International Journal of Analysis and Applications

Recursive Computation of Rayleigh- Rayleigh Distribution via Ordered Random Variates

M. I. Khan[*]

Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Kingdom of Saudi Arabia

[*]*Corresponding author:* izhar.stats@gmail.com, khanizhar@iu.edu.sa

Abstract. Ordered random variables (ORVs) are of great importance in statistical science. These random variables are organized in increasing order called generalized order statistics (GOS). It has tremendous applications in engineering and science due to the inclusion of ordered random variables. This article addresses recursive moments of Rayleigh-Rayleigh distribution using order random variables. Such moments are applicable in studying the characteristics of random variables in increasing order such as time to failure of an electronic devices. The characterization result is also obtained by simple moments.

1. Introduction

The ORVs such as order statistics (OS) and record values play a significant role in applied probability and many branches of statistics. [1] introduced the concept of GOS and documented that OS, record values, and some other ORVs can be taken as special cases of GOS. The GOS have been of interest in the past thirty years because they are more flexible in statistical modeling, reliability theory and inference. The is defined as follows.
Suppose $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in R^{n-1}, n \in N, M_u = \sum_{u=1}^{n-1} m_u, 1 < u < n-1$, be the parameters such that $\zeta_u = k + (n-u) + M_u > 0, k > 1$. Then their joint density of $f_{u:n,\tilde{m},k}(x_1, x_2, \cdots, x_n)$ takes the form as

$$k\left(\prod_{\mu=1}^{n-1} xi_{\mu}\right)\left(\prod_{\mu=1}^{n-1} [\bar{F}(x_{\mu})]^{m_{\mu}} f(x_{\mu})\right)[\bar{F}(x_n)]^{k-1} f(x_n). \tag{1.1}$$

Marginal density of simple and joint densities of two GOS are presented by two ways symbolically $\xi_t \neq \xi_s$ and $\xi_1 = \xi_s$, $(s \neq t,\ s, t = 1, 2, \cdots, n-1)$.

Received: May 13, 2024.

2020 *Mathematics Subject Classification.* 62G30, 62E10, 60E05.

Key words and phrases. Rayleigh- Rayleigh distribution; Moments; Recurrence relations; Order random variables.

https://doi.org/10.28924/2291-8639-22-2024-190

ISSN: 2291-8639

Case I: The marginal density of simple GOS presented by [2] for $(1 \leq \mu \leq n)$, when $\xi_t \neq \xi_s$

$$f_{u:n,\tilde{m},k}(x) = C_{u-1} \sum_{t=1}^{u} a_t(u)[\bar{F}(x)]^{\xi_t - 1} f(x). \tag{1.2}$$

The joint densities of two GOS for $1 \leq u \leq v \leq n$ are.

$$f_{u,v:n,\tilde{m},k}(x,y) = C_{v-1} \sum_{s=u+1}^{v} a_s^{(u)}(v) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\xi_s} \left[\sum_{t=1}^{u} a_t(u)[\bar{F}(x)]^{\xi_t}\right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \tag{1.3}$$

where

$$\begin{aligned} C_{u-1} &= \prod_{s=1}^{u} \xi_s, u = 1, 2, \cdots, n. \\ \xi_t &= k + n - t + M_t, t = 1, 2, \cdots, n, \\ a_t(u) &= \prod_{s=1, s \neq t}^{u} (\xi_s - \xi_t)^{-1}, 1 \leq t \leq u \leq n, \end{aligned}$$

and

$$a_t^{(u)}(v) = \prod_{s=u+,s \neq t}^{v} (\xi_s - \xi_t)^{-1}, u + 1 \leq t \leq v \leq n.$$

Case II. The marginal density of simple GOS presented by [1] for $(1 \leq u \leq n)$, when $\xi_t = \xi_s$

$$f_{u:n,m,k}(x) = \frac{C_{u-1}}{(u-1)!}[\bar{F}(x)]^{\xi_u - 1} g_m^{u-1}[F(x)] f(x). \tag{1.4}$$

The joint densities of two GOS are.

$$f_{u,v:n,\tilde{m},k}(x,y) = \frac{c_{v-1}}{(u-1)!(v-u-1)!}[\bar{F}(x)]^m f(x) g_m^{u-1}[F(x)][h_m F(y) - h_m F(x)]^{v-u-1} \tag{1.5}$$

where $g_m(x)$ and $h_m(x)$ are defined in [1].

GOS is a combined framework for ORVs and has extensive relevance in many domains of life. Moments of GOS appears in several areas of allied sciences. It allows calculates the basis to evaluate higher moments of GOS from lower-order moments analogously. The OS and record values are noteworthy and more popular cases of GOS which is defined as follows.

