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# On Pair of Compatible Mappings and Coincidence Point Theorems in $b$-Metric Spaces

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**Abstract.** The main goal of this study is to give two different results for a couple of compatible self-mappings. The first result provides the necessary condition for the existence of a coincidence point for a pair of mappings that are partially weakly increasing in partially ordered $b$-metric spaces. Additionally, we establish a fixed point result in order to guarantee the uniqueness of common fixed points for pair of maps satisfying the $b$ - (E.A.) Property. Our findings extends and improve well established results of existing literature. In order highlight the distinctiveness of our main theorems, two discrete examples with Tabular and Graphical representations are also presented.

## 1. Introduction

Maurice Frechet [1] founded the well renowned concept of metric space as a generalization of traditional distance. In the field of metric space theory, specifically in the realm of non-linear analysis, many writers have investigated non-contraction mappings. It is widely recognized that nonlinear differential and integral equations are typically involved in the processes of solving physical problems. It is important to note that the contraction principle proposed by Banach [2] plays a significant role in dealing with physical problems of this kind and serves as an effective means for finding the solutions to these equations. In general, contraction mappings are continuous. It has several applications and extensions. In 1968, Kannan [3] demonstrated a generalization of Banach's [2] theorem that does not need the assumption of map continuity. Since then, there have been other extensions and generalizations of the contraction principle. One such expansion was presented by Jungck [4] for two pairs of self-maps that possess a unique common fixed point.

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Jungck ( [5], [6]) led the idea of commuting, compatible mapping and weak compatible mappings to deduce the fixed point results for pair of self mappings in complete metric spaces. Sessa [7] and Pant [8] have further developed and modified the concept of commuting maps by introducing the notion of R-weak commuting, see also ( [9], [10], [11]). Morales and Rojas ( [12], [13]) have established the existence and uniqueness of fixed points and common fixed points for a broad category of contraction mappings with rational expressions. The contractive inequality of these mappings is regulated by functions that remain stable at zero.

The concept of metric spaces has been extensively investigated in several ways in the literature, in addition to the study of contraction mappings. A well-known extension of metric spaces is the concept of $b$ - metric spaces. This concept was originated from the research of Bourbaki [14] and Bakhtin [15]. Subsequently, Czerwik [16] presented and effectively defined the concept of $b$ - metric space in 1993, and further, in 1998, author [17] extended the contraction mapping theorem in sense of $b$ - metric spaces, see also ( [18], [19], [20]). It is important to note that the class of $b$-metric spaces is much more significant than the class of metric spaces. For more examples, fixed point results, coincidence point results and their applications, definitions of notions as $b$ - convergence, $b$ - Cauchy, $b$ - completeness and related result in the setting of partially ordered $b$ - metric spaces, we refer ( [21], [22]).

Numerous authors have carried out research on the existence of fixed points for weak contractions and generalized contractions within the context of partially ordered sets. In 2004, Ran and Reurings [23] provided the first outcome in this field. In continuation, Nieto and Lopez ( [24], [25]) further refined and extended above results with the help of non-decreasing functions and then proved some fixed-point results in such spaces. Recently, Gupta et al. ( [26], [27]) proved several fixed point theorems under partially ordered settings by defining some generalized contractions. Mani [28] and Gupta et al. [29] have also presented a class of generalized contraction involving control functions and proved some fixed and common fixed point results in the setting of partially ordered metric spaces. Aamri and Moutawakil [30] introduced the notion of $(E.A)$ - property in metric space. Later in 2015, Ozturk and Turkoglu [21] extended this idea in the setting of $b-$ metric space and give the notion of $b-(E.A)$ property.

## 2. Fundamental Notions and Relevant Literature

Before proceeding to the main results of this paper, lets recall some basic definition, examples and fundamental lemmas that will be quite useful in proving our main theorem. Authors in ( [15], [16]) defined $b-$ metric space as follows:

**Definition 2.1.** *[16] Consider $\Delta$ as a space and let $\mathbb{R}^+$ represent the set of all nonnegative integers. A function $\rho : \Delta \times \Delta \to \mathbb{R}^+$ is said to be a b-metric on $\Delta$ if it satisfies the following properties, for any $\xi, \eta, q$ in $\Delta$ and $s \geq 1$,*

*(1)* $\rho(\xi, \eta) = 0$ *if and only if* $\xi = \eta$

*(2)* $\rho(\xi, \eta) = \rho(\eta, \xi)$

*(3)* $\rho(\xi, q) \leq s[\rho(\xi, \eta) + \rho(\eta, q)]$

*The pair* $(\Delta, \rho)$ *is called a* $b$ *- metric space.*

Further details on topological characteristics and examples of $b$ - metric may be found in ( [16], [17]).

**Definition 2.2.** *[21] Let* $(\Delta, \rho)$ *is a* $b$ *- metric space and* $\mathcal{U}, \mathcal{V}$ *be the self - mappings defined on* $\Delta$*. Then*

*(1)* $\mathcal{U}, \mathcal{V}$ *are said to be compatible [5] if whenever a sequence* $\{\xi_n\}$ *in* $\Delta$ *is such that* $\{\mathcal{U}\xi_n\}$ *and* $\{\mathcal{V}\xi_n\}$ *are* $b$ *- convergent to some* $t \in \Delta$ *then*

$$\lim_{n \to \infty} \rho\left(\mathcal{U}\mathcal{V}\xi_n, \mathcal{V}\mathcal{U}\xi_n\right) = 0.$$

*(2)* $\mathcal{U}, \mathcal{V}$ *are said to be non-compatible [5], if at least one sequence in* $\Delta$ *is such that* $\{\mathcal{U}\xi_n\}$ *and* $\{\mathcal{V}\xi_n\}$ *are* $b$ *- convergent to some* $t \in \Delta$ *but*

$$\lim_{n \to \infty} \rho\left(\mathcal{U}\mathcal{V}\xi_n, \mathcal{V}\mathcal{U}\xi_n\right)$$

*is either nonzero or does not exist.*

*(3)* $\mathcal{U}, \mathcal{V}$ *are said to satisfy the b- (E.A) property [30] if there exists a sequence* $\{\xi_n\}$ *in* $\Delta$ *is such that* $\lim_{n \to \infty} \mathcal{U}\xi_n = \lim_{n \to \infty} \mathcal{V}\xi_n = t$ *for some* $t \in \Delta$*.*

*(4) A pair of maps* $\mathcal{U}$ *and* $\mathcal{V}$ *are said to be weakly compatible pair, if they commute at points where they coincide.*

**Definition 2.3.** *[25] Suppose* $\Delta$ *is a non-empty set and* $\preceq$ *is a partially ordered relation on set* $\Delta$*. Then a map* $\mathcal{U} : \Delta \to \Delta$ *is said to be non-decreasing if each* $\xi, \eta \in \Delta$*,*

$$\xi \preceq \eta \quad \textit{implies} \quad \mathcal{U}(\xi) \preceq \mathcal{U}(\eta).$$

