International Journal of Analysis and Applications
ISSN 2291-8639
Volume 5, Number 2 (2014), 198-211
http://www.etamaths.com

PROPERTIES OF MEROMORPHIC SOLUTIONS OF A CLASS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the growth of meromorphic solutions of some second order linear differential equations, where it is assumed that the coefficients are meromorphic functions. Our results extend the previous results due to Chen and Shon, Xu and Zhang, Peng and Chen and others.

1. **Introduction and statement of result**

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [13] , [20]). In addition, we will use notations $\rho(f)$, $\rho_2(f)$ to denote respectively the order and the hyper-order of growth of a meromorphic function $f(z)$.

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of equation (1.1) is an entire function, and that if f_1 and f_2 are two linearly independent solutions of (1.1) , then by [7], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But equation (1.1) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [8], Ozawa [16], Amemiya-Ozawa [1] and Gundersen [10], Langley [14] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order.

In 2002, Chen [3] considered the question: What conditions on $B(z)$ when $\rho(B) = 1$ will guarantee that every nontrivial solution of (1.1) has infinite order? He proved the following result, which improved results of Frei, Amemiya-Ozawa, Ozawa, Langley and Gundersen.

2010 *Mathematics Subject Classification.* 34M10, 30D35.
Key words and phrases. Linear differential equations, meromorphic solutions, order of growth, hyper-order.

Theorem A [3] *Let* $A_j(z)\,(\not\equiv 0)$ $(j=0,1)$ *be entire functions with* $\max\{\rho(A_j)\ (j=0,1)\}<1$, *and let* a,b *be complex constants that satisfy* $ab\neq 0$ *and* $a\neq b$. *Then every solution* $f\not\equiv 0$ *of the differential equation*

$$f''+A_1(z)\,e^{az}f'+A_0(z)\,e^{bz}f=0 \tag{1.2}$$

is of infinite order.

In [4], Chen and Shon have considered equation (1.2) when $A_j(z)$ $(j=0,1)$ are meromorphic functions and have proved the following result.

Theorem B ([4]) *Let* $A_j(z)\,(\not\equiv 0)$ $(j=0,1)$ *be meromorphic functions with* $\rho(A_j)<1$ $(j=0,1)$, *and let* a, b *be complex numbers such that* $ab\neq 0$ *and* $\arg a\neq \arg b$ *or* $a=cb$ $(0<c<1)$. *Then every meromorphic solution* $f(z)\not\equiv 0$ *of equation* (1.2) *has infinite order.*

In [17], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem C [17] *Let* $A_j(z)$ $(\not\equiv 0)$ $(j=1,2)$ *be entire functions with* $\rho(A_j)<1$, a_1, a_2 *be complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leq|a_2|$*). If* $\arg a_1\neq\pi$ *or* $a_1<-1$, *then every solution* $f\,(\not\equiv 0)$ *of the differential equation*

$$f''+e^{-z}f'+\left(A_1e^{a_1z}+A_2e^{a_2z}\right)f=0$$

has infinite order and $\rho_2(f)=1$.

Recently in [2], the authors extend and improve the results of Theorem C to some second order linear differential equations as follows.

Theorem D [2] *Let* $n\geq 2$ *be an integer,* $A_j(z)$ $(\not\equiv 0)$ $(j=1,2)$ *be entire functions with* $\max\{\rho(A_j):j=1,2\}<1$, $Q(z)=q_mz^m+\cdots+q_1z+q_0$ *be nonconstant polynomial and* a_1, a_2 *be complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$. *If* (1) $\arg a_1\neq\pi$ *and* $\arg a_1\neq\arg a_2$ *or* (2) $\arg a_1\neq\pi$, $\arg a_1=\arg a_2$ *and* $|a_2|>n\,|a_1|$ *or* (3) $a_1<0$ *and* $\arg a_1\neq\arg a_2$ *or* (4) $-\frac{1}{n}(|a_2|-m)<a_1<0$, $|a_2|>m$ *and* $\arg a_1=\arg a_2$, *then every solution* $f\not\equiv 0$ *of the differential equation*

$$f''+Q\left(e^{-z}\right)f'+\left(A_1e^{a_1z}+A_2e^{a_2z}\right)^nf=0$$

satisfies $\rho(f)=+\infty$ *and* $\rho_2(f)=1$.

Recently Xu and Zhang have investigated the order and the hyper-order of meromorphic solutions of some second order linear differential equations and have proved the following result.

Theorem E [19] *Suppose that* $A_j(z)$ $(\not\equiv 0)$ $(j=0,1,2)$ *are meromorphic functions and* $\rho(A_j)<1$, *and* a_1, a_2 *are two complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leq|a_2|$*). Let* a_0 *be a constant satisfying* $a_0<0$. *If* $\arg a_1\neq\pi$ *or* $a_1<a_0$, *then every meromorphic solution* $f\,(\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of the equation*

$$f''+A_0e^{a_0z}f'+\left(A_1e^{a_1z}+A_2e^{a_2z}\right)f=0$$

has infinite order and $\rho_2(f) = 1$.

The main purpose of this paper is to extend and improve the results of Theorems A-E to some second order linear differential equations. In fact we will prove the following result.