1.1. **Order Statistics.** A sequence of random variables (RV) are organized in their magnitude of increasing order referred to OS. Let $X_{1:n} \leq X_{2:n} \leq \cdots X_{n:n}$ denote the corresponding OS. Then the density of u^{th} OS ($X_{u:n}$) is given by [3].

$$f_{u:n}(x) = C_{u:n}[F(x)]^{u-1}[1 - F(x)]^{n-u} f(x), -\infty < x < \infty, \tag{1.6}$$

where

$$C_{u:n} = \frac{n!}{((u-1)!(n-u)!}.$$

The density of minimal ($u = 1$) and maximal ($u = n$) OS can be easily obtained from (1.6).

The OS deals with many theoretical and practical problems like, estimation of parameters, goodness of fit, characterizations, censored samples, reliability theory and entropy, see [4]- [7]. Putting $m_\mu = 0$ and $k = 1$ in (1.1), the formula agrees with (1.6).

1.2. **Record Values.** The theme of record values was coined by [8]. An observation which exhibits greater than all the previous observation, called records. For a survey on important results in this area one may refer to Ref. [9]- [12].
Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed with cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. Let $Y = \max(\min)\{X_1, X_2, \cdots, , X_n\}$ for $n \geq 1$. We say X_j is a upper (lower) record values of this sequence if $Y_j > (<) Y_{j-1}$ for all j.
The marginal PDF of nth upper record value is given as

$$f_{X_{U(n)}}(x) = \frac{1}{(n-1)!}[-\ln(\bar{F}(x))]^{n-1} f(x), \ -\infty < x < \infty. \tag{1.7}$$

Letting $m_\mu \to -1$ in (1.1), the resultant PDF agrees with (1.7).
A voluminous work on computation recurrence relations for different distribution via GOS are considered in the literature. Some pioneering works initiated with [13–15]. Further, several authors contributed to this direction, see [16] to [25] and many more. The recursive moment techniques execute competitively with the best available statistical methods, which makes recursive moments now worthy of the attention of researchers.
The theme of this article is to set the moments of GOS derived from RRD. The outline of the paper is as. Section 2 presents RRD and its moments properties. The simple moments are addressed in Section 3. The relation for joint moments is dealt in Section 4. The characterization result is proved in Section 4. The cumulative residual entropy is given in Section 5. The article concludes in Section 6.

2. Rayleigh- Rayleigh Distribution

Rayleigh distribution (RD) proposed by [26] signifies a pivotal role in modelling and examining lifetime data e.g., physical sciences, communication theory, life testing experiments, medical imaging science, clinical studies, reliability analysis and engineering. Due to relevance of RD in many fields, a comprehensive extension of RD has been established. One of them is transformed transformer (TT) technique proposed by [27]. According to TT technique the function form CDF of a RV is taken to transform PDF of other RV into a new distribution.
On using TT technique [28] generalized RD named Rayleigh- RD. The generalized distribution provides more flexible lifetime data analysis over several lifetime distributions (see, RD, generalized RD , exponentiated RD, Weibull-RD and alpha power RD). This leads to tremendous applications of Rayleigh- RD (RRD) in medical statistics, survival analysis, industrial engineering, wind energy and hydrology.
A RV Xassumes RRD with shape and scale parameters (σ, β). Then its CDF is given in (2.1).

$$F(x) = 1 - e^{-\frac{x^4}{8\beta^4\sigma^2}}, x > 0, \sigma, \beta > 0, \tag{2.1}$$

and PDF is given in (2.2).

$$f(x) = \frac{x^3}{2\beta^4\sigma^2} e^{-\frac{x^4}{8\beta^4\sigma^2}}, x > 0, \sigma, \beta > 0. \tag{2.2}$$

The k^{th} moments of RRD is given in (2.3).

$$E[x^k] = 8^{\frac{k}{4}} \beta(\sigma)^{\frac{k}{2}} gamma\left(\frac{k}{4} + 1\right). \tag{2.3}$$

The RRD satisfies the following properties.

(i) It is a unimodal, increasing and decreasing shapes.
(ii) The reliability function is monotonically decreasing.
(iii) The hazard rate function is exponentially increasing.

These features enable it to be used in several areas, such as reliability, survival analysis, life testing and others. For more details, of RRD, (see, [28]).

TABLE 1. First four moments and variance of RRD.

