right]=\frac{hB_\alpha}{\alpha},\ \ \alpha=\overline{2,m-2},$$

$$\sum_{k=1}^{m-1} m_k\left[\frac{(\lambda_k+\lambda_k^{N+1})[\sinh(\omega-h\omega)+\sinh(\omega h)]-(\lambda_k^2+\lambda_k^N)\sinh\omega}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)\sinh\omega}-\frac{\lambda_k-\lambda_k^N}{1-\lambda_k}\right]$$
$$=\frac{(\cosh\omega-1)\ [2(1-\cosh h\omega)+h\omega\sinh h\omega]}{2\omega\ \sinh\omega\ (1-\cosh h\omega)},$$

$$\sum_{k=1}^{m-1} m_k\left[\frac{\lambda_k+\lambda_k^{N+1}}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\right]=\frac{h\omega\sinh h\omega-2(\cosh h\omega-1)}{2\omega\ \sinh h\omega\ (\cosh h\omega-1)},$$

here λ_k are given in Theorem 3, $|\lambda_k|<1$, $E_{\alpha-1}(x)$ is the Euler-Frobenius polynomial of degree $\alpha-1$, B_α is the Bernoulli numbers.

Proof. Firstly, the convolution in the left-hand side of (5.29) we write as a difference of two expressions

$$\mathring{C}_\beta * G_m(h\beta)=g_1(h\beta)-g_2(h\beta), \tag{5.43}$$

where

$$g_1(h\beta)=\frac{1}{2\omega^{2m-1}}\sum_{\gamma=0}^{\beta}\mathring{C}_\gamma\Big[(h\omega\beta-h\omega\gamma)\cosh(h\omega\beta-h\omega\gamma)-(2m-3)\sinh(h\omega\beta-h\omega\gamma)$$
$$+2\sum_{k=1}^{m-2}\frac{(m-k-1)(h\omega\beta-h\omega\gamma)^{2k-1}}{(2k-1)!}\Big]$$

and

$$g_2(h\beta)=\frac{1}{4\omega^{2m-1}}\sum_{\gamma=0}^{N}\mathring{C}_\gamma\Big[(h\omega\beta-h\omega\gamma)\cosh(h\omega\beta-h\omega\gamma)-(2m-3)\sinh(h\omega\beta-h\omega\gamma)$$
$$+2\sum_{k=1}^{m-2}\frac{(m-k-1)(h\omega\beta-h\omega\gamma)^{2k-1}}{(2k-1)!}\Big].$$

Hence, taking into account (5.10), using (5.12)-(5.21), after some calculations and simplifications for $g_1(h\beta)$ and $g_2(h\beta)$ we have

$$g_1(h\beta)=\frac{1}{2\omega^{2m-1}}$$
$$\times\Bigg\{\Bigg[\mathring{C}_0-\frac{T}{2}+\sum_{k=1}^{m-1}\Bigg(m_k\frac{\lambda_k(\cosh h\omega-\lambda_k)}{\lambda_k^2+1-2\lambda_k\cosh h\omega}+n_k\frac{\lambda_k^N(\lambda_k\cosh h\omega-1)}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg)\Bigg](h\omega\beta)\cosh(h\omega\beta)$$
$$+\Bigg[\frac{T\sinh h\omega}{2(\cosh h\omega-1)}+\sum_{k=1}^{m-1}\Bigg(m_k\frac{-\lambda_k\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}+n_k\frac{-\lambda_k^{N+1}\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg)\Bigg](h\omega\beta)\sinh(h\omega\beta)$$

$$+\Bigg[-\frac{(2m-3)T\sinh h\omega+Th\omega}{2(\cosh h\omega-1)}+\sum_{k=1}^{m-1}\Bigg[m_k\Bigg(-\frac{\lambda_k h\omega(\lambda_k^2\cosh h\omega-2\lambda_k+\cosh h\omega)}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)^2}+\frac{(2m-3)\lambda_k\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg)$$

$$+n_k\Bigg(-\frac{\lambda_k^{N+1}h\omega(\lambda_k^2\cosh h\omega-2\lambda_k+\cosh h\omega)}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)^2}+\frac{(2m-3)\lambda_k^{N+1}\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg)\Bigg]\Bigg]\cosh(h\omega\beta)$$

$$+\Bigg[(2m-3)(\frac{T}{2}-\mathring{C}_0)+\sum_{k=1}^{m-1}\Bigg[m_k\Bigg(\frac{(2m-3)\lambda_k(\lambda_k-\cosh h\omega)}{\lambda_k^2+1-2\lambda_k\cosh h\omega}-\frac{\lambda_k h\omega(\lambda_k^2-1)\sinh h\omega}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)^2}\Bigg)$$