**Definition 2.4.** *[31] Let us denote* $\psi$ *as the set of all altering distance function* $\psi : [0, \infty) \to [0, \infty)$ *which satisfies the following property:*

*(1)* $\psi$ *is continuous and not decreasing;*

*(2)* $\psi(t) = 0$ *if and only if* $t = 0$*;*

**Lemma 2.1.** *[21] Let* $(\Delta, \rho)$ *is a metric space and* $s \geq 1$*. Suppose the sequence* $\{\xi_m\}$ *satisfies the following condition:*

$$\rho\left(\xi_m, \xi_{m+1}\right) \leq k\rho\left(\xi_{m-1}, \xi_m\right)$$

*for some* $0 < k < \frac{1}{s}$ *and* $m = 1, 2, 3 \ldots$*. Then* $\{\xi_m\}$ *is a* $b$ *- Cauchy sequence in*$(\Delta, \rho)$*.*

## 3. Coincidence Point for Pair of Compatible Mappings

This section includes two theorems to determine the coincidence point a given pair of compatible self mappings in partially ordered $b$-metric spaces as well as in $b$ - metric spaces.
Lets state and prove our first result.

**Theorem 3.1.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{T} : \Delta \to \Delta$ be continuous mappings on a partially ordered complete b -metric space $(\Delta, \preceq, \rho)$ and with $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$, and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$. Assume that the compatible pairs $(\mathcal{U}, \mathcal{T})$ and $(\mathcal{V}, \mathcal{S})$, and the comparable elements $\mathcal{S}\eta$ and $\mathcal{T}\xi$ satisfies the condition:*

$$s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) \leq \psi(\mathcal{N}(\xi, \eta)), \text{ for all } \xi, \eta \in \Delta, \tag{3.1}$$

*where*

$$\mathcal{N}(\xi, \eta) = \max \left\{ \begin{array}{c} \rho(\mathcal{T}\xi, \mathcal{S}\eta), \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{T}\xi) + \rho(\mathcal{V}\eta, \mathcal{S}\eta)], \\ \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{S}\eta) + \rho(\mathcal{T}\xi, \mathcal{V}\eta)] \end{array} \right\}, \tag{3.2}$$

*$s \geq 1$, $\epsilon > 1$ is constant and function $\psi : [0, \infty) \to [0, \infty)$ is such that $\psi(n) \leq n$ for all $n > 0$ with $\psi(0) = 0$.*
*Further, if the pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{U})$ are partially weakly increasing with respect to $\mathcal{S}$ and $\mathcal{T}$, then $(\mathcal{U}, \mathcal{T})$ and $(\mathcal{V}, \mathcal{S})$ have a coincidence point.*
*Moreover, if $\eta$ is a coincidence point for comparable elements $\mathcal{S}\eta$ and $\mathcal{T}\eta$, then $\mathcal{U}\eta = \mathcal{V}\eta = \mathcal{T}\eta = \mathcal{S}\eta$.*

*Proof.* Let $\xi_0 \in \Delta$, as $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$ and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$ there exists $\xi_1, \xi_2$ in $\Delta$, such that $\mathcal{U}\xi_0 = \mathcal{T}\xi_1$ and $\mathcal{V}\xi_1 = \mathcal{S}\xi_2$.
Thus, we can construct a sequence $\{\eta_m\}$ as follow:

$$\eta_{2m+1} = \mathcal{U}\xi_{2m} = \mathcal{T}\xi_{2m+1}$$
$$\eta_{2m+2} = \mathcal{V}\xi_{2m+1} = \mathcal{S}\xi_{2m+2},$$

where $m = 0, 1, 2 \ldots$
Since the pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{U})$ are partially weakly increasing with respect to maps $\mathcal{T}$ and $\mathcal{S}$ respectively, therefore

$$\eta_1 = \mathcal{T}\xi_1 = \mathcal{U}\xi_0 \preceq \mathcal{V}\xi_1 = \eta_2 = \mathcal{S}\xi_2 \preceq \mathcal{U}\xi_2 = \mathcal{T}\xi_3 = \eta_3 = \cdots$$

for all $\xi_1 \in \mathcal{T}^{-1}(\mathcal{U}\xi_0), \xi_2 \in \mathcal{S}^{-1}(\mathcal{V}\xi_1)$.
By repeating the process, we deduce that

$$\eta_1 \preceq \eta_2 \preceq \cdots \preceq \eta_{2m} \preceq \eta_{2m+1} \preceq \eta_{2m+2} \preceq \cdots$$

for all $m \in N \cup \{0\}$.
**Claim:** The sequence $\{\eta_{2m}\}$ is a Cauchy sequence.
Since $\eta_{2m}$ and $\eta_{2m+1}$ are comparable, therefore from Eq. (3.1), we have

$$\begin{aligned} s^{\epsilon}\rho(\eta_{2m+1}, \eta_{2m}) &= s^{\epsilon}\rho(\mathcal{U}\xi_{2m}, \mathcal{V}\xi_{2m-1}) \\ &\leq \psi(\mathcal{N}(\xi_{2m}, \xi_{2m-1})), \end{aligned} \tag{3.3}$$

where

$$\mathcal{N}(\xi_{2m}, \xi_{2m-1}) = \max \left\{ \begin{array}{c} \rho(\mathcal{T}\xi_{2m}, \mathcal{S}\xi_{2m-1}), \\ \frac{1}{2s}[\rho(\mathcal{U}\xi_{2m}, \mathcal{T}\xi_{2m}) + \rho(\mathcal{V}\xi_{2m-1}, \mathcal{S}\xi_{2m-1})], \\ \frac{1}{2s}[\rho(\mathcal{T}\xi_{2m}, \mathcal{V}\xi_{2m-1}) + \rho(\mathcal{S}\xi_{2m-1}, \mathcal{U}\xi_{2m})]. \end{array} \right\}$$

Since $\mathcal{S}\xi_{2m-1} = \eta_{2m-1}$ and $\mathcal{T}\xi_{2m} = \eta_{2m}$ and $\mathcal{U}\xi_{2m} = \eta_{2m+1}$, therefore

$$\begin{aligned}
\mathcal{N}(\xi_{2m}, \xi_{2m-1}) &= \max\left\{\begin{array}{c} \rho(\eta_{2m}, \eta_{2m-1}), \frac{1}{2s}\left[\rho(\eta_{2m+1}, \eta_{2m}) + \rho(\eta_{2m}, \eta_{2m-1})\right], \\ \frac{1}{2s}\left[\rho(\eta_{2m+1}, \eta_{2m-1}) + \rho(\eta_{2m}, \eta_{2m})\right] \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} \rho(\eta_{2m}, \eta_{2m-1}), \frac{1}{2s}\left[\rho(\eta_{2m+1}, \eta_{2m}) + \rho(\eta_{2m}, \eta_{2m-1})\right], \\ \frac{1}{2s}\left[s\rho(\eta_{2m+1}, \eta_{2m}) + s\rho(\eta_{2m}, \eta_{2m-1})\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \rho(\eta_{2m}, \eta_{2m-1}), \frac{1}{2s}\left[\rho(\eta_{2m+1}, \eta_{2m}) + \rho(\eta_{2m}, \eta_{2m-1})\right], \\ \frac{1}{2}\left[\rho(\eta_{2m+1}, \eta_{2m}) + \rho(\eta_{2m}, \eta_{2m-1})\right] \end{array}\right\} \\
&= \max\left\{\rho(\eta_{2m}, \eta_{2m-1}), \frac{1}{2}\left[\rho(\eta_{2m+1}, \eta_{2m}) + \rho(\eta_{2m}, \eta_{2m-1})\right]\right\}.
\end{aligned}$$