E(X)						$E(X^2)$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	1.077	1.523	1	0.761	1.524	1	1.253	2.506	1	0.627	2.507
2	2.154	3.046	2	1.077	2.155	2	5.013	10.026	2	1.253	5.013
3	3.231	4.569	3	1.319	2.639	3	11.280	22.559	3	1.880	7.520
4	4.308	6.092	4	1.524	3.047	4	20.052	40.105	4	2.507	10.026
5	5.385	7.650	5	1.703	3.407	5	31.332	62.664	5	3.133	12.533
$E(X^3)$						$E(X^4)$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	1.545	4.372	1	2.186	4.372	1	2	8	1	2	8
2	3.091	8.744	2	6.182	12.365	2	4	16	2	8	32
3	4.636	13.116	3	11.358	22.717	3	6	24	3	18	72
4	6.182	17.488	4	17.486	34.975	4	8	32	4	32	128
5	7.727	21.860	5	24.437	48.878	5	10	40	5	50	200
Var(X)											
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
0.5	0.93	0.373	0.841	1.493	2.333	0.5	0.048	0.093	0.140	0.184	0.233
1.0	0.186	0.747	1.683	2.992	4.141	1.0	0.184	0.369	0.556	0.742	0.925

The next we evaluate the numerical characteristics through ORVs for RRD.
The moments of u^{th} OS for RRD is.

$$\mu_{u:n}^{(k)} = C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i [8\beta^4\sigma^2]^{\frac{k}{4}} \left[\frac{1}{i + (n-u) + 1}\right]^{\frac{k}{4}+1} gamma\left(\frac{k}{4} + 1\right). \tag{2.4}$$

The moments of k^{th} record values for RRD is.

$$E(X_{U(n)}^k) = \frac{(8\beta^4\sigma^2)^{\frac{k}{4}}}{(n-1)!} gamma\left(\frac{k}{4} + n\right). \tag{2.5}$$

In the next two sections, we address the relationship of recursive moments for RRD through GOS.

Table 2. The characteristics of means and variances by minimal OS.

$E(X_{1:1}) = \mu_{1:1}$						$E(X_{1:2}) = \mu_{1:2}$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	1.082	1.53	1	0.765	1.53	1	0.91	1.287	1	0.644	1.287
2	2.164	3.06	2	1.082	2.164	2	1.82	2.574	2	0.911	1.820
3	3.246	4.59	3	1.324	2.649	3	2.73	3.861	3	1.115	2.229
4	4.328	6.12	4	1.53	3.06	4	3.64	5.148	4	1.288	2.574
5	5.41	7.65	5	1.712	3.421	5	4.55	6.435	5	1.44	2.878
$E(X_{1:3}) = \mu_{1:3}$						$E(X_{1:4}) = \mu_{1:4}$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	0.824	1.165	1	0.683	1.165	1	0.764	1.08	1	0.54	1.08
2	1.648	2.33	2	0.824	1.6475	2	1.528	1.53	2	0.764	1.527
3	2.472	3.495	3	1.009	2.0177	3	2.292	1.871	3	0.935	1.871
4	3.296	4.66	4	1.166	2.33	4	3.056	2.16	4	1.08	2.16
5	4.12	5.825	5	1.303	2.604	5	3.763	2.414	5	1.207	2.415
$E(X_{1:5}) = \mu_{1:5}$						$Var(X_{1:1}) = Var(\mu_{1:1}))$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	0.724	1.024	1	0.512	1.024	1	0.089	0.179	1	0.044	0.179
2	1.448	2.048	2	0.724	1.448	2	0.278	0.716	2	0.089	0.357
3	2.172	3.072	3	0.886	1.773	3	0.803	1.612	3	0.137	0.542
4	2.896	4.096	4	1.024	2.048	4	1.428	2.866	4	0.179	0.716
5	3.62	5.12	5	1.144	2.289	5	2.232	4.477	5	0.219	0.896
$Var(X_{1:2}) = Var(\mu_{1:2})$						$Var(X_{1:3}) = Var(\mu_{1:3})$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	0.061	0.123	1	0.040	0.123	1	0.058	0.092	1	0.113	0.411
2	0.248	0.494	2	0.072	0.248	2	0.204	0.371	2	0.221	0.844
3	0.557	1.113	3	0.117	0.367	3	0.459	0.835	3	0.331	1.259
4	0.990	1.978	4	0.136	0.544	4	0.816	1.391	4	0.438	1.691
5	1.548	3.090	5	0.176	0.617	5	1.275	0.552	5	0.552	2.119
$Var(X_{1:4}) = Var(\mu_{1:4})$						$Var(X_{1:5}) = Var(\mu_{1:5})$					
β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$	β	$\alpha = 0.5$	$\alpha = 1.0$	α	$\beta = 0.5$	$\beta = 1.0$
1	0.052	0.093	1	0.028	0.093	1	0.042	0.079	1	0.017	0.071
2	0.239	2.699	2	0.062	0.179	2	0.138	0.285	2	0.042	0.928
3	0.425	7.843	3	0.076	0.283	3	0.322	0.642	3	0.047	0.227
4	0.740	6.13	4	0.114	0.374	4	0.550	1.142	4	0.071	0.286
5	1.161	9.922	5	0.143	0.443	5	0.895	1.786	5	0.100	1.355