Theorem 1.1 *Let* $A_j(z)\,(\not\equiv 0)\,(j = 1, \cdots, l_1)\,(l_1 \geq 3)$ *and* $B_j(z)\,(\not\equiv 0)\,(j = 1, \cdots, l_2)$ $(l_2 \geq 1)$ *be meromorphic functions with*

$$\max\{\rho(A_j) \quad (j = 1, \cdots, l_1), \rho(B_j) \quad (j = 1, \cdots, l_2)\} < 1$$

and $a_j \neq 0\,(j = 1, \cdots, l_1)$ *be distinct complex numbers and* $b_j\,(j = 1, \cdots, l_2)$ *be distinct real numbers such that* $b_j < 0$. *Suppose that there exist* α_j, β_j $(j = 3, \cdots, l_1)$ *where* $0 < \alpha_j < 1$, $0 < \beta_j < 1$ *and* $a_j = \alpha_j a_1 + \beta_j a_2$. *Set* $\alpha = \max\{\alpha_j : j = 3, \cdots, l_1\}$, $\beta = \max\{\beta_j : j = 3, \cdots, l_1\}$ *and* $b = \min\{b_j : j = 1, \cdots, l_2\}$. *If*
(1) $\arg a_1 \neq \pi$ *and* $\arg a_1 \neq \arg a_2$
or
(2) $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ *and* (i) $|a_2| > \frac{|a_1|}{1-\beta}$ *or* (ii) $|a_2| < (1-\alpha)|a_1|$
or
(3) $a_1 < 0$ *and* $\arg a_1 \neq \arg a_2$
or
(4) (i) $(1-\beta)a_2 - b < a_1 < 0$, $a_2 < \frac{b}{1-\beta}$ *or* (ii) $a_1 < \frac{a_2+b}{1-\alpha}$ *and* $a_2 < 0$, *then every meromorphic solution* $f\,(\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of the differential equation*

$$f'' + \left(\sum_{j=1}^{l_2} B_j e^{b_j z}\right) f' + \left(\sum_{j=1}^{l_1} A_j e^{a_j z}\right) f = 0 \tag{1.3}$$

satisfies $\rho(f) = +\infty$ *and* $\rho_2(f) = 1$.

2. Preliminary lemmas

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where χ_H is the characteristic function of a set H.

Lemma 2.1 [11] *Let* f *be a transcendental meromorphic function with* $\rho(f) = \rho < +\infty$. *Let* $\varepsilon > 0$ *be a given constant, and let* k, j *be integers satisfying* $k > j \geq 0$. *Then, there exists a set* $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *with linear measure zero, such that, if* $\psi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, *then there is a constant* $R_0 = R_0(\psi) > 1$, *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geq R_0$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([4], [15]) *Consider* $g(z) = A(z)e^{az}$, *where* $A(z) \not\equiv 0$ *is a meromorphic function with order* $\rho(A) = \alpha < 1$, a *is a complex constant,* $a = |a|e^{i\varphi}$ $(\varphi \in [0, 2\pi))$. *Set* $E_2 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, *then* E_2 *is a finite set. Then for any given* ε $(0 < \varepsilon < 1 - \alpha)$ *there is a set* $E_3 \subset [0, 2\pi)$ *that has linear measure zero such that if* $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3)$, *then we have when* r *is sufficiently large:*

(i) *If* $\cos(\varphi+\theta)>0$, *then*

$$\exp\{(1-\varepsilon)r\delta(az,\theta)\}\le|g(z)|\le\exp\{(1+\varepsilon)r\delta(az,\theta)\}. \tag{2.2}$$

(ii) *If* $\cos(\varphi+\theta)<0$, *then*

$$\exp\{(1+\varepsilon)r\delta(az,\theta)\}\le|g(z)|\le\exp\{(1-\varepsilon)r\delta(az,\theta)\}, \tag{2.3}$$

where $\delta(az,\theta)=|a|\cos(\varphi+\theta)$.

Lemma 2.3 [17] *Suppose that* $n\ge1$ *is a natural number. Let* $P_j(z)=a_{jn}z^n+\cdots$ $(j=1,2)$ *be nonconstant polynomials, where* a_{jq} $(q=1,\cdots,n)$ *are complex numbers and* $a_{1n}a_{2n}\ne0$. *Set* $z=re^{i\theta}$, $a_{jn}=|a_{jn}|e^{i\theta_j}$, $\theta_j\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)$, $\delta(P_j,\theta)=|a_{jn}|\cos(\theta_j+n\theta)$, *then there is a set* $E_4\subset\left[-\frac{\pi}{2n},\frac{3\pi}{2n}\right)$ *that has linear measure zero such that if* $\theta_1\ne\theta_2$, *then there exists a ray* $\arg z=\theta$ *with* $\theta\in\left(-\frac{\pi}{2n},\frac{\pi}{2n}\right)\backslash(E_4\cup E_5)$ *satisfying either*

$$\delta(P_1,\theta)>0,\ \delta(P_2,\theta)<0 \tag{2.4}$$

or

$$\delta(P_1,\theta)<0,\ \delta(P_2,\theta)>0, \tag{2.5}$$

where $E_5=\left\{\theta\in\left[-\frac{\pi}{2n},\frac{3\pi}{2n}\right):\delta(P_j,\theta)=0\right\}$ *is a finite set, which has linear measure zero.*

Remark 2.1 [17] We can obtain, in Lemma 2.3, if $\theta\in\left(-\frac{\pi}{2n},\frac{\pi}{2n}\right)\setminus(E_4\cup E_5)$ is replaced by $\theta\in\left(\frac{\pi}{2n},\frac{3\pi}{2n}\right)\setminus(E_4\cup E_5)$, then it has the same result.

Lemma 2.4 [4] *Let* $f(z)$ *be a transcendental meromorphic function of order* $\rho(f)=\alpha<+\infty$. *Then for any given* $\varepsilon>0$, *there is a set* $E_6\subset\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)$ *that has linear measure zero such that if* $\theta\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)\diagdown E_6$, *then there is a constant* $R_1=R_1(\theta)>1$, *such that for all* z *satisfying* $\arg z=\theta$ *and* $|z|\ge R_1$, *we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\}\le|f(z)|\le\exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.6}$$

Lemma 2.5 [11] *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha>1$ *be a given constant. Then there exist a set* $E_7\subset(1,\infty)$ *with finite logarithmic measure and a constant* $B>0$ *that depends only on* α *and* i,j $(0\le i<j\le k)$, *such that for all* z *satisfying* $|z|=r\notin[0,1]\cup E_7$, *we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right|\le B\left\{\frac{T(\alpha r,f)}{r}(\log^\alpha r)\log T(\alpha r,f)\right\}^{j-i}. \tag{2.7}$$

Lemma 2.6 [12] *Let* $\varphi:[0,+\infty)\to\mathbb{R}$ *and* $\psi:[0,+\infty)\to\mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r)\le\psi(r)$ *for all* $r\notin E_8\cup[0,1]$, *where* $E_8\subset(1,+\infty)$ *is a set of finite logarithmic measure. Let* $\gamma>1$ *be a given constant. Then there exists an* $r_1=r_1(\gamma)>0$ *such that* $\varphi(r)\le\psi(\gamma r)$ *for all* $r>r_1$.