This gives that

$$\mathcal{N}(\xi_{2m}, \xi_{2m-1}) = \begin{cases} \rho(\eta_{2m}, \eta_{2m-1}), \text{ if } \rho(\eta_{2m}, \eta_{2m-1}) > \rho(\eta_{2m+1}, \eta_{2m}) \\ \rho(\eta_{2m+1}, \eta_{2m}), \text{ if } \rho(\eta_{2m}, \eta_{2m-1}) < \rho(\eta_{2m+1}, \eta_{2m}). \end{cases}$$

Let us discuss both possible cases of $\mathcal{N}(\xi_{2m}, \xi_{2m-1})$. Also, assume that either $\eta_{2m} \neq \eta_{2m+1}$ for all $m \in N$ or $\eta_{2m} = \eta_{2m+1}$ for no $m \in N$.

**Case 1:-** First suppose that $\mathcal{N}(\xi_{2m}, \xi_{2m-1}) = \rho(\eta_{2m}, \eta_{2m-1})$, then from equation (3.3),

$$s^{\epsilon}\rho(\eta_{2m+1}, \eta_{2m}) \leq \psi(\rho(\eta_{2m}, \eta_{2m-1})).$$

Since $\psi(n) \leq n$ for all $n > 0$.

$$s^{\epsilon}\rho(\eta_{2m+1}, \eta_{2m}) \leq \rho(\eta_{2m}, \eta_{2m-1}).$$

Thus

$$\begin{aligned}
\rho(\eta_{2m+1}, \eta_{2m}) &\leq \frac{1}{s^{\epsilon}}\rho(\eta_{2m}, \eta_{2m-1}) \\
\rho(\eta_{2m+1}, \eta_{2m}) &\leq K\rho(\eta_{2m}, \eta_{2m-1}),
\end{aligned}$$

where, $K = \frac{1}{s^{\epsilon}} \in (0, \frac{1}{s})$. Therefore, by using Lemma 2.1, we get our claim that the sequence $\{\eta_{2m}\}$ is a Cauchy sequence.

**Case 2:-** Secondly, assume that $\mathcal{N}(\xi_{2m}, \xi_{2m-1}) = \rho(\eta_{2m}, \eta_{2m+1})$, and also let $\eta_{2m} = \eta_{2m+1}$ for no $m \in N$. Then again from Eq.(3.3), we have

$$s^{\epsilon}\rho(\eta_{2m+1}, \eta_{2m}) \leq \psi(\rho(\eta_{2m}, \eta_{2m+1}))$$

Since $\psi(n) \leq n$ for $n > 0$, therefore

$$s^{\epsilon}\rho(\eta_{2m+1}, \eta_{2m}) \leq \rho(\eta_{2m}, \eta_{2m+1})$$

implies

$$\rho(\eta_{2m+1}, \eta_{2m}) \leq \frac{1}{s^{\epsilon}}\rho(\eta_{2m}, \eta_{2m+1}) \leq K\rho(\eta_{2m}, \eta_{2m+1}).$$

where, $K = \frac{1}{s^{\epsilon}} \in (0, \frac{1}{s})$. This is a contradiction. Thus our assumption is wrong. Hence $\eta_{2m} = \eta_{2m+1}$ for some $m \in N$. Let us say $m = k$. That is for some $m = k$, we have $\eta_{2k} = \eta_{2k+1}$. Similarly, for $m = k+1$, we have $\eta_{2k+2} = \eta_{2k+3}$. Thus we get a sequence $\{\eta_{2m}\}$ which is a constant sequence with for all $k > m$.

This prove that $\{\eta_{2m}\}$ is a $b$ -Cauchy sequence. The completeness of space $\Delta$ implies that there exists a $\eta \in \Delta$ such that

$$\begin{cases} \mathcal{U}\xi_{2m} \to \eta \\ \mathcal{V}\xi_{2m+1} \to \eta \\ \mathcal{S}\xi_{2m+1} \to \eta \\ \mathcal{T}\xi_{2m} \to \eta \end{cases} \tag{3.4}$$

implies that

$$\lim_{m\to\infty} \rho\left(\mathcal{U}\xi_{2m}, \eta\right) = \lim_{m\to\infty} \rho\left(\mathcal{T}\xi_{2m+1}, \eta\right) = \lim_{m\to\infty} \rho\left(\eta_{2m+1}, \eta\right) = 0$$

$$\lim_{m\to\infty} \rho\left(\mathcal{V}\xi_{2m+1}, \eta\right) = \lim_{m\to\infty} \rho\left(\mathcal{S}\xi_{2m+2}, \eta\right) = \lim_{m\to\infty} \rho\left(\eta_{2m+2}, \eta\right) = 0.$$

**To verify:-** $\eta$ is a coincidence point of maps $\mathcal{U}$ and $\mathcal{T}$.

Since the pair $(\mathcal{U}, \mathcal{T})$ is compatible, gives that

$$\lim_{m\to\infty} \rho\left(\mathcal{T}\mathcal{U}\xi_{2m}, \mathcal{U}\mathcal{T}\xi_{2m}\right) = 0. \tag{3.5}$$

Since the maps $\mathcal{U}$ and $\mathcal{T}$ are continuous mapping, we have

$$\begin{cases} \lim_{m\to\infty} \rho\left(\mathcal{T}\mathcal{U}\xi_{2m}, \mathcal{T}\eta\right) = 0, \\ \lim_{m\to\infty} \rho\left(\mathcal{U}\mathcal{T}\xi_{2m}, \mathcal{U}\eta\right) = 0. \end{cases} \tag{3.6}$$

On using triangle inequality twice, we have

$$\rho(\mathcal{T}\eta, \mathcal{U}\eta) \leq s\rho\left(\mathcal{T}\eta, \mathcal{T}\mathcal{U}\xi_{2m}\right) + s^2\rho\left(\mathcal{T}\mathcal{U}\xi_{2m}, \mathcal{U}\mathcal{T}\xi_{2m}\right) + s^2\rho\left(\mathcal{U}\mathcal{T}\xi_{2m}, \mathcal{U}\eta\right) \tag{3.7}$$

Take $\lim m \to \infty$ and also from Eq. (3.5) and Eq. (3.6) in Eq. (3.7), we have

$$\lim_{m\to\infty} \rho(\mathcal{T}\eta, \mathcal{U}\eta) \leq 0.$$

Possible only if $\mathcal{T}\eta = \mathcal{U}\eta$. Thus $\eta$ is coincidence point of $\mathcal{U}$ and $\mathcal{T}$.