Table 3. The values of mean and variance are based on the upper record

$E(X_{U(1)})$						$E(X_{U(2)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	1.077	1.523	1	0.761	1.524	1	1.347	1.905	1	0.960	1.905
2	2.154	3.046	2	1.077	2.155	2	2.694	3.810	2	1.347	2.695
3	3.231	4.569	3	1.319	2.639	3	4.041	5.715	3	1.650	3.300
4	4.308	6.092	4	1.524	3.047	4	5.388	7.620	4	1.905	3.811
5	5.385	7.650	5	1.703	3.407	5	6.735	9.525	5	2.130	4.261
$E(X_{U(3)})$						$E(X_{U(4)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	1.515	2.144	1	1.072	2.143	1	1.642	2.322	1	1.1611	2.322
2	3.030	4.287	2	1.516	3.032	2	3.283	4.644	2	1.642	3.284
3	4.544	6.431	3	1.856	3.713	3	4.925	6.966	3	2.011	4.022
4	6.059	8.574	4	2.143	4.287	4	6.567	9.288	4	2.322	4.644
5	7.574	10.718	5	2.397	4.793	5	8.209	11.61	5	2.596	5.193
$E(X^2_{U(1)})$						$E(X^2_{U(2)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	1.253	2.506	1	0.627	2.507	1	1.880	3.760	1	0.940	3.760
2	5.013	10.026	2	1.253	5.013	2	7.520	15.093	2	1.880	7.520
3	11.280	22.559	3	1.880	7.520	3	16.919	33.838	3	2.819	11.279
4	20.052	40.105	4	2.507	10.026	4	30.079	60.157	4	3.760	15.039
5	31.332	62.664	5	3.133	12.533	5	46.999	93.996	5	4.700	18.799
$E(X^2_{U(3)})$						$E(X^2_{U(4)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	2.350	4.700	1	1.175	4.6700	1	2.742	5.483	1	1.371	5.483
2	9.400	18.800	2	2.350	9.400	2	10.966	21.933	2	2.742	10.966
3	21.150	42.300	3	3.525	14.100	3	24.674	49.349	3	4.112	16.449
4	37.600	75.200	4	4.700	18.800	4	43.865	87.731	4	5.483	21.933
5	58.750	117.500	5	5.875	23.500	5	68.540	137.080	5	6.854	27.416
$Var(X_{U(1)})$						$Var(X_{U(2)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	0.093	0.186	1	0.048	0.184	1	0.065	0.131	1	0.018	0.131
2	0.373	0.747	2	0.093	0.369	2	0.262	0.576	2	0.066	0.257
3	0.841	1.683	3	0.140	0.556	3	0.589	1.177	3	0.097	0.389
4	1.493	2.992	4	0.184	0.742	4	1.048	2.992	4	0.131	0.515
5	2.333	4.141	5	0.233	0.925	5	1.638	3.270	5	0.163	0.643
$Var(X_{U(3)})$						$Var(X_{U(4)})$					
β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$	β	$\alpha=0.5$	$\alpha=1.0$	α	$\beta=0.5$	$\beta=1.0$
1	0.055	0.103	1	0.026	0.078	1	0.046	0.914	1	0.023	0.913
2	0.219	0.423	2	0.064	0.206	2	0.188	0.366	2	0.046	0.181
3	0.502	0.942	3	0.080	0.314	3	0.418	0.823	3	0.068	0.323
4	0.889	1.687	4	0.108	0.423	4	0.739	1.4640	4	0.091	0.366
5	1.385	2.625	5	0.129	0.527	5	1.152	2.288	5	0.115	0.448

3. Recursive Simple Moments

The a^{th} moments of GOS for a random sample from $F(x)$ is.

$$\varphi^{a}_{u:n,\tilde{m},k} = C_(u-1)\int_{-\infty}^{\infty} x^{a} f(x)\sum_{t=1}^{u} a_t(u)[\bar{F}(x)]^{\xi_t - 1}dx.$$