Lemma 2.7 [5] *Let* $k\ge2$ *and* $A_0,A_1,\cdots,A_{k-1}$ *be meromorphic functions. Let* $\rho=\max\{\rho(A_j):j=0,\cdots,k-1\}$ *and all poles of* f *are of uniformly bounded multiplicities. Then every transcendental meromorphic solution* f *of the differential equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0$$

satisfies $\rho_2(f)\le\rho$.

Lemma 2.8 ([9], [20]) *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)\,(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *are entire functions satisfying the following conditions:*
(i) $\sum_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(iii) For $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\left\{T\left(r, e^{g_h(z) - g_k(z)}\right)\right\}$ $(r \to \infty,$ $r \notin E_9)$, *where* E_9 *is a set with finite linear measure.*
Then $f_j(z) \equiv 0$ $(j = 1, \cdots, n)$.

Lemma 2.9 [18] *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)\,(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *are entire functions satisfying the following conditions:*
(i) $\sum_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv f_{n+1}$;
(ii) *If* $1 \leq j \leq n+1, 1 \leq k \leq n$, *the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \geq 2$, $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, *and the order of* f_j *is less than the order of* $e^{g_h - g_k}$.
Then $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, n+1)$.

3. Proof of Theorem 1.1

First of all we prove that equation (1.3) can't have a meromorphic solution $f \not\equiv 0$ with $\rho(f) < 1$. Assume a meromorphic solution $f \not\equiv 0$ with $\rho(f) < 1$. We can rewrite (1.3) in the following form

$$\sum_{j=1}^{l_2} B_j f' e^{b_j z} + \sum_{j=1}^{l_1} A_j f e^{a_j z} = -f''. \tag{3.1}$$

Obviously, $\rho(B_j f') < 1$ $(j = 1, \cdots, l_2)$ and $\rho(A_j f) < 1$ $(j = 1, \cdots, l_1)$. Set $I = \{a_j \ (j = 1, \cdots, l_1),\ b_j \ (j = 1, \cdots, l_2)\}$.
1) By the conditions (1) or (2) or (4) (ii) of Theorem 1.1, we can see that $a_1 \neq a_2, a_3, \cdots, a_{l_1}, b_1, \cdots, b_{l_2}$. Then, we can rewrite (3.1) in the following form

$$A_1 f e^{a_1 z} + \sum_{\lambda \in \Gamma_1} f_\lambda e^{\lambda z} = -f'', \tag{3.2}$$

where $\Gamma_1 \subseteq I \setminus \{a_1\}$ and f_λ $(\lambda \in \Gamma_1)$ are meromorphic functions with order less than 1 and a_1, λ $(\lambda \in \Gamma_1)$ are distinct numbers. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, which is a contradiction.
2) By the conditions (3) or (4) (i) of Theorem 1.1, we can see that $a_2 \neq a_1, a_3, \cdots, a_{l_1}$, $b_1, \cdots, b_{l_2}$. Then, we can rewrite (3.1) in the following form

$$A_2 f e^{a_2 z} + \sum_{\lambda \in \Gamma_2} f_\lambda e^{\lambda z} = -f'', \tag{3.3}$$

where $\Gamma_2 \subseteq I \setminus \{a_2\}$ and f_λ $(\lambda \in \Gamma_2)$ are meromorphic functions with order less than 1 and a_2, λ $(\lambda \in \Gamma_2)$ are distinct numbers. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, which is a contradiction. Therefore $\rho(f) \geq 1$.

First step. We prove that $\rho(f) = +\infty$. Assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.3) with $1 \le \rho(f) = \sigma_1 < +\infty$. From equation (1.3), we know that the poles of $f(z)$ can occur only at the poles of A_j $(j = 1, \cdots, l_1)$ and B_j $(j = 1, \cdots, l_2)$. Note that the multiplicities of poles of f are uniformly bounded, and thus we have [6]

$$N(r,f) \le M_1 \overline{N}(r,f) \le M_1 \left(\sum_{j=1}^{l_1} \overline{N}(r,A_j) + \sum_{j=1}^{l_2} \overline{N}(r,B_j) \right)$$

$$\le M \max\{N(r,A_j)(j=1,\cdots,l_1), N(r,B_j)(j=1,\cdots,l_2)\},$$

where M_1 and M are some suitable positive constants. This gives

$$\lambda\left(\frac{1}{f}\right) \le \gamma = \max\{\rho(A_j)\quad (j=1,\cdots,l_1), \rho(B_j)\quad (j=1,\cdots,l_2)\} < 1.$$

Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\rho(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) = \sigma_2 \le \gamma < 1$, g is an entire function and $1 \le \rho(g) = \rho(f) = \sigma_1 < \infty$. Substituting $f = g/d$ into (1.3), we can get

$$\frac{g''}{g} + \left[\left(\sum_{j=1}^{l_2} B_j e^{b_j z}\right) - 2\frac{d'}{d}\right]\frac{g'}{g} + 2\left(\frac{d'}{d}\right)^2 - \frac{d''}{d} - \left(\sum_{j=1}^{l_2} B_j e^{b_j z}\right)\frac{d'}{d}$$

$$+A_1 e^{a_1 z} + A_2 e^{a_2 z} + \sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1 + \beta_j a_2) z} = 0. \tag{3.4}$$

By Lemma 2.4, for any given ε $(0 < \varepsilon < 1 - \gamma)$, there is a set $E_6 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ that has linear measure zero such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_6$, then there is a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \ge R_1$, we have

$$|B_j(z)| \le \exp\left\{r^{\gamma+\varepsilon}\right\} \quad (j = 1, \cdots, l_2). \tag{3.5}$$

By Lemma 2.1, for any given ε $(0 < \varepsilon < 1 - \gamma)$, there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \ge R_0$, we have

$$\left|\frac{g^{(j)}(z)}{g(z)}\right| \le r^{j(\sigma_1 - 1 + \varepsilon)} \quad (j = 1, 2), \tag{3.6}$$

$$\left|\frac{d^{(j)}(z)}{d(z)}\right| \le r^{j(\sigma_2 - 1 + \varepsilon)} \quad (j = 1, 2). \tag{3.7}$$

Let $z = re^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. We know that $\delta(\alpha_j a_1 z, \theta) = \alpha_j \delta(a_1 z, \theta)$, $\delta(\beta_j a_2 z, \theta) = \beta_j \delta(a_2 z, \theta)$ $(j = 3, \cdots, l_1)$ and $\alpha < 1$, $\beta < 1$.