$$+n_k\Bigg(\frac{\lambda_k^{N+1}h\omega(\lambda_k^2-1)\sinh h\omega}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)^2}-\frac{(2m-3)\lambda_k^{N}(\lambda_k\cosh h\omega-1)}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg)\Bigg]\Bigg]\sinh(h\omega\beta)$$

$$+\frac{(2m-3)T\sinh h\omega+Th\omega}{2(\cosh h\omega-1)}+2\mathring{C}_0\sum_{l=1}^{m-2}\frac{(m-l-1)(h\omega\beta)^{2l-1}}{(2l-1)!}+\frac{2}{\omega}\sum_{l=1}^{m-2}\frac{(m-l-1)Th^{-1}(h\omega\beta)^{2l}}{(2l)!}$$

$$+2\sum_{l=1}^{m-2}(m-l-1)T(h\omega)^{2l-1}\sum_{j=1}^{2l-1}\frac{B_{2l-j}}{j!\,(2l-j)!}\beta^j$$

$$-2\sum_{l=1}^{m-2}\frac{(m-l-1)(h\omega)^{2l-1}}{(2l-1)!}\sum_{j=0}^{2l-1}\binom{2l-1}{j}\beta^j\sum_{k=1}^{m-1}m_k\frac{\lambda_k}{\lambda_k-1}\sum_{i=0}^{2l-1}\frac{\Delta^i 0^{2l-1-j}}{(\lambda_k-1)^i}$$

$$-2\sum_{l=1}^{m-2}\frac{(m-l-1)(h\omega)^{2l-1}}{(2l-1)!}\sum_{j=0}^{2l-1}\binom{2l-1}{j}\beta^j\sum_{k=1}^{m-1}n_k\frac{\lambda_k^N}{1-\lambda_k}\sum_{i=0}^{2l-1}(\frac{\lambda_k}{1-\lambda_k})^i\Delta^i 0^{2l-1-j}\Bigg\}$$

and

$$\begin{aligned}
g_2(h\beta) &= \tfrac{1}{4\omega^{2m-1}}\\
&\times\Bigg\{\Big[(h\omega\beta)\cosh(h\omega\beta)-(2m-3)\sinh(h\omega\beta)\Big]\tfrac{\sinh\omega}{\omega}\\
&-\Big[(h\omega\beta)\sinh(h\omega\beta)-(2m-3)\cosh(h\omega\beta)\Big]\tfrac{\cosh\omega-1}{\omega}\\
&-\cosh(h\omega\beta)\sum_{\gamma=0}^{N}\mathring{C}_\gamma\,(h\omega\gamma)\cosh(h\omega\gamma)+\sinh(h\omega\beta)\sum_{\gamma=0}^{N}\mathring{C}_\gamma\,(h\omega\gamma)\sinh(h\omega\gamma)\\
&+2\sum_{k=1}^{[\frac{m}{2}]-1}\sum_{\alpha=0}^{2k-1}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k-1}}{(2k-1-\alpha)!\,(\alpha+1)!}(h\beta)^{2k-1-\alpha}\\
&+2\sum_{k=[\frac{m}{2}]}^{m-2}\sum_{\alpha=0}^{m-3}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k-1}}{(2k-1-\alpha)!\,(\alpha+1)!}(h\beta)^{2k-1-\alpha}\\
&+2\sum_{k=[\frac{m}{2}]}^{m-2}\sum_{\alpha=m-2}^{2k-1}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k-1}}{(2k-1-\alpha)!\,\alpha!}(h\beta)^{2k-1-\alpha}\sum_{\gamma=0}^{N}\mathring{C}_\gamma\,(h\gamma)^\alpha\Bigg\}.
\end{aligned}$$