The characterizing differential equation for RRD from (8) and (9) is related as

$$1 - F(x) = 2\beta^4\sigma^2 x^{-3} f(x). \tag{3.1}$$

Theorem 3.1. *For RRD, the simple moments of GOS is related as*

$$\varphi^{a}_{u:n,\tilde{m},k} = \varphi^{a}_{u-1:n,\tilde{m},k} + \frac{2\beta^4\sigma^2 a}{\xi_u}\varphi^{a-4}_{r:n,\tilde{m},k}. \tag{3.2}$$

Proof. For any distribution, [15] proved that the recursive moments of simple GOS as

$$\varphi^{a}_{u:n,\tilde{m},k} - \varphi^{a}_{u-1:n,\tilde{m},k} = aC_{u-2}\int_{-\infty}^{\infty} x^{a-1}\sum_{t=1}^{u} a_t(u)[\bar{F}(x)]^{\xi_t}dx. \tag{3.3}$$

Or,

$$\varphi^{a}_{u:n,\tilde{m},k} - \varphi^{a}_{u-1:n,\tilde{m},k} = aC_{u-2}\int_{0}^{\infty} x^{a-1}\bar{F}(x)\sum_{t=1}^{u} a_t(u)[\bar{F}(x)]^{\xi_t - 1}dx. \tag{3.4}$$

By using (3.1), the relation in (3.2) is obtained. □

Remark 1. Recursive inverse moments of GOS for RRD is obtained by replacing a with $(-a)$ in (3.2) i.e.

$$\varphi^{-a}_{u:n,\tilde{m},k} = \varphi^{-a}_{u-1:n,\tilde{m},k} - \frac{2\beta^4\sigma^2 a}{\xi_u}\varphi^{-(a+4)}_{r:n,\tilde{m},k}.$$

Remark 2. The recursive moments of OS for RRD is.

$$\varphi^{a}_{u:n} = \varphi^{a}_{u-1:n} + \frac{2\beta^4\sigma^2 a}{n-u+1}\varphi^{a-4}_{u:n},$$

and recursive inverse moments as

$$\varphi^{-a}_{u:n} = \varphi^{-a}_{u-1:n} - \frac{2\beta^4\sigma^2 a}{n-u+1}\varphi^{-(a+4)}_{u:n}.$$

Remark 3. The recursive moments of k^{th} record values for RRD is.

$$\varphi^{a}_{K(u)} = \varphi^{a}_{k(u-1)} + \frac{2\beta^4\sigma^2 a}{k}\varphi^{a-4}_{k(u)},$$

and recursive inverse moments of k^{th} record is,

$$\varphi^{-a}_{K(u)} = \varphi^{-a}_{k(u-1)} - \frac{2a\beta^4\sigma^2}{k}\varphi^{-(a+4)}_{k(u)}.$$

4. Recursive Joint Moments

The $(a,b)^{th}$ moments of GOS for a random sample from $F(x)$ is.

$$\varphi^{(a,b)}_{u,v:n,\tilde{m},k} = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^a x_2^b C_{v-1}\left\{\sum_{t=u+1}^{v} a_t^{(u)}(v)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\xi_t}\right\}\sum_{t=1}^{u} a_t(u)[\bar{F}(x_1)]^{\xi_t} \times\frac{f(x_1)}{\bar{F}(x_1)}\frac{f(x_2)}{\bar{F}(x_2)}dx_2dx_1.$$

Theorem 4.1. *The joint moments of for RRD is stated as*

$$\varphi^{a,b}_{u,v:n,\tilde{m},k} = \varphi^{a,b}_{u,v-1:n,\tilde{m},k} + \frac{b}{\xi_v}2\beta^4\sigma^2\varphi^{a,b-4}_{u,v:n,\tilde{m},k}. \tag{4.1}$$

Proof. For any distribution, recursive relation for product moment is.

$$\varphi^{a,b}_{u,v:n,\tilde{m},k} - \varphi^{a,b}_{u,v-1:n,\tilde{m},k} = bC_{v-2}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^a x_2^{b-1}\sum_{t=u+1}^{v} a_t^{(u)}(v)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\xi_t} \times\sum_{t=1}^{u} a_t(u)[\bar{F}(x_1)]^{\xi_t}\frac{f(x_1)}{\bar{F}(x_1)}dx_2dx_1. \tag{4.2}$$

Or,

$$= bC_{s-2}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^a x_2^{b-1}\sum_{t=u+1}^{v} a_t^{(u)}(v)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\xi_t} \times\sum_{t=1}^{u} a_t(u)[\bar{F}(x_1)]^{\xi_t}\frac{f(x_1)}{\bar{F}(x_1)}\frac{\bar{F}(x_2)}{\bar{F}(x_2)}dx_2dx_1. \tag{4.3}$$