**To verify:-** $\eta$ is a coincidence point of maps $\mathcal{V}$ and $\mathcal{S}$.

Continuity and compatible property of pair $(\mathcal{V}, \mathcal{S})$ give that

$$\lim_{m\to\infty} \rho\left(\mathcal{S}\mathcal{V}\xi_{2m+1}, \mathcal{V}\mathcal{S}\xi_{2m+1}\right) = 0$$

and

$$\lim_{m\to\infty} \rho\left(\mathcal{V}\mathcal{S}\xi_{2m+1}, \mathcal{V}\eta\right) = 0;$$

$$\lim_{m\to\infty} \rho\left(\mathcal{S}\mathcal{V}\xi_{2m+1}, \mathcal{S}\eta\right) = 0.$$

Further, on using triangle inequality, we have

$$\rho(\mathcal{S}\eta, \mathcal{V}\eta) \leq s\rho\left(\mathcal{S}\eta, \mathcal{S}\mathcal{V}\xi_{2m+1}\right) + s^2\rho\left(\mathcal{S}\mathcal{V}\xi_{2m+1}, \mathcal{V}\mathcal{S}\xi_{2m+1}\right) + s^2\rho\left(\mathcal{V}\mathcal{S}\xi_{2m+1}, \mathcal{V}\eta\right)$$

On taking $\lim m \to \infty$ in above inequality, we have

$$\rho(\mathcal{S}\eta, \mathcal{V}\eta) = 0.$$

Thus, $\mathcal{S}\eta = \mathcal{V}\eta$. This proves that $\eta$ is a coincidence point of $\mathcal{V}$ and $\mathcal{S}$.

**Claim:-** $\mathcal{U}\eta = \mathcal{V}\eta = \mathcal{T}\eta = \mathcal{S}\eta$.

Since the elements $\mathcal{T}\eta$ and $\mathcal{S}\eta$ are comparable, and hence from Eq. (3.1), we obtain

$$s^{\epsilon}\rho(\mathcal{U}\eta, \mathcal{V}\eta) \leq \psi(\mathcal{N}(\eta, \eta)), \tag{3.8}$$

where

$$\mathcal{N}(\eta, \eta) = \max\left\{\rho(\mathcal{U}\eta, \mathcal{V}\eta), \frac{1}{2s}\left[\rho(\mathcal{U}\eta, \mathcal{T}\eta) + \rho(\mathcal{V}\eta, \mathcal{S}\eta)\right], \frac{1}{2s}\left[\rho(\mathcal{U}\eta, \mathcal{S}\eta) + \rho(\mathcal{T}\eta, \mathcal{V}\eta)\right]\right\}.$$

Since $\mathcal{S}\eta = \mathcal{V}\eta$ and $\mathcal{T}\eta = \mathcal{U}\eta$, implies that

$$\begin{aligned}\mathcal{N}(\eta, \eta) &= \max\left\{\rho(\mathcal{T}\eta, \mathcal{S}\eta), \frac{1}{s}\rho(\mathcal{T}\eta, \mathcal{S}\eta)\right\} \\ &= \rho(\mathcal{T}\eta, \mathcal{S}\eta).\end{aligned}$$

Thus from Eq. (3.8), we get

$$\begin{aligned}s^{\epsilon}\rho(\mathcal{U}\eta, \mathcal{V}\eta) &\leq \psi(\mathcal{N}(\eta, \eta)) \\ &\leq \psi(\rho(\mathcal{S}\eta, \mathcal{T}\eta)) \\ &\leq \rho(\mathcal{S}\eta, \mathcal{T}\eta) = \rho(\mathcal{U}\eta, \mathcal{V}\eta) \quad [\text{Using } \mathcal{S}\eta = \mathcal{V}\eta \text{ and } \mathcal{T}\eta = \mathcal{U}\eta.\end{aligned}$$

Gives that $\mathcal{U}\eta = \mathcal{V}\eta$. Thus $\mathcal{U}\eta = \mathcal{V}\eta = \mathcal{T}\eta = \mathcal{S}\eta$. Hence proved the result. □

Now we extend the previous theorem (Theorem 3.1) to guarantee the uniqueness of common fixed points for four self maps satisfying the property that:

(1) one of the subspace of $\Delta$ is to be closed and
(2) one of the pair of self maps satisfies $b$ - (E.A.) property and weakly compatible property.

**Theorem 3.2.** *Let $(\Delta, \rho)$ is a $b$ - metric space with $s \geq 1$, and let $\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{T} : \Delta \to \Delta$ be self mappings on $\Delta$ with $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$ and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$ such that*

$$s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) \leq \psi(\mathcal{A}(\xi, \eta)) \text{ for all } \xi, \eta \in \Delta, \tag{3.9}$$

*where*

$$\mathcal{A}(\xi, \eta) = \max\left\{\begin{array}{c}\rho(\mathcal{S}\xi, \mathcal{T}\eta), \frac{1}{2\,s}[\,\rho(\mathcal{U}\xi, \mathcal{S}\xi) + \rho(\mathcal{V}\eta, \mathcal{T}\eta)], \\ \frac{1}{2\,s}[\,\rho(\mathcal{U}\xi, \mathcal{T}\eta) + \rho(\mathcal{S}\xi, \mathcal{V}\eta)]\end{array}\right\}, \tag{3.10}$$

*$\epsilon > 1$ is a constant and function $\psi : [0, \infty) \to [0, \infty)$ is such that $\psi(n) \leq n$ for all $n > 0$ with $\psi(0) = 0$. Further, suppose that*

*(1) one of the sub spaces $\mathcal{U}(\Delta), \mathcal{V}(\Delta), \mathcal{S}(\Delta)$ and $\mathcal{T}(\Delta)$ is $b$ - closed in $\Delta$,*
*(2) one of the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{T})$ satisfy the $b$ - (E.A.) property.*

*Then the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{T})$ have a point of coincidence in $\Delta$.*

*Moreover, if the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{T})$ are weakly compatible, then $\mathcal{U}, \mathcal{V}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.*

*Proof.* Since the pair $(\mathcal{U}, \mathcal{S})$ satisfy the $b$ - (E.A.) property then for some $q$ in $\Delta$, there exist a sequence $\{\xi_n\}$ in $\Delta$ satisfying

$$\lim_{n\to\infty} \mathcal{U}\xi_n = \lim_{n\to\infty} \mathcal{S}\xi_n = q. \tag{3.11}$$