Case 1. Assume that $\arg a_1 \ne \pi$ and $\arg a_1 \ne \arg a_2$, which is $\theta_1 \ne \pi$ and $\theta_1 \ne \theta_2$. By Lemma 2.2 and Lemma 2.3, for any given ε

$$0 < \varepsilon < \min\left\{1 - \gamma, \frac{1-\alpha}{2(1+\alpha)}, \frac{1-\beta}{2(1+\beta)}\right\},$$

there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ (where E_4 and E_5 are defined as in Lemma 2.3, $E_1 \cup E_4 \cup E_5 \cup E_6$ is of linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0,\ \delta(a_2 z, \theta) < 0$$

or

$$\delta(a_1 z, \theta) < 0,\ \delta(a_2 z, \theta) > 0.$$

(a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1 e^{a_1 z}| \geq \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.8}$$

$$|A_2 e^{a_2 z}| \leq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} < 1, \tag{3.9}$$

$$\begin{aligned} |A_j e^{\alpha_j a_1 z}| &\leq \exp\{(1+\varepsilon)\alpha_j \delta(a_1 z, \theta) r\} \\ &\leq \exp\{(1+\varepsilon)\alpha \delta(a_1 z, \theta) r\} \ \ (j = 3, \cdots, l_1), \end{aligned} \tag{3.10}$$

$$\left|e^{\beta_j a_2 z}\right| \leq \exp\{(1-\varepsilon)\beta_j \delta(a_2 z, \theta) r\} < 1 \ \ (j = 3, \cdots, l_1). \tag{3.11}$$

By (3.10) and (3.11), we get

$$\begin{aligned} \left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1 + \beta_j a_2) z}\right| &\leq \sum_{j=3}^{l_1} |A_j e^{\alpha_j a_1 z}| \left|e^{\beta_j a_2 z}\right| \\ &\leq (l_1 - 2) \exp\{(1+\varepsilon)\alpha \delta(a_1 z, \theta) r\}. \end{aligned} \tag{3.12}$$

For $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by (3.5) we have

$$\begin{aligned} \left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| &\leq \sum_{j=1}^{l_2} |B_j| \left|e^{b_j z}\right| \leq \exp\left\{r^{\gamma+\varepsilon}\right\} \sum_{j=1}^{l_2} \left|e^{b_j z}\right| \\ &= \exp\left\{r^{\gamma+\varepsilon}\right\} \sum_{j=1}^{l_2} e^{b_j r \cos\theta} \leq l_2 \exp\left\{r^{\gamma+\varepsilon}\right\} \end{aligned} \tag{3.13}$$

because $b_j < 0$ and $\cos\theta > 0$. By (3.4), we obtain

$$\begin{aligned} |A_1 e^{a_1 z}| \leq& \left|\frac{g''}{g}\right| + \left[\left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| + 2\left|\frac{d'}{d}\right|\right] \left|\frac{g'}{g}\right| + 2\left|\frac{d'}{d}\right|^2 + \left|\frac{d''}{d}\right| \\ &+ \left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| \left|\frac{d'}{d}\right| + |A_2 e^{a_2 z}| + \left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1 + \beta_j a_2) z}\right|. \end{aligned} \tag{3.14}$$

Substituting $(3.6) - (3.9), (3.12)$ and (3.13) into (3.14), we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} &\leq |A_1 e^{a_1 z}| \\ \leq r^{2(\sigma_1 - 1 + \varepsilon)} &+ \left[l_2 \exp\left\{r^{\gamma+\varepsilon}\right\} + 2r^{\sigma_2 - 1 + \varepsilon}\right] r^{\sigma_1 - 1 + \varepsilon} + 3r^{2(\sigma_2 - 1 + \varepsilon)} \\ + l_2 \exp\left\{r^{\gamma+\varepsilon}\right\} & r^{\sigma_2 - 1 + \varepsilon} + 1 + (l_1 - 2) \exp\{(1+\varepsilon)\alpha \delta(a_1 z, \theta) r\} \\ &\leq M_1 r^{M_2} \exp\left\{r^{\gamma+\varepsilon}\right\} \exp\{(1+\varepsilon)\alpha \delta(a_1 z, \theta) r\}, \end{aligned} \tag{3.15}$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. By $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)}$ and (3.15), we have

$$\exp\left\{\frac{1-\alpha}{2}\delta\left(a_1 z,\theta\right)r\right\} \le M_1 r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}. \tag{3.16}$$

By $\delta\left(a_1 z,\theta\right) > 0$ and $\gamma+\varepsilon < 1$ we know that (3.16) is a contradiction.