Now using (3.1) in (4.3), we have

$$\mu^{a,b}_{r:s,n,\tilde{m},k} - \mu^{a,b}_{r:s-1,n,\tilde{m},k} = bC_{s-2}\int_{0}^{\infty}\int_{x_1}^{\infty} x_1^a x_2^{b-1}\sum_{t=u+1}^{v} a_t^{(u)}(v)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\xi_t} \times\sum_{t=1}^{u} a_t(u)[\bar{F}(x_1)]^{\xi_t}\frac{f(x_1)}{\bar{F}(x_1)}\frac{f(x_2)}{\bar{F}(x_2)}[2\beta^4\sigma^2x_2^{-3}]dx_2dx_1.$$

It leads to (4.1).

□

Remark 4. Ratio moments of GOS for RRD is noted as

$$\varphi^{a,-b}_{u,v:n,\tilde{m},k} = \varphi^{a,-b}_{u,v-1:n,\tilde{m},k} - \frac{b}{\xi_v}2\beta^4\sigma^2\varphi^{a,-(b+4)}_{u,v,n,\tilde{m},k}$$

Remark 5. The joint and ratio moments of O.S. for RRD are noted as

$$\varphi^{a,b}_{u,v:n} = \varphi^{a,b}_{u,v-1:n} + \frac{b}{n-v+1}2\beta^4\sigma^2\varphi^{a,b-4}_{u,v:n},$$

and

$$\varphi^{a,-b}_{u,v:n} = \varphi^{a,-b}_{u,v-1:n} - \frac{b}{(n-v+1)}2\beta^4\sigma^2\varphi^{a,-(b+4)}_{u,v:n}.$$

Remark 6. The product and ratio moments of k^{th} record values for RRD are given as

$$\varphi^{a,b}_{k(u,v)} = \varphi^{a,b}_{k(u,v-1)} + \frac{b}{k} 2\beta^4\sigma^2 \varphi^{a,b-4}_{k(u,v)},$$

and

$$\varphi^{a,-b}_{k(u,v)} = \varphi^{a,-b}_{k(u,v-1)} + \frac{b}{k} 2\beta^4\sigma^2 \varphi^{a,-(b+4)}_{k(u,v)}.$$

5. Characterization

Characterization of a probability distribution confirms a unique property of that probability distribution. Various techniques are available in the literature. In this section, characterization for RRD by simple moments of GOS is proved, when $\xi_t = \xi$. i.e.,

$$\varphi^{a}_{u:n,m,k} = \varphi^{a}_{u-1:n,m,k} + \frac{aC_{u-1}}{\xi_u(u-1)!} \int_{-\infty}^{\infty} x^{a-1} [\bar{F}(x)]^{\xi_u} g_m^{u-1}[F(x)] dx.$$

Theorem 5.1. *If $X \sim RRD(\sigma, \beta)$, then necessary and sufficient condition for RV X is.*

$$\varphi^{a}_{u:n,m,k} = \varphi^{a}_{u-1:n,m,k} + 2\beta^4\sigma^2 \frac{a}{\xi_u} \varphi^{a-4}_{u:n,m,k}. \tag{5.1}$$

Proof. Theorem 3.1 suffices the necessary part. Consider the relation in (5.1) to prove the sufficient part, and hence we get.

$$\frac{aC_{u-1}}{\xi_u(u-1)!} \int_{0}^{\infty} x^{a-1} [\bar{F}(x)]^{\xi_u} g_m^{u-1}[F(x)] dx = 2\beta^4\sigma^2 \frac{a}{\xi_u} \varphi^{a-4}_{u:n,m,k},$$

which yields

$$\frac{aC_{u-1}}{\xi_u(u-1)!} \int_{0}^{\infty} x^{a-1} [\bar{F}(x)]^{\xi_u} g_m^{u-1}[F(x)] dx = \frac{2\beta^4\sigma^2 aC_{u-1}}{\xi_u(u-1)!} \int_{0}^{\infty} x^{a-4} [\bar{F}(x)]^{\xi_u - 1} g_m^{u-1}[F(x)] f(x) dx.$$

Or,

$$\frac{aC_{u-1}}{\xi_u(u-1)!} \int_{0}^{\infty} x^{a-1} [\bar{F}(x)]^{\xi_u} g_m^{u-1}[F(x)] \{\bar{F}(x) - 2\beta^4\alpha^2 x^{-3} f(x)\} dx = 0. \tag{5.2}$$

By employing generalization of [29] to (5.2), we get.

$$\bar{F}(x) = 2\beta^4\alpha^2 x^{-3} f(x).$$

It approves the relation given in (3.1) and completes proof. □

6. Cumulative Residual Entropy

[30] presented another measure of uncertainty called cumulative residual entropy (CRE) based on distribution function. It generalizes the Shannon entropy and holds more general mathematical properties. For more details see [31–33]. The CRE is defined as

$$CRE(x) = -\int_{0}^{\infty} \bar{F}(x) \ln \bar{F}(x) dx \tag{6.1}$$

The CRE from RRD is.

$$CRE(X) = \frac{1}{4} (8\beta^4\sigma^2)^{\frac{1}{4}} gamma\left(\frac{5}{4}\right). \tag{6.2}$$

Table 4. Pattern of CRE for RRD.