As $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$, there exist a sequence $\{\eta_n\}$ in $\Delta$ such that $\mathcal{U}\xi_n = \mathcal{T}\eta_n$. Hence $\lim_{n\to\infty} \mathcal{T}\eta_n = q$. To prove that $\lim_{n\to\infty} \mathcal{V}\eta_n = q$.
From Eq. (3.9), we have

$$s^{\epsilon}\rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right) \le \psi\left(\mathcal{A}\left(\xi_n, \eta_n\right)\right), \tag{3.12}$$

where

$$\begin{aligned}
\mathcal{A}\left(\xi_n, \eta_n\right) &= \max\left\{\begin{array}{c} \rho\left(\mathcal{S}\xi_n, \mathcal{T}\eta_n\right), \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho\left(\mathcal{V}\eta_n, \mathcal{T}\eta_n\right)\right], \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{T}\eta_n\right) + \rho\left(\mathcal{S}\xi_n, \mathcal{V}\eta_n\right)\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right), \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho\left(\mathcal{V}\eta_n, \mathcal{U}\xi_n\right)\right], \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{U}\xi_n\right) + \rho\left(\mathcal{S}\xi_n, \mathcal{V}\eta_n\right)\right] \end{array}\right\} \text{[Using the fact that} \mathcal{T}\eta_n = \mathcal{U}\xi_n \\
&= \max\left\{\begin{array}{c} \rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right), \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho\left(\mathcal{V}\eta_n, \mathcal{U}\xi_n\right)\right], \\ \frac{1}{2}\left[\rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right) + \rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right)\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right), \\ \frac{1}{2}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho\left(\mathcal{V}\eta_n, \mathcal{U}\xi_n\right)\right] \end{array}\right\}.
\end{aligned} \tag{3.13}$$

Taking limit superior as $n \to \infty$ in Eq. (3.12) and in Eq.(3.13), we have

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, s^{\epsilon}\rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right) &\le \overline{\lim_{n\to\infty}}\, \psi\left[\mathcal{A}\left(\xi_n, \eta_n\right)\right] \\
&\le \overline{\lim_{n\to\infty}}\, \psi\left[\max\left\{\rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right), \frac{1}{2}\left[\rho\left(\mathcal{S}\xi_n, \mathcal{U}\xi_n\right) + \rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right)\right]\right\}\right] \\
&\le \overline{\lim_{n\to\infty}}\, \psi\left(\rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right)\right) \quad \text{[on using Eq.(3.11)} \\
&< \overline{\lim_{n\to\infty}}\, \rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right).
\end{aligned}$$

Which is a contradiction, and hence

$$\lim_{n\to\infty} \rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right) = 0. \tag{3.14}$$

On using triangle inequality, one can write

$$\rho\left(q, \mathcal{V}\eta_n\right) \le s\left[\rho\left(q, \mathcal{U}\xi_n\right) + \rho\left(\mathcal{U}\xi_n, \mathcal{V}\eta_n\right)\right].$$

Make use of Eq. (3.14) in above inequality (on taking limit as $n \to \infty$), we have

$$\rho\left(q, \mathcal{V}\eta_n\right) = 0.$$

This implies that $\mathcal{V}\eta_n \to q$ as $n \to \infty$.
Since $\mathcal{T}(\Delta)$ is closed subspace of $\Delta$, then there exist $r \in \Delta$ such that $\mathcal{T}r = q$.
Next, we prove that $\mathcal{V}r = q$.
Further on using triangle inequality, one can have

$$\frac{1}{s}\rho(q, \mathcal{V}r) \leq \rho\left(q, \mathcal{U}\xi_n\right) + \rho\left(\mathcal{U}\xi_n, \mathcal{V}r\right). \tag{3.15}$$

Lets recall Eq. (3.9) once again, we have

$$\begin{aligned} s^{\epsilon}\rho\left(\mathcal{U}\xi_n, \mathcal{V}r\right) &\leq \psi\left(\mathcal{A}\left(\xi_n, r\right)\right) \\ \rho\left(\mathcal{U}\xi_n, \mathcal{V}r\right) &\leq \frac{1}{s^{\epsilon}}\mathcal{A}\left(\xi_n, r\right). \end{aligned} \tag{3.16}$$

Hence from Eq. (3.15), we have

$$\frac{1}{s}\rho(q, \mathcal{V}r) \leq \left[\rho\left(q, \mathcal{U}\xi_n\right) + \frac{1}{s^{\epsilon}}\mathcal{A}\left(\xi_n, r\right)\right], \tag{3.17}$$

where

$$\begin{aligned} \mathcal{A}\left(\xi_n, r\right) &= \max\left\{ \begin{array}{c} \rho\left(\mathcal{S}\xi_n, \mathcal{T}r\right), \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho(\mathcal{T}r, \mathcal{V}r)\right], \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{T}r\right) + \rho\left(\mathcal{S}\xi_n, \mathcal{V}r\right)\right] \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \rho\left(\mathcal{S}\xi_n, q\right), \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, \mathcal{S}\xi_n\right) + \rho(\mathcal{V}r, q)\right], \\ \frac{1}{2s}\left[\rho\left(\mathcal{U}\xi_n, q\right) + s\rho\left(\mathcal{S}\xi_n, q\right) + s\rho(q, \mathcal{V}r)\right] \end{array} \right\}. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$ we have,

$$\begin{aligned} \lim_{n\to\infty} \mathcal{A}\left(\xi_n, r\right) &\leq \max\left\{0, \frac{1}{2s}\left[0 + \rho(\mathcal{V}r, q)\right], \frac{1}{2s}\left[0 + 0 + s\rho(q, \mathcal{V}r)\right]\right\} \\ &= \frac{1}{2}\rho(\mathcal{V}r, q) < \rho(\mathcal{V}r, q). \end{aligned}$$

Thus from Eq. (3.17), we have

$$\frac{1}{s}\rho(q, \mathcal{V}r) \leq \frac{1}{s^{\epsilon}}\rho(\mathcal{V}r, q).$$

This is possible only if

$$\rho(q, \mathcal{V}r) = 0 \text{ that is } \mathcal{V}r = q.$$

Thus we get

$$\mathcal{T}r = \mathcal{V}r = q.$$

This proves $r$ is the coincidence point of the pair $(\mathcal{V}, \mathcal{T})$.
As $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$, there exists a point $\zeta$ in $\Delta$ such that $q = \mathcal{S}\zeta$.
Next we claim that $\mathcal{S}\zeta = \mathcal{U}\zeta$.
Once again from Eq. (3.9), we have

$$s^{\epsilon}\rho(\mathcal{U}\zeta, \mathcal{V}r) \leq \psi(\mathcal{A}(\zeta, r)), \tag{3.19}$$

where

$$\mathcal{A}(\zeta, r) = \max\left\{\rho(\mathcal{S}\zeta, \mathcal{T}r), \frac{1}{2s}\left[\rho(\mathcal{U}\zeta, \mathcal{S}\zeta) + \rho(\mathcal{T}r, \mathcal{V}r)\right], \frac{1}{2s}\left[\rho(\mathcal{U}\zeta, \mathcal{T}r) + \rho(\mathcal{S}\zeta, \mathcal{V}r)\right]\right\}.$$

Since $\mathcal{T}r = \mathcal{V}r = \mathcal{S}\zeta = q$, therefore

$$\mathcal{A}(\zeta, r) = \max\left\{0, \frac{1}{2s}[\rho(\mathcal{U}\zeta, q) + 0], \frac{1}{2s}[\rho(\mathcal{U}\zeta, q) + 0]\right\} = \frac{1}{2s}\rho(\mathcal{U}\zeta, q).$$