(b) When $\delta\left(a_1 z,\theta\right) < 0$, $\delta\left(a_2 z,\theta\right) > 0$, for sufficiently large r, we get

$$\left|A_2 e^{a_2 z}\right| \ge \exp\left\{(1-\varepsilon)\,\delta\left(a_2 z,\theta\right)r\right\}, \tag{3.17}$$

$$\left|A_1 e^{a_1 z}\right| \le \exp\left\{(1-\varepsilon)\,\delta\left(a_1 z,\theta\right)r\right\} < 1, \tag{3.18}$$

$$\left|A_j e^{\alpha_j a_1 z}\right| \le \exp\left\{(1-\varepsilon)\,\alpha_j\delta\left(a_1 z,\theta\right)r\right\} < 1 \ \ (j=3,\cdots,l_1), \tag{3.19}$$

$$\begin{aligned}\left|e^{\beta_j a_2 z}\right| &\le \exp\left\{(1+\varepsilon)\,\beta_j\delta\left(a_2 z,\theta\right)r\right\}\\ &\le \exp\left\{(1+\varepsilon)\,\beta\delta\left(a_2 z,\theta\right)r\right\} \ \ (j=3,\cdots,l_1).\end{aligned} \tag{3.20}$$

By (3.19) and (3.20), we get

$$\begin{aligned}\left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1+\beta_j a_2)z}\right| &\le \sum_{j=3}^{l_1}\left|A_j e^{\alpha_j a_1 z}\right|\left|e^{\beta_j a_2 z}\right|\\ &\le (l_1-2)\exp\left\{(1+\varepsilon)\,\beta\delta\left(a_2 z,\theta\right)r\right\}.\end{aligned} \tag{3.21}$$

By (3.4), we obtain

$$\begin{aligned}\left|A_2 e^{a_2 z}\right| \le{}& \left|\frac{g''}{g}\right| + \left[\left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| + 2\left|\frac{d'}{d}\right|\right]\left|\frac{g'}{g}\right| + 2\left|\frac{d'}{d}\right|^2 + \left|\frac{d''}{d}\right|\\ &+\left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right|\left|\frac{d'}{d}\right| + \left|A_1 e^{a_1 z}\right| + \left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1+\beta_j a_2)z}\right|.\end{aligned} \tag{3.22}$$

Substituting $(3.6), (3.7), (3.13), (3.17), (3.18)$ and (3.21) into (3.22), we have

$$\begin{aligned}\exp\left\{(1-\varepsilon)\,\delta\left(a_2 z,\theta\right)r\right\} &\le \left|A_2 e^{a_2 z}\right|\\ \le r^{2(\sigma_1-1+\varepsilon)} &+ \left[l_2\exp\left\{r^{\gamma+\varepsilon}\right\} + 2r^{\sigma_2-1+\varepsilon}\right]r^{\sigma_1-1+\varepsilon} + 3r^{2(\sigma_2-1+\varepsilon)}\\ +l_2\exp\left\{r^{\gamma+\varepsilon}\right\}&r^{\sigma_2-1+\varepsilon} + 1 + (l_1-2)\exp\left\{(1+\varepsilon)\,\beta\delta\left(a_2 z,\theta\right)r\right\}\\ &\le M_1 r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,\beta\delta\left(a_2 z,\theta\right)r\right\}.\end{aligned} \tag{3.23}$$

By $0 < \varepsilon < \frac{1-\beta}{2(1+\beta)}$ and (3.23), we obtain

$$\exp\left\{\frac{1-\beta}{2}\delta\left(a_2 z,\theta\right)r\right\} \le M_1 r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}. \tag{3.24}$$

By $\delta\left(a_2 z,\theta\right) > 0$ and $\gamma+\varepsilon < 1$ we know that (3.24) is a contradiction.

Case 2. Assume that $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$, which is $\theta_1 \neq \pi$, $\theta_1 = \theta_2$. By Lemma 2.3, for any given ε

$$0 < \varepsilon < \min\left\{1-\gamma, \frac{(1-\alpha)\left|a_1\right| - \left|a_2\right|}{2\left[(1+\alpha)\left|a_1\right| + \left|a_2\right|\right]}, \frac{(1-\beta)\left|a_2\right| - \left|a_1\right|}{2\left[(1+\beta)\left|a_2\right| + \left|a_1\right|\right]}\right\},$$

there is a ray $\arg z=\theta$ such that $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\backslash(E_1\cup E_4\cup E_5\cup E_6)$ and $\delta(a_1z,\theta)>0$. Since $\theta_1=\theta_2$, then $\delta(a_2z,\theta)>0$.

(i) $|a_2|>\frac{|a_1|}{1-\beta}$. For sufficiently large r, we have $(3.10),(3.17),(3.20)$ hold and we get

$$|A_1e^{a_1z}|\le\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}. \tag{3.25}$$

By (3.10) and (3.20), we obtain

$$\left|\sum_{j=3}^{l_1}A_je^{(\alpha_ja_1+\beta_ja_2)z}\right|\le\sum_{j=3}^{l_1}|A_je^{\alpha_ja_1z}|\left|e^{\beta_ja_2z}\right|$$
$$\le(l_1-2)\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}. \tag{3.26}$$

Substituting $(3.6),(3.7),(3.13),(3.17),(3.25)$ and (3.26) into (3.22), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(a_2z,\theta)r\}&\le|A_2e^{a_2z}|\\ \le r^{2(\sigma_1-1+\varepsilon)}&+\left[l_2\exp\left\{r^{\gamma+\varepsilon}\right\}+2r^{\sigma_2-1+\varepsilon}\right]r^{\sigma_1-1+\varepsilon}+3r^{2(\sigma_2-1+\varepsilon)}\\ &+l_2\exp\left\{r^{\gamma+\varepsilon}\right\}r^{\sigma_2-1+\varepsilon}+\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}\\ &+(l_1-2)\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}\end{aligned}$$
$$\le M_1r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}. \tag{3.27}$$

From (3.27), we obtain

$$\exp\{\eta_1r\}\le M_1r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}, \tag{3.28}$$

where

$$\eta_1=(1-\varepsilon)\delta(a_2z,\theta)-(1+\varepsilon)\delta(a_1z,\theta)-(1+\varepsilon)\beta\delta(a_2z,\theta).$$

Since $0<\varepsilon<\frac{(1-\beta)|a_2|-|a_1|}{2[(1+\beta)|a_2|+|a_1|]},\theta_1=\theta_2$ and $\cos(\theta_1+\theta)>0$, then

$$\begin{aligned}\eta_1&=[1-\beta-\varepsilon(1+\beta)]\delta(a_2z,\theta)-(1+\varepsilon)\delta(a_1z,\theta)\\ &=[1-\beta-\varepsilon(1+\beta)]|a_2|\cos(\theta_1+\theta)-(1+\varepsilon)|a_1|\cos(\theta_1+\theta)\\ &=\{[1-\beta-\varepsilon(1+\beta)]|a_2|-(1+\varepsilon)|a_1|\}\cos(\theta_1+\theta)\\ &=\{(1-\beta)|a_2|-|a_1|-\varepsilon[(1+\beta)|a_2|+|a_1|]\}\cos(\theta_1+\theta)\\ &>\frac{(1-\beta)|a_2|-|a_1|}{2}\cos(\theta_1+\theta)>0.\end{aligned}$$

By $\eta_1>0$ and $\gamma+\varepsilon<1$ we know that (3.28) is a contradiction.