α	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$	β	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$
0.5	0.272	0.543	0.816	1.088	1.360	0.5	0.192	0.270	0.33	0.385	0.425
1.0	0.383	0.765	1.149	1.532	1.915	1.0	0.383	0.540	0.662	0.766	0.854
1.5	0.467	0.935	1.402	1.869	2.336	1.5	0.575	0.810	0.990	1.155	1.275
2.0	0.540	1.080	1.620	2.160	2.700	2.0	0.776	1.08	1.32	1.54	1.70
2.5	0.605	1.210	1.815	2.421	3.026	2.5	0.958	1.35	1.65	1.925	2.125
3.0	0.663	1.326	1.988	2.650	3.312	3.0	1.149	1.62	1.98	2.31	2.55
3.5	0.716	1.432	2.149	2.865	3.581	3.5	1.341	1.89	2.31	2.695	2.975
4.0	0.766	1.532	2.298	3.064	3.83	4.0	1.52	2.16	2.64	3.08	3.40
4.5	0.812	1.624	2.436	3.248	4.060	4.5	1.723	2.43	2.97	3.465	3.825
5.0	0.854	1.708	2.562	3.416	4.270	5.0	1.915	2.70	3.30	3.85	4.25

7. Conclusion

Moments of ORVs play an important role in mathematical statistics. It calculates higher moments from lower order analogously. The estimation and hypothesis testing of RRD parameters based on GOS are open for future work.

Acknowledgement: The author wishes to extend his sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for support provided to the Post-Publication Program.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] U. Kamps, A Concept of Generalized Order Statistics, J. Stat. Plan. Inference 48 (1995), 1–23. https://doi.org/10.1016/0378-3758(94)00147-N.

[2] U. Kamps, E. Cramer, On Distributions Of Generalized Order Statistics, Statistics 35 (2001), 269–280. https://doi.org/10.1080/02331880108802736.

[3] H.A. David, H.N. Nagaraja, Order Statistics, Wiley, 2003. https://doi.org/10.1002/0471722162.

[4] P. Prescott, N. Balakrishnan, A.C. Cohen, Order Statistics and Inference–Estimation Methods, J. R. Stat. Soc. Ser. A (Stat. Soc.) 155 (1992), 307–312. https://doi.org/10.2307/2982964.

[5] J.R.M. Hosking, L-Moments: Analysis and Estimation of Distributions Using Linear Combinations of Order Statistics, J. R. Stat. Soc. Ser. B: Stat. Methodol. 52 (1990), 105–124. https://doi.org/10.1111/j.2517-6161.1990.tb01775.x.

[6] A.G. Glen, L.M. Leemis, D.R. Barr, Order Statistics in Goodness-of-Fit Testing, IEEE Trans. Rel. 50 (2001), 209–213. https://doi.org/10.1109/24.963129.

[7] N. Balakrishnan, F. Buono, M. Longobardi, On Cumulative Entropies in Terms of Moments of Order Statistics, Methodol. Comput. Appl. Prob. 24 (2022), 345–359. https://doi.org/10.1007/s11009-021-09850-0.

[8] K.N. Chandler, The Distribution and Frequency of Record Values, J. R. Stat. Soc. Ser. B: Stat. Methodol. 14 (1952), 220–228. https://doi.org/10.1111/j.2517-6161.1952.tb00115.x.

[9] M. Ahsanullah, Record Statistics, Nova Science Publishers, New York, 1995.

[10] B.C. Arnold, N. Balakrishnan, H.N. Nagaraja, Records, Wiley, New York, 2011.

[11] M. Ahsanullah, V.B. Nevzorov, Record via Probability Theory, Atlantis Press, Paris, 2015.

[12] A. Asgharzadeh, M. Abdi, S. Nadarajah, Interval Estimation for Gumbel Distribution Using Climate Records, Bull. Malays. Math. Sci. Soc. 39 (2016), 257–270. https://doi.org/10.1007/s40840-015-0185-2.