Hence from Eq. (3.19), we have

$$\begin{aligned} s^{\epsilon}\rho(\mathcal{U}\zeta, \mathcal{V}r) &\leq \psi\left(\frac{1}{2s}\rho(\mathcal{U}\zeta, q)\right) \\ &\leq \frac{1}{2s}\rho(\mathcal{U}\zeta, q) \end{aligned}$$

In what it follows that $\rho(\mathcal{U}\zeta, \mathcal{V}r) = 0$. Thus

$$\mathcal{S}\zeta = \mathcal{U}\zeta = q.$$

Henceforth $\zeta$ is the coincidence point of pair $(\mathcal{U}, \mathcal{S})$

$$\mathcal{S}\zeta = \mathcal{U}\zeta = \mathcal{T}r = \mathcal{V}r = q.$$

By the weak compatibility of the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{T})$, we obtain that

$$\mathcal{U}q = \mathcal{S}q \text{ and } \mathcal{V}q = \mathcal{T}q. \tag{3.20}$$

Next, we prove that $q$ is a common fixed point of $\mathcal{U}, \mathcal{V}, \mathcal{S}$ and $\mathcal{T}$.
On using Eq. (3.9), we can have

$$s^{\epsilon}\rho(\mathcal{U}q, q) = s^{\epsilon}\rho(\mathcal{U}q, \mathcal{V}r) \leq \psi(\mathcal{A}(q, r)) \tag{3.21}$$

where

$$\mathcal{A}(q, r) = \max\left\{\begin{array}{c} \rho(\mathcal{S}q, \mathcal{T}r), \frac{1}{2s}[\rho(\mathcal{U}q, \mathcal{S}q) + \rho(\mathcal{T}r, \mathcal{V}r)], \\ \frac{1}{2s}[\rho(\mathcal{U}q, \mathcal{T}r) + \rho(\mathcal{S}q, \mathcal{V}r)] \end{array}\right\}.$$

On using Eq. (3.20) and the fact that $\mathcal{T}r = \mathcal{V}r = q$, we obtain

$$\begin{aligned} \mathcal{A}(q, r) &= \max\left\{\rho(\mathcal{U}q, q), \frac{1}{2s}\left[\rho\left(\mathcal{U}q, \mathcal{U}q\right) + \rho(q, q)\right], \frac{1}{2s}[\rho(\mathcal{U}q, q) + \rho(\mathcal{U}q, q)]\right\} \\ &= \rho(\mathcal{U}q, q). \end{aligned}$$

On using it in Eq. (3.21), we have

$$s^{\epsilon}\rho(\mathcal{U}q, q) \leq \psi[\rho(\mathcal{U}q, q)] \leq \rho(\mathcal{U}q, q).$$

It is possible only if $\rho(\mathcal{U}q, q) = 0$, that is $\mathcal{U}q = q$. Consequently from Eq. (3.20), we get $\mathcal{U}q = \mathcal{S}q = q$.

Next we claim that $\mathcal{V}q = \mathcal{T}q = q$. It is sufficient to show that $\mathcal{V}q = q$. Consider

$$s^{\epsilon}\rho(q, \mathcal{V}q) = s^{\epsilon}\rho(\mathcal{U}\zeta, \mathcal{V}q) \leq \psi(\mathcal{A}(\zeta, q)), \tag{3.22}$$

where

$$\mathcal{A}(\zeta, q) = \max\left\{\begin{array}{c} \rho(\mathcal{S}\zeta, \mathcal{T}q), \frac{1}{2s}[\rho(\mathcal{U}\zeta, \mathcal{S}\zeta) + \rho(\mathcal{T}q, \mathcal{V}q)], \\ \frac{1}{2s}[\rho(\mathcal{U}\zeta, \mathcal{T}q) + \rho(\mathcal{S}\zeta, \mathcal{V}q)] \end{array}\right\}.$$

Gain on using Eq. (3.20) and the fact that $S\zeta = \mathcal{U}\zeta = \mathcal{T}r = \mathcal{V}r = q$, we get

$$\mathcal{A}(\zeta, q) = \max\left\{\rho(q, \mathcal{V}q), \frac{1}{s}\rho(q, \mathcal{V}q)\right\} = \rho(q, \mathcal{V}q).$$

Hence from Eq. (3.22), we have

$$s^{\epsilon}\rho(q, \mathcal{V}q) \leq \psi(\rho(q, \mathcal{V}q)) < \rho(q, \mathcal{V}q).$$

This is possible only if $\rho(q, \mathcal{V}q) = 0$ i.e. $\mathcal{V}q = q$. Thus we have $\mathcal{V}q = \mathcal{T}q = q$.
This proves that $q$ is the coincidence point of map $\mathcal{U}, \mathcal{V}, \mathcal{S}$ and $\mathcal{T}$.

Suppose that $p$ is another fixed point of $\mathcal{U}, \mathcal{V}, \mathcal{S}$ and $\mathcal{T}$, then from Eq. (3.9), we have

$$s^{\epsilon}\rho(q, p) = s^{\epsilon}\rho(\mathcal{U}q, \mathcal{V}p) \leq \psi(\mathcal{A}(q, p), \tag{3.23}$$

where

$$\begin{aligned}\mathcal{A}(q, p) &= \max\left\{\begin{array}{c}\rho(\mathcal{S}q, \mathcal{T}p), \frac{1}{2s}[\rho(\mathcal{U}q, \mathcal{S}q) + \rho(\mathcal{T}p, \mathcal{V}p)], \\ \frac{1}{2s}[\rho(\mathcal{U}q, \mathcal{T}p) + \rho(\mathcal{S}q, \mathcal{V}p)]\end{array}\right\} \\ &= \max\left\{\rho(q, p), \frac{1}{2s}[\rho(q, q) + \rho(p, p)], \frac{1}{2s}[\rho(q, p) + \rho(q, p)]\right\} \\ &= \rho(q, p).\end{aligned}$$

On using property of $\psi$ and above value of $\mathcal{A}(q, p)$ in Eq. (3.23), we obtain

$$s^{\epsilon}\rho(q, p) \leq \psi(\rho(q, p)) \leq \rho(q, p).$$

From which it follows that $\rho(q, p) = 0$. So $q = p$.
This proves our result. □

## 4. Corollaries and Examples

In this section, we review the noteworthy outcomes of our main findings and give only a couple of examples with graphic representations that demonstrate the validity of our findings.

By substituting $\mathcal{S}$ for $\mathcal{T}$ in Theorem 3.1, we get the following outcome.