(ii) $|a_2|<(1-\alpha)|a_1|$. For sufficiently large r, we have $(3.8),(3.10),(3.20),(3.26)$ hold and we obtain

$$|A_2e^{a_2z}|\le\exp\{(1+\varepsilon)\delta(a_2z,\theta)r\}. \tag{3.29}$$

Substituting $(3.6),(3.7),(3.8),(3.13),(3.26)$ and (3.29) into (3.14), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(a_1z,\theta)r\}&\le|A_1e^{a_1z}|\\ \le r^{2(\sigma_1-1+\varepsilon)}&+\left[l_2\exp\left\{r^{\gamma+\varepsilon}\right\}+2r^{\sigma_2-1+\varepsilon}\right]r^{\sigma_1-1+\varepsilon}+3r^{2(\sigma_2-1+\varepsilon)}\\ &+l_2\exp\left\{r^{\gamma+\varepsilon}\right\}r^{\sigma_2-1+\varepsilon}+\exp\{(1+\varepsilon)\delta(a_2z,\theta)r\}\\ &+(l_1-2)\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}\end{aligned}$$
$$\le M_1r^{M_2}\exp\left\{r^{\gamma+\varepsilon}\right\}\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\delta(a_2z,\theta)r\}. \tag{3.30}$$

From (3.30), we obtain

$$\exp\{\eta_2 r\} \le M_1 r^{M_2} \exp\{r^{\gamma+\varepsilon}\}, \tag{3.31}$$

where

$$\eta_2 = (1-\varepsilon)\,\delta(a_1 z,\theta) - (1+\varepsilon)\,\alpha\delta(a_1 z,\theta) - (1+\varepsilon)\,\delta(a_2 z,\theta).$$

Since $0<\varepsilon<\frac{(1-\alpha)|a_1|-|a_2|}{2[(1+\alpha)|a_1|+|a_2|]}$, $\theta_1=\theta_2$ and $\cos(\theta_1+\theta)>0$, then we get

$$\begin{aligned}\eta_2 &= \{(1-\alpha)|a_1| - |a_2| - \varepsilon[(1+\alpha)|a_1|+|a_2|]\}\cos(\theta_1+\theta)\\ &> \frac{(1-\alpha)|a_1|-|a_2|}{2}\cos(\theta_1+\theta) > 0.\end{aligned}$$

By $\eta_2>0$ and $\gamma+\varepsilon<1$ we know that (3.31) is a contradiction.

Case 3. Assume that $a_1<0$ and $\arg a_1 \ne \arg a_2$, which is $\theta_1=\pi$ and $\theta_2\ne\pi$. By Lemma 2.2, for the above ε, there is a ray $\arg z=\theta$ such that $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$ and $\delta(a_2 z,\theta)>0$. Because $\cos\theta>0$, we have $\delta(a_1 z,\theta)=|a_1|\cos(\theta_1+\theta)=-|a_1|\cos\theta<0$. Using the same reasoning as in Case 1(b), we can get a contradiction.

Case 4. Assume that (i) $(1-\beta)a_2-b<a_1<0$ and $a_2<\frac{b}{1-\beta}$ or (ii) $a_1<\frac{a_2+b}{1-\alpha}$ and $a_2<0$, which is $\theta_1=\theta_2=\pi$. By Lemma 2.2, for any given ε

$$0<\varepsilon<\min\left\{1-\gamma, \frac{(1-\alpha)|a_1|-|a_2|+b}{2[(1+\alpha)|a_1|+|a_2|]}, \frac{(1-\beta)|a_2|-|a_1|+b}{2[(1+\beta)|a_2|+|a_1|]}\right\},$$

there is a ray $\arg z=\theta$ such that $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$, then $\cos\theta<0$, $\delta(a_1 z,\theta)=|a_1|\cos(\theta_1+\theta)=-|a_1|\cos\theta>0$, $\delta(a_2 z,\theta)=|a_2|\cos(\theta_2+\theta)=-|a_2|\cos\theta>0$.

(i) $(1-\beta)a_2-b<a_1<0$ and $a_2<\frac{b}{1-\beta}$. For sufficiently large r, we get $(3.10), (3.17), (3.20), (3.25)$ and (3.26) hold. For $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ by (3.5) we have

$$\begin{aligned}\left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| &\le \sum_{j=1}^{l_2}|B_j|\left|e^{b_j z}\right| \le \exp\{r^{\gamma+\varepsilon}\}\sum_{j=1}^{l_2}\left|e^{b_j z}\right|\\ &= \exp\{r^{\gamma+\varepsilon}\}\sum_{j=1}^{l_2} e^{b_j r\cos\theta} \le l_2\exp\{r^{\gamma+\varepsilon}\}e^{br\cos\theta}\end{aligned} \tag{3.32}$$

because $b\le b_j<0$ and $\cos\theta<0$. Substituting $(3.6), (3.7), (3.17), (3.25), (3.26)$ and (3.32) into (3.22), we obtain

$$\begin{aligned}&\exp\{(1-\varepsilon)\,\delta(a_2 z,\theta)\,r\} \le |A_2 e^{a_2 z}|\\ &\le M_1 r^{M_2} e^{br\cos\theta}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\,\delta(a_1 z,\theta)\,r\}\exp\{(1+\varepsilon)\,\beta\delta(a_2 z,\theta)\,r\}.\end{aligned} \tag{3.33}$$