[13] M. Ahsanullah, Generalized Order Statistics from Exponential Distribution, J. Stat. Plan. Inference 85 (2000), 85–91. https://doi.org/10.1016/S0378-3758(99)00068-3.

[14] A.E.B.A. Ahmad, Single and Product Moments of Generalized Order Statistics from Linear Exponential Distribution, Comm. Stat. - Theory Meth. 37 (2008), 1162–1172. https://doi.org/10.1080/03610920701713344.

[15] H. Athar, H. M. Islam, Recurrence Relations between Single and Product Moments of Generalized Order Statistics from a General Class of Distribution, Metron, 42(2004), 327-337.

[16] R. U. Khan, Z, Anwar, and H. Athar, Recurrence Relations for Single and Product Moments of Dual Generalized Order Statistics from Exponentiated Weibull Distribution, Aligarh J. Stat. 28 (2008), 37–45.

[17] M. Mohsin, M.Q. Shahbaz, G. Kibria, Recurrence Relations for Single and Product Moments of Generalized Order Statistics for Rayleigh Distribution, Appl. Math. Inf. Sci. 4 (2010), 273-279.

[18] M.D.I. Khan, M.A.R. Khan, Moments of Generalized Order Statistics From New Weibull-Pareto Distribution and Its Characterization, Int. J. Appl. Math. 29 (2016), 641–653. https://doi.org/10.12732/ijam.v29i6.1.

[19] M.I. Khan, M.A.R. Khan, Recurrence Relations of Moments of Generalized Order Statistics in a Compound Rayleigh Distribution, Thailand Statistician, 15(2017), 11–16.

[20] M. Faizan, M.I. Khan, Some Characterization Results Based on Expected Values of Generalized Order Statistics, Appl. Math. E-Notes 17 (2017), 106–113.

[21] B. Singh, R.U. Khan, M.A.R. Khan, Generalized Order Statistics from Kumaraswamy-Burr Distribution and Related Inference, J. Stat.: Adv. Theory Appl. 19 (2018), 1–16.

[22] M.I. Khan, Generalized Order Statistics from Power Lomax Distribution and Characterization, Appl. Math. E-Notes 18 (2018), 148–155.

[23] M.I. Khan, Generalized Order Statistics from Power-Linear Hazard Rate Distribution and Characterizations, Thailand Statistician, 20 (2022), 532–544.

[24] F.A.M. Khan, S.H. Shahbaz, M.Q. Shahbaz, On Relations for Moments of Generalized Order Statistics for Linley-Weibull Distribution, Comp. Syst. Sci. Eng. 41 (2022), 197–208.

[25] M.I. Khan, Moments of Ordered Random Variates for Transmuted Power Hazard Distribution, J. Appl. Math. Inf. 41 (2023), 1047–1056.

[26] J.W.S. Rayleigh, On the Resultant of a Large Number of Vibrations of the Same Pitch and of Arbitrary Phase, Phil. Mag. 5th Ser. 10 (1880), 73–78.

[27] A. Alzaatreh, C. Lee, F. Famoye, A New Method for Generating Families of Continuous Distributions, Metron 71 (2013), 63–79. https://doi.org/10.1007/s40300-013-0007-y.

[28] K. Ateeq, T.B. Qasim, A.R. Alvi, An Extension of Rayleigh Distribution and Applications, Cogent Math. Stat. 6 (2019), 1622191. https://doi.org/10.1080/25742558.2019.1622191.

[29] J.S. Hwang, G.D. Lin, Extensions of Müntz-Szász Theorem and Applications, Analysis 4 (1984), 143–160. https://doi.org/10.1524/anly.1984.4.12.143.

[30] M. Rao, Y. Chen, B.C. Vemuri, F. Wang, Cumulative Residual Entropy: A New Measure of Information, IEEE Trans. Inf. Theory 50 (2004), 1220–1228. https://doi.org/10.1109/TIT.2004.828057.

[31] C. Calì, M. Longobardi, J. Ahmadi, Some Properties of Cumulative Tsallis Entropy, Physica A: Stat. Mech. Appl. 486 (2017), 1012–1021. https://doi.org/10.1016/j.physa.2017.05.063.

[32] C. Calì, M. Longobardi, J. Navarro, Properties for Generalized Cumulative Past Measures of Information, Prob. Eng. Inf. Sci. 34 (2020), 92–111. https://doi.org/10.1017/S0269964818000360.

[33] M. Longobardi, Cumulative Measures of Information and Stochastic Orders, Ric. Mat. 63 (2014), 209–223. https://doi.org/10.1007/s11587-014-0212-x.