**Corollary 4.1.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{S} : \Delta \to \Delta$ be continuous mappings on a partially ordered complete b -metric space $(\Delta, \preceq, \rho)$ and with $\mathcal{U}(\Delta) \subseteq \mathcal{S}(\Delta)$, and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$. Assume that the compatible pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{S})$, and the comparable elements $\mathcal{S}\eta$ and $\mathcal{S}\xi$ satisfies the condition:*

$$s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) \leq \psi(\mathcal{N}(\xi, \eta)), \text{ for all } \xi, \eta \in \Delta,$$

*where*

$$\mathcal{N}(\xi, \eta) = \max\left\{\begin{array}{c}\rho(\mathcal{S}\xi, \mathcal{S}\eta), \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{S}\xi) + \rho(\mathcal{V}\eta, \mathcal{S}\eta)], \\ \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{S}\eta) + \rho(\mathcal{S}\xi, \mathcal{V}\eta)]\end{array}\right\},$$

*$s \geq 1$, $\epsilon > 1$ is constant and function $\psi : [0, \infty) \to [0, \infty)$ is such that $\psi(n) \leq n$ for all $n > 0$ with $\psi(0) = 0$.*

*Further, if the pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{U})$ are partially weakly increasing with respect to $\mathcal{S}$, then $(\mathcal{U}, \mathcal{S})$ and*

*$(\mathcal{V}, \mathcal{S})$ have a coincidence point.*
*Moreover, if $\eta$ is a coincidence point for comparable element $\mathcal{S}\eta$, then $\mathcal{U}\eta = \mathcal{V}\eta = \mathcal{S}\eta$.*

By substituting $\mathcal{U}$ with $\mathcal{V}$ in Corollary 4.1, we get a significant result that extends and generalizes the findings of Jungck [4].

**Corollary 4.2.** *Let $\mathcal{U}, \mathcal{S} : \Delta \to \Delta$ be continuous mappings on a partially ordered complete b - metric space $(\Delta, \preceq, \rho)$ and with $\mathcal{U}(\Delta) \subseteq \mathcal{S}(\Delta)$. Assume that the pair $(\mathcal{U}, \mathcal{S})$ is compatible, and the comparable elements $\mathcal{S}\eta$ and $\mathcal{S}\xi$ satisfies the condition:*

$$s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{U}\eta) \leq \psi(\mathcal{N}(\xi, \eta)), \text{ for all } \xi, \eta \in \Delta,$$

*where*

$$\mathcal{N}(\xi, \eta) = \max\left\{ \begin{array}{c} \rho(\mathcal{S}\xi, \mathcal{S}\eta), \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{S}\xi) + \rho(\mathcal{U}\eta, \mathcal{S}\eta)], \\ \frac{1}{2s}[\rho(\mathcal{U}\xi, \mathcal{S}\eta) + \rho(\mathcal{S}\xi, \mathcal{U}\eta)] \end{array} \right\},$$

*$s \geq 1$, $\epsilon > 1$ is constant and function $\psi : [0, \infty) \to [0, \infty)$ is such that $\psi(n) \leq n$ for all $n > 0$ with $\psi(0) = 0$.*
*Further, if the map $\mathcal{U}$ is partially weakly increasing with respect to $\mathcal{S}$, then $(\mathcal{U}, \mathcal{S})$ has a coincidence point.*
*Moreover, if $\eta$ is a coincidence point for comparable element $\mathcal{S}\eta$, then $\mathcal{U}\eta = \mathcal{S}\eta$.*

In letting $\mathcal{T} = \mathcal{S}$ in Theorem 3.2, we obtain the following result.

**Corollary 4.3.** *Let $(\Delta, \rho)$ is a b - metric space with $s \geq 1$, and let $\mathcal{U}, \mathcal{V}, \mathcal{S} : \Delta \to \Delta$ be self mappings on $\Delta$ with $\mathcal{U}(\Delta) \subseteq \mathcal{S}(\Delta)$ and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$ such that*

$$s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) \leq \psi(\mathcal{A}(\xi, \eta)) \text{ for all } \xi, \eta \in \Delta,$$

*where*

$$\mathcal{A}(\xi, \eta) = \max\left\{ \begin{array}{c} \rho(\mathcal{S}\xi, \mathcal{S}\eta), \frac{1}{2\,s}[\,\rho(\mathcal{U}\xi, \mathcal{S}\xi) + \rho(\mathcal{V}\eta, \mathcal{S}\eta)], \\ \frac{1}{2\,s}[\,\rho(\mathcal{U}\xi, \mathcal{S}\eta) + \rho(\mathcal{S}\xi, \mathcal{V}\eta)] \end{array} \right\},$$

*$\epsilon > 1$ is a constant and function $\psi : [0, \infty) \to [0, \infty)$ is such that $\psi(n) \leq n$ for all $n > 0$ with $\psi(0) = 0$.*
*Further, suppose that*

*(i). one of the sub spaces $\mathcal{U}(\Delta), \mathcal{V}(\Delta)$, and $\mathcal{S}(\Delta)$ is b - closed in $\Delta$,*
*(ii). one of the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{S})$ satisfy the b - (E.A.) property.*

*Then the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{S})$ have a point of coincidence in $\Delta$.*
*Moreover, if the pairs $(\mathcal{U}, \mathcal{S})$ and $(\mathcal{V}, \mathcal{S})$ are weakly compatible, then $\mathcal{U}, \mathcal{V}$ and $\mathcal{S}$ have a unique common fixed point.*

Following two examples are based on our findings.

**Example 4.1.** *Let $\Delta = [0, 1)$. Define a metric $\rho : \Delta \times \Delta \to \mathbb{R}^{+}$ by*

$$\rho(\xi, \eta) = \begin{cases} 0, & \text{if } \xi = \eta \\ (\xi + \eta)^2, & \text{if } \xi \neq \eta \end{cases}$$

*Then clearly the pair $(\rho, \Delta)$ is a complete b - metric space.*
*Define four continuous self maps $\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{T} : \Delta \to \Delta$ by*

$$\mathcal{U}(\xi) = \frac{\xi}{2}, \quad \mathcal{T}(\xi) = \xi, \quad \mathcal{V}(\xi) = \frac{\xi}{3}, \quad \mathcal{S}(\xi) = \frac{2\xi}{3}$$

*on a partially ordered complete b - metric space $(\Delta, \preceq, \rho)$.*
*Further define a sequence $\{\xi_n\} = \frac{1}{n^2}$ and also assume that $s = 2, \epsilon = 1.1$ is constant.*
*Let us define $\psi$ as $\psi(\xi) = \frac{2\xi}{3}$ and $\psi(0) = 0$.*

*We will now proceed to show that the above defined metric and mappings under given assumptions satisfies all the conditions of Theorem 3.1.*

***Verification:***

(1) *By definitions of maps, its clear that $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$ and $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$.*

(2) *Also,*

$$\lim_{n\to\infty} \rho\left(\mathcal{U}\mathcal{T}\xi_n, \mathcal{T}\mathcal{U}\xi_n\right) = \lim_{n\to\infty}\left(\rho\left(\frac{\xi_n}{2}, \frac{\xi_n}{2}\right)\right)^2 = 0$$

$$\lim_{n\to\infty} \rho\left(\mathcal{V}\mathcal{S}\xi_n, \mathcal{S}\mathcal{V}\xi_n\right) = \lim_{n\to\infty}\left(\rho\left(\frac{2\xi_n}{9}, \frac{2\xi_n}{9}\right)\right)^2 = 0$$