And for the right-hand side of equation (5.29) we get the following

$$
\begin{aligned}
&f_m(h\beta) = \tfrac{1}{4\omega^{2m}}\\
&\times\Bigg(\bigg[\omega\sinh\omega - (2m-2)(1+\cosh\omega)\bigg]\cosh(h\omega\beta)\\
&+\bigg[(2m-2)\sinh\omega - \omega\cosh\omega\bigg]\sinh(h\omega\beta)\\
&-\sinh\omega\cdot(h\omega\beta)\cosh(h\omega\beta) + (1+\cosh\omega)(h\omega\beta)\sinh(h\omega\beta) + (4m-4)\\
&+4\sum_{k=1}^{m-2}\tfrac{(m-k-1)(h\omega\beta)^{2k}}{(2k)!} - 2\sum_{k=1}^{[\frac{m}{2}]-1}\sum_{\alpha=0}^{2k-1}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k}}{(2k-1-\alpha)!\,(\alpha+1)!}(h\beta)^{2k-1-\alpha}\\
&-2\sum_{k=[\frac{m}{2}]}^{m-2}\sum_{\alpha=0}^{m-3}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k}}{(2k-1-\alpha)!\,(\alpha+1)!}(h\beta)^{2k-1-\alpha}\\
&-2\sum_{k=[\frac{m}{2}]}^{m-2}\sum_{\alpha=m-2}^{2k-1}\tfrac{(-1)^\alpha(m-k-1)\omega^{2k}}{(2k-1-\alpha)!\,(\alpha+1)!}(h\beta)^{2k-1-\alpha}\Bigg),
\end{aligned}
$$

where $[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$.

Now, putting the last three expressions of $g_1(h\beta)$, $g_2(h\beta)$ and $f_m(h\beta)$ into equation (5.29), we come to the following identity with respect to $(h\beta)$

$$
g_1(h\beta) - g_2(h\beta) + \mathring{d}_1\sinh(h\omega\beta) + \mathring{d}_2\cosh(h\omega\beta) + \sum_{k=0}^{m-3}\mathring{p}_k(h\beta)^k = f_m(h\beta). \tag{5.44}
$$

From here, equating the terms $(h\beta)^{2m-4}$ of both sides for $m \geq 3$, we obtain that $T = D_m(h\beta) * f_m(h\beta) = h$ and, accordingly, for the optimal coefficients (5.37) we get the following formula

$$
\mathring{C}_\beta = h + \sum_{k=1}^{m-1}\left(m_k\lambda_k^\beta + n_k\lambda_k^{N-\beta}\right),\ \ \beta = \overline{1, N-1}. \tag{5.45}
$$

Then from the identity (5.44) equating the coefficients of like terms $(h\omega\beta)\cosh(h\omega\beta)$, $(h\omega\beta)\sinh(h\omega\beta)$ and the terms which consist of $(h\beta)^\alpha$, $\alpha = \overline{m-2, 2m-5}$ in both sides, we get the following system of equations for m_k and n_k

$$
\mathring{C}_0 - \frac{T}{2} + \sum_{k=1}^{m-1}\left(m_k\frac{\lambda_k(\cosh h\omega - \lambda_k)}{\lambda_k^2 + 1 - 2\lambda_k\cosh h\omega} + n_k\frac{\lambda_k^N(\lambda_k\cosh h\omega - 1)}{\lambda_k^2 + 1 - 2\lambda_k\cosh h\omega}\right) = 0, \tag{5.46}
$$

$$
\frac{T\sinh h\omega}{2(\cosh h\omega - 1)} - \frac{1}{\omega} - \sum_{k=1}^{m-1}\left(m_k\frac{\lambda_k\sinh h\omega}{\lambda_k^2 + 1 - 2\lambda_k\cosh h\omega} + n_k\frac{\lambda_k^{N+1}\sinh h\omega}{\lambda_k^2 + 1 - 2\lambda_k\cosh h\omega}\right) = 0, \tag{5.47}
$$

$$
\begin{aligned}
&\sum_{l=[\frac{m}{2}]}^{m-2}\Bigg[\mathring{C}_0\frac{(m-l-1)(h\omega\beta)^{2l-1}}{(2l-1)!} + \sum_{j=m-2}^{2l-1}\frac{(m-l-1)T(h\omega)^{2l-1}B_{2l-j}}{j!\,(2l-j)!}\beta^j\\
&-\sum_{j=m-2}^{2l-1}\frac{(m-l-1)(h\omega)^{2l-1}}{(2l-1-j)!\,j!}\beta^j\sum_{k=1}^{m-1}m_k\sum_{i=0}^{2l-1-j}\frac{\lambda_k\Delta^i 0^{2l-1-j}}{(\lambda_k-1)^{i+1}}
\end{aligned}
$$

$$-\sum_{j=m-2}^{2l-1}\frac{(m-l-1)(h\omega)^{2l-1}}{(2l-1-j)!\,j!}\beta^j\sum_{k=1}^{m-1}n_k\sum_{i=0}^{2l-1-j}\frac{\lambda_k^{N+i}\Delta^i 0^{2l-1-j}}{(1-\lambda_k)^{i+1}}\Bigg]=0. \tag{5.48}$$