From (3.33) we have

$$\exp\{\eta_3 r\} \le M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}, \tag{3.34}$$

where

$$\eta_3 = (1-\varepsilon)\,\delta(a_2 z,\theta) - (1+\varepsilon)\,\delta(a_1 z,\theta) - (1+\varepsilon)\,\beta\delta(a_2 z,\theta) - b\cos\theta.$$

Since $(1-\beta) a_2 - b < a_1$, $a_2 = -|a_2|$ and $a_1 = -|a_1|$, then we get $(1-\beta)|a_2| - |a_1| + b > 0$. We can see that $0 < (1-\beta)|a_2| - |a_1| + b < (1-\beta)|a_2| - |a_1| < 2[(1+\beta)|a_2| + |a_1|]$. Therefore

$$0 < \frac{(1-\beta)|a_2| - |a_1| + b}{2[(1+\beta)|a_2| + |a_1|]} < 1.$$

By $0 < \varepsilon < \frac{(1-\beta)|a_2| - |a_1| + b}{2[(1+\beta)|a_2| + |a_1|]}, \theta_1 = \theta_2 = \pi$ and $\cos\theta < 0$, we obtain

$$\begin{aligned}\eta_3 &= [1-\beta-\varepsilon(1+\beta)]\,\delta(a_2 z,\theta) - (1+\varepsilon)\,\delta(a_1 z,\theta) - b\cos\theta \\ &= -[1-\beta-\varepsilon(1+\beta)]\,|a_2|\cos\theta + (1+\varepsilon)|a_1|\cos\theta - b\cos\theta \\ &= (-\cos\theta)\{[1-\beta-\varepsilon(1+\beta)]\,|a_2| - (1+\varepsilon)|a_1| + b\} \\ &= (-\cos\theta)\{(1-\beta)|a_2| - |a_1| + b - \varepsilon[(1+\beta)|a_2| + |a_1|]\} \\ &> \frac{-1}{2}[(1-\beta)|a_2| - |a_1| + b]\cos\theta > 0.\end{aligned}$$

By $\eta_3 > 0$ and $\gamma + \varepsilon < 1$ we know that (3.34) is a contradiction.

(ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. For sufficiently large r, we get (3.8), (3.10), (3.20), (3.26) and (3.29) hold. Substituting (3.6), (3.7), (3.8), (3.26), (3.29) and (3.32) into (3.14), we obtain

$$\exp\{(1-\varepsilon)\,\delta(a_1 z,\theta)\,r\} \le |A_1 e^{a_1 z}|$$

(3.35)

$$\le M_1 r^{M_2} e^{br\cos\theta} \exp\{r^{\gamma+\varepsilon}\} \exp\{(1+\varepsilon)\,\alpha\delta(a_1 z,\theta)\,r\} \exp\{(1+\varepsilon)\,\delta(a_2 z,\theta)\,r\}.$$

From (3.35) we have

$$\exp\{\eta_4 r\} \le M_1 r^{M_2} \exp\{r^{\gamma+\varepsilon}\}, \tag{3.36}$$

where

$$\eta_4 = (1-\varepsilon)\,\delta(a_1 z,\theta) - (1+\varepsilon)\,\alpha\delta(a_1 z,\theta) - (1+\varepsilon)\,\delta(a_2 z,\theta) - b\cos\theta.$$

Since $a_1 < \frac{a_2+b}{1-\alpha}, a_2 = -|a_2|$ and $a_1 = -|a_1|$, then we get $(1-\alpha)|a_1| - |a_2| + b > 0$. We can see that $0 < (1-\alpha)|a_1| - |a_2| + b < (1-\alpha)|a_1| - |a_2| < 2[(1+\alpha)|a_1| + |a_2|]$. Therefore

$$0 < \frac{(1-\alpha)|a_1| - |a_2| + b}{2[(1+\alpha)|a_1| + |a_2|]} < 1.$$

By $0 < \varepsilon < \frac{(1-\alpha)|a_1| - |a_2| + b}{2[(1+\alpha)|a_1| + |a_2|]}, \theta_1 = \theta_2 = \pi$ and $\cos\theta < 0$, we get

$$\begin{aligned}\eta_4 &= (-\cos\theta)\{(1-\alpha)|a_1| - |a_2| + b - \varepsilon[(1+\alpha)|a_1| + |a_2|]\} \\ &> \frac{-1}{2}[(1-\alpha)|a_1| - |a_2| + b]\cos\theta > 0.\end{aligned}$$

By $\eta_4 > 0$ and $\gamma + \varepsilon < 1$ we know that (3.36) is a contradiction. Concluding the above proof, we obtain $\rho(f) = \rho(g) = +\infty$.

Second step. We prove that $\rho_2(f) = 1$. By

$$\max\left\{\rho\left(\sum_{j=1}^{l_2} B_j e^{b_j z}\right), \rho\left(\sum_{j=1}^{l_1} A_j e^{a_j z}\right)\right\} = 1$$

and Lemma 2.7, we obtain $\rho_2(f) \le 1$. By Lemma 2.5, we know that there exists a set $E_7 \subset (1, +\infty)$ with finite logarithmic measure and a constant $C > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_7$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le C\,[T(2r,f)]^{j+1} \quad (j = 1,2)\,. \tag{3.37}$$

By (1.3), we have

$$|A_1 e^{a_1 z}| \le \left|\frac{f''}{f}\right| + \left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| \left|\frac{f'}{f}\right| + |A_2 e^{a_2 z}| + \left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1 + \beta_j a_2) z}\right|, \tag{3.38}$$

$$|A_2 e^{a_2 z}| \le \left|\frac{f''}{f}\right| + \left|\sum_{j=1}^{l_2} B_j e^{b_j z}\right| \left|\frac{f'}{f}\right| + |A_1 e^{a_1 z}| + \left|\sum_{j=3}^{l_1} A_j e^{(\alpha_j a_1 + \beta_j a_2) z}\right|. \tag{3.39}$$