From (5.48), collecting similar terms for m_k and n_k we get the following $(m-2)$ linear equations

$$\begin{aligned}&\sum_{k=1}^{m-1}m_k\Bigg[\sum_{l=1}^{j}\frac{(j-l+1)}{(2l-2)!}(h\omega)^{2l-2}\sum_{i=0}^{2l-2}\frac{\lambda_k\Delta^i0^{2l-2}}{(\lambda_k-1)^{i+1}}+j\frac{\lambda_k^{N+1}\sinh h\omega-\lambda_k^2\sinh\omega+\lambda_k\sinh(\omega-h\omega)}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)\sinh\omega}\Bigg]\\&\quad+\sum_{k=1}^{m-1}n_k\Bigg[\sum_{l=1}^{j}\frac{(j-l+1)}{(2l-2)!}(h\omega)^{2l-2}\sum_{i=0}^{2l-2}\frac{\lambda_k^{N+i}\Delta^i0^{2l-2}}{(1-\lambda_k)^{i+1}}+j\frac{\lambda_k^{N+1}\sinh(\omega-h\omega)-\lambda_k^N\sinh\omega+\lambda_k\sinh h\omega}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)\sinh\omega}\Bigg]\\&\quad=j\frac{2(\cosh\omega-1)(1-\cosh h\omega)-h\omega[\sinh(\omega-h\omega)+\sinh h\omega-\sinh\omega]}{2\omega\sinh\omega\cdot(1-\cosh h\omega)}-hB_1,\quad j=\overline{1,[\tfrac{m+1}{2}]-1},\end{aligned} \tag{5.49}$$

$$\begin{aligned}&\sum_{k=1}^{m-1}m_k\Bigg[\sum_{l=1}^{j}\frac{(j-l+1)}{(2l-2)!}(h\omega)^{2l-1}\sum_{i=0}^{2l-1}\frac{\lambda_k\Delta^i0^{2l-1}}{(\lambda_k-1)^{i+1}}\Bigg]+\sum_{k=1}^{m-1}n_k\Bigg[\sum_{l=1}^{j}\frac{(j-l+1)}{(2l-1)!}(h\omega)^{2l-1}\sum_{i=0}^{2l-1}\frac{\lambda_k^{N+i}\Delta^i0^{2l-1}}{(1-\lambda_k)^{i+1}}\Bigg]\\&\quad=\sum_{l=1}^{j}\frac{(j-l+1)}{(2l)!}\frac{(h\omega)^{2l}B_{2l}}{\omega},\quad j=\overline{1,[\tfrac{m}{2}]-1},\end{aligned} \tag{5.50}$$

Further, from (5.9) for $\alpha=0,1,...,m-3$, we obtain the following $m-2$ linear equations

$$\begin{aligned}&\sum_{k=1}^{m-1}(m_k+n_k)\Bigg[\frac{(\lambda_k+\lambda_k^{N+1})[\sinh(\omega-h\omega)+\sinh h\omega]-(\lambda_k^2+\lambda_k^N)\sinh\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}-\frac{(\lambda_k-\lambda_k^N)\sinh\omega}{1-\lambda_k}\Bigg]\\&\qquad=\frac{(\cosh\omega-1)\ [2(\cosh h\omega-1)-h\omega\ \sinh h\omega]}{\omega\ (\cosh h\omega-1)},\end{aligned} \tag{5.51}$$

$$\begin{aligned}&\sum_{k=1}^{m-1}m_k\Bigg[h^\alpha\sum_{i=0}^{\alpha}\frac{(\lambda_k^i-\lambda_k^{N+i})\Delta^i0^\alpha}{(1-\lambda_k)^{i+1}}-\sum_{j=0}^{\alpha-1}h^jC_\alpha^j\sum_{i=0}^{j}\frac{\lambda_k^{N+i}\Delta^i0^j}{(1-\lambda_k)^{i+1}}-\frac{\lambda_k^{N+1}\sinh(\omega-h\omega)+\lambda_k\sinh h\omega-\lambda_k^N\sinh\omega}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)\sinh\omega}\Bigg]\\&+\sum_{k=1}^{m-1}n_k\Bigg[h^\alpha\sum_{i=0}^{\alpha}\frac{(\lambda_k^{N+1}-\lambda_k)\Delta^i0^\alpha}{(\lambda_k-1)^{i+1}}-\sum_{j=0}^{\alpha-1}h^jC_\alpha^j\sum_{i=0}^{j}\frac{\lambda_k\Delta^i0^j}{(\lambda_k-1)^{i+1}}-\frac{\lambda_k^{N+1}\sinh h\omega-\lambda_k^2\sinh\omega+\lambda_k\sinh(\omega-h\omega)}{(\lambda_k^2+1-2\lambda_k\cosh h\omega)\sinh\omega}\Bigg]\\&\quad=-\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+1-j}}{j!\ (\alpha+1-j)!}h^{\alpha+1-j}-\frac{2(\cosh\omega-1)(1-\cosh h\omega)-h\omega[\sinh(\omega-h\omega)+\sinh h\omega-\sinh\omega]}{2\omega\sinh\omega\ (1-\cosh h\omega)},\\&\alpha=\overline{1,m-3}.\end{aligned} \tag{5.52}$$