Case 1. $\arg a_1 \ne \pi$ and $\arg a_1 \ne \arg a_2$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying

$$\delta(a_1 z, \theta) > 0,\ \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0,\ \delta(a_2 z, \theta) > 0.$$

(a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get (3.8) – (3.12) hold. Substituting (3.8), (3.9), (3.12), (3.13) and (3.37) into (3.38), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_7$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\begin{aligned} \exp\{(1-\varepsilon)\,\delta(a_1 z, \theta)\, r\} &\le |A_1 e^{a_1 z}| \\ &\le M \exp\left\{r^{\gamma+\varepsilon}\right\} \exp\{(1+\varepsilon)\,\alpha\delta(a_1 z, \theta)\, r\}\,[T(2r,f)]^3, \end{aligned} \tag{3.40}$$

where $M > 0$ is a some constant. From (3.40) and $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)}$, we get

$$\exp\left\{\frac{1-\alpha}{2}\delta(a_1 z, \theta)\, r\right\} \le M \exp\left\{r^{\gamma+\varepsilon}\right\}[T(2r,f)]^3. \tag{3.41}$$

Since $\delta(a_1 z, \theta) > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.6 and (3.41), we obtain $\rho_2(f) \ge 1$. Hence $\rho_2(f) = 1$.

(b) When $\delta(a_1 z, \theta) < 0$, $\delta(a_2 z, \theta) > 0$, for sufficiently large r, we get (3.17) – (3.21) hold. By using the same reasoning as above, we can get $\rho_2(f) = 1$.

Case 2. $\arg a_1 \ne \pi$, $\arg a_1 = \arg a_2$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta(a_1 z, \theta) > 0$ and $\delta(a_2 z, \theta) > 0$.

(i) $|a_2| > \frac{|a_1|}{1-\beta}$. For sufficiently large r, we have (3.10), (3.17), (3.20), (3.25) and (3.26) hold. Substituting (3.13), (3.17), (3.25), (3.26) and (3.37) into (3.39), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_7$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\begin{aligned} &\exp\{(1-\varepsilon)\,\delta(a_2 z, \theta)\, r\} \le |A_2 e^{a_2 z}| \\ &\le M \exp\left\{r^{\gamma+\varepsilon}\right\} \exp\{(1+\varepsilon)\,\delta(a_1 z, \theta)\, r\} \exp\{(1+\varepsilon)\,\beta\delta(a_2 z, \theta)\, r\}\,[T(2r,f)]^3. \end{aligned} \tag{3.42}$$

From (3.42), we obtain

$$\exp\{\eta_1 r\} \le M \exp\left\{r^{\gamma+\varepsilon}\right\}[T(2r,f)]^3, \tag{3.43}$$

where

$$\eta_1 = (1-\varepsilon)\,\delta\,(a_2 z,\theta) - (1+\varepsilon)\,\delta\,(a_1 z,\theta) - (1+\varepsilon)\,\beta\delta\,(a_2 z,\theta)\,.$$

Since $\eta_1 > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.6 and (3.43), we obtain $\rho_2(f) \geq 1$. Hence $\rho_2(f) = 1$.

(ii) $|a_2| < (1-\alpha)\,|a_1|$. For sufficiently large r, we have $(3.8)\,,\,(3.10)\,,\,(3.20)\,,\,(3.26)$ and (3.29) hold. By using the same reasoning as above, we can get $\rho_2(f) = 1$.

Case 3. $a_1 < 0$ and $\arg a_1 \neq \arg a_2$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta\,(a_2 z,\theta) > 0$ and $\delta\,(a_1 z,\theta) < 0$. Using the same reasoning as in the second step (**Case 1** (b)), we can get $\rho_2(f) = 1$.

Case 4. (i) $(1-\beta)\,a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$ or (ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta\,(a_2 z,\theta) > 0$ and $\delta\,(a_1 z,\theta) > 0$.

(i) $(1-\beta)\,a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$. For sufficiently large r, we get $(3.10)\,,$ $(3.17)\,,\,(3.20)\,,\,(3.25)$ and (3.26) hold. Substituting $(3.17)\,,\,(3.25)\,,\,(3.26)\,,\,(3.32)$ and (3.37) into (3.39), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_7$, $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\exp\{(1-\varepsilon)\,\delta\,(a_2 z,\theta)\,r\} \leq |A_2 e^{a_2 z}|$$

$$\leq Me^{br\cos\theta} \exp\left\{r^{\gamma+\varepsilon}\right\} \exp\{(1+\varepsilon)\,\delta\,(a_1 z,\theta)\,r\}$$

$$\times \exp\{(1+\varepsilon)\,\beta\delta\,(a_2 z,\theta)\,r\}\,[T\,(2r,f)]^3\,. \tag{3.44}$$

From (3.44) we obtain

$$\exp\{\eta_3 r\} \leq M \exp\left\{r^{\gamma+\varepsilon}\right\}[T\,(2r,f)]^3\,, \tag{3.45}$$

where

$$\eta_3 = (1-\varepsilon)\,\delta\,(a_2 z,\theta) - (1+\varepsilon)\,\delta\,(a_1 z,\theta) - (1+\varepsilon)\,\beta\delta\,(a_2 z,\theta) - b\cos\theta.$$

Since $\eta_3 > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.6 and (3.45), we obtain $\rho_2(f) \geq 1$. Hence $\rho_2(f) = 1$.

(ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. For sufficiently large r, we get $(3.8)\,,\,(3.10)\,,\,(3.20)\,,$ (3.26) and (3.29) hold. By using the same reasoning as above, we can get $\rho_2(f) = 1$. Concluding the above proof, we obtain that every meromorphic solution $f\,(\not\equiv 0)$ whose poles are of uniformly bounded multiplicities of (1.3) satisfies $\rho(f) = \infty$ and $\rho_2(f) = 1$. The proof of Theorem 1.1 is complete.

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