From equations (5.46)–(5.52), after some simplifications, for m_k and n_k we get the following system of $(2m-2)$ linear equations

$$\begin{aligned}&\sum_{k=1}^{m-1}(m_k-n_k)\Bigg[\frac{(\lambda_k^{N+1}-\lambda_k)[\sinh(\omega-h\omega)-\sinh h\omega]-(\lambda_k^N-\lambda_k^2)\sinh\omega}{(\lambda_k^2+1-2\cosh h\omega)\sinh\omega}-\frac{\lambda_k^N+\lambda_k}{\lambda_k-1}\Bigg]=0,\\&\sum_{k=1}^{m-1}(m_k-n_k)\Bigg[\frac{\lambda_k^{N+1}-\lambda_k}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\Bigg]=0,\\&\sum_{k=1}^{m-1}(m_k-n_k)\Bigg[\sum_{j=1}^{\alpha}h^jC_\alpha^j\frac{\lambda_k^{N+1}+(-1)^j\lambda_k}{(1-\lambda_k)^{j+1}}E_{j-1}(\lambda_k)\Bigg]=0,\quad\alpha=\overline{1,m-3},\\&\sum_{k=1}^{m-1}m_k\Bigg[\frac{\lambda_k}{(\lambda_k-1)^\alpha}E_{\alpha-2}(\lambda_k)\Bigg]+\sum_{k=1}^{m-1}n_k\Bigg[\frac{\lambda_k^{N+1}}{(1-\lambda_k)^\alpha}E_{\alpha-2}(\lambda_k)\Bigg]=\frac{hB_\alpha}{\alpha},\quad\alpha=\overline{2,m-2},\end{aligned} \tag{5.53}$$

$$\sum_{k=1}^{m-1}(m_k+n_k)\left[\frac{(\lambda_k+\lambda_k^{N+1})[\sinh(\omega-h\omega)+\sinh h\omega]-(\lambda_k^2+\lambda_k^N)\sinh\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}-\frac{(\lambda_k^N-\lambda_k)\sinh\omega}{\lambda_k-1}\right]$$
$$=\frac{(\cosh\omega-1)\ [2(\cosh h\omega-1)-h\omega\ \sinh h\omega]}{\omega\ (\cosh h\omega-1)},$$

$$\sum_{k=1}^{m-1} m_k\left[\frac{\lambda_k\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\right]+\sum_{k=1}^{m-1} n_k\left[\frac{\lambda_k^{N+1}\sinh h\omega}{\lambda_k^2+1-2\lambda_k\cosh h\omega}\right]=\frac{h\omega\sinh h\omega-2(\cosh h\omega-1)}{2\omega\ (\cosh h\omega-1)}.$$

The first $(m-1)$ equations of the last system give us that

$$m_k=n_k,\quad k=1,2,...,m-1.$$

Keeping in mind the latter equalities, from (5.41), (5.42) and (5.45) we get the expressions for optimal coefficients $\mathring{C}_\beta$, $\beta=0,\ldots,N$, and the system of $(m-1)$ linear equations for m_k $(k=1,2,...,m-1)$ which are given in the assertion of the Theorem. Theorem 7 is proved.

6. Conclusion

Thus, the present work is devoted to construction of the weighted optimal quadrature formulas of the form (1.1) in Hilbert spaces. Here, we obtained The following main results:

- In order to get the upper bound of the weighted quadrature formulas in the Hilbert space we have found the Riesz element for the error functional.
- We calculated the norm of the error functional (1.2) for the quadrature formulas.
- We have got the system of linear equations for the coefficients of weighted optimal quadrature formulas of the form (1.1).
- We developed a new algorithm for analytic solution of the system of linear equations.
- Using the algorithm we constructed the optimal quadrature formulas which are exact for hyperbolic functions and polynomials.

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