*Hence the pairs $(\mathcal{U}, \mathcal{T})$ and $(\mathcal{V}, \mathcal{S})$ are the compatible pairs and the elements $\mathcal{S}\eta$ and $\mathcal{T}\xi$ are comparable.*

(3) *Lets verify the inequality (3.1).*
*Consider*

$$L.H.S. = s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) = (2)^{1.1}\left[\frac{\xi}{2} + \frac{\eta}{3}\right]^2 = 0.0595 \times (3\xi + 2\eta)^2.$$

*and*

$$R.H.S. = \psi(\mathcal{A}(\xi, \eta)) = 0.1111(3\xi + 2\eta)^2$$

*We consider three possible cases as follows:*

*Case I: If $\xi = 0$ and $\eta \in [0, 1)$, then*

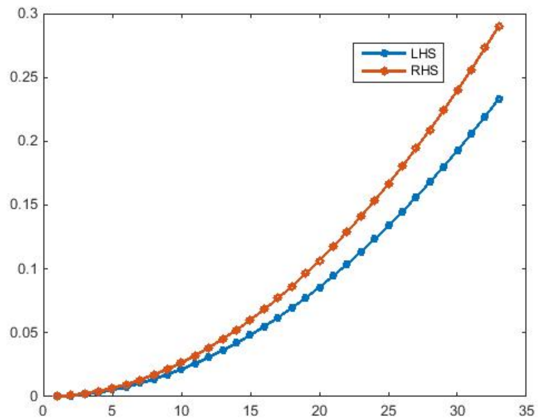
$$\begin{aligned} LHS = 0.238\eta^2 &\leq 0.2962\eta^2 \\ &= \psi\left(0.4444\eta^2\right) = RHS. \end{aligned}$$

*Case II: If $\eta = 0$ and $\xi \in [0, 1)$, then we have*

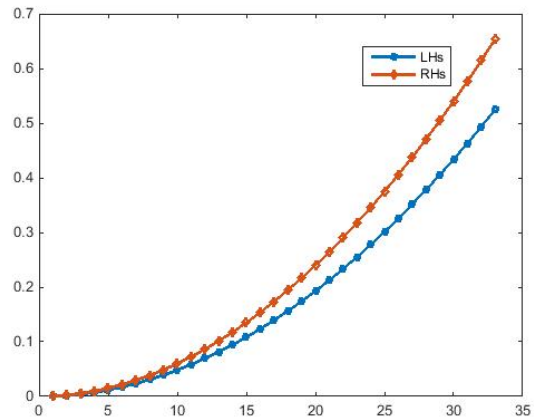
$$\begin{aligned} LHS = 0.0595 \times 9\xi^2 &= 0.5355\xi^2 \\ &\leq 0.6666\xi^2 \\ &= \psi\left(0.9999\xi^2\right) \quad = RHS. \end{aligned}$$

*Case III: If $\xi = \eta$, then*

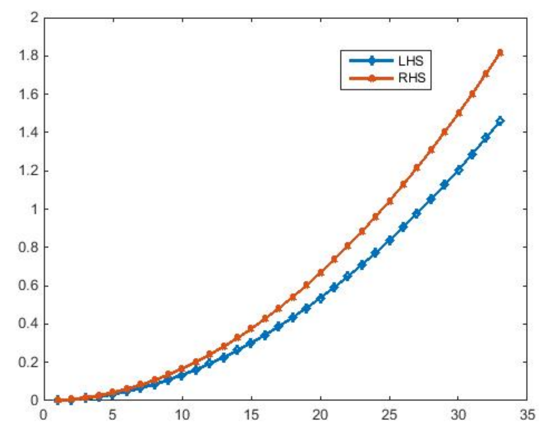
$$\begin{aligned} LHS &= 1.4875\eta^2 \\ &\leq 1.8516\eta^2 \\ &= \psi(2.7775\eta^2) = RHS. \end{aligned}$$




(A) for the case when $\xi = 0$ and $\eta \in [0, 1)$




(B) for the case when $\eta = 0$ and $\xi \in [0, 1)$




(C) for the case when $\xi = \eta$

FIGURE 1. Graphical behavior of inequality (3.1) of the Example 4.1

*From above all three cases, we conclude that L.H.S. $\leq$ R.H.S. for all $\xi, \eta \in [0, 1)$.*
*Thus all the condition of Theorem 3.1 are satisfied.*
*The inequality (3.9) behaviour is shown graphically in Figure 1 respectively. Moreover, "0" is the unique common fixed point of the maps.*

**Example 4.2.** *Let $\Delta = [0, 1)$. Define a metric $\rho : \Delta \times \Delta \to \mathbb{R}^+$ by*

$$\rho(\xi, \eta) = \begin{cases} 0, \ \text{if } \xi = \eta \\ (\xi + \eta)^2, \ \text{if } \xi \neq \eta \end{cases}$$

*Then clearly the pair $(\rho, \Delta)$ is a complete b- metric space.*
*Define four continuous self maps $\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{T} : \Delta \to \Delta$ by*

$$\mathcal{U}(\xi) = \frac{\xi}{2}, \quad \mathcal{T}(\xi) = \xi, \quad \mathcal{V}(\xi) = \frac{\xi^2}{2}, \quad \mathcal{S}(\xi) = \xi^2$$

*on a partially ordered complete b - metric space* $(\Delta, \preceq, \rho)$
*Further define a sequence* $\{\xi_n\} = \frac{1}{n^3}$ *and also assume that* $s = 2, \epsilon = 1.2$ *is constant.*
*Let us define* $\psi$ *as* $\psi(\xi) = \frac{2\xi}{3}$ *and* $\psi(0) = 0$.

*We will now proceed to show that the above defined metric and mappings under given assumptions satisfies all the conditions of Theorem 3.2.*

***Verification:***

(1) *By definitions of maps, its clear that* $\mathcal{U}(\Delta) \subseteq \mathcal{T}(\Delta)$ *and* $\mathcal{V}(\Delta) \subseteq \mathcal{S}(\Delta)$.

(2) *Also, we have*

$$\lim_{n\to\infty} \mathcal{U}\xi_n = \lim_{n\to\infty} \frac{\xi_n}{2} = \lim_{n\to\infty} \frac{1}{2n^3} = 0$$

$$\lim_{n\to\infty} \mathcal{S}\xi_n = \lim_{n\to\infty} (\xi_n)^2 = \lim_{n\to\infty} \left(\frac{1}{n^3}\right)^2 = 0$$

$$\lim_{n\to\infty} \mathcal{V}\xi_n = \lim_{n\to\infty} \frac{(\xi_n)^2}{2} = \frac{1}{2} \lim_{n\to\infty} \left(\frac{1}{n^3}\right)^2 = 0$$

$$\lim_{n\to\infty} \mathcal{T}\xi_n = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$$

*Thus the pair* $(\mathcal{U}, \mathcal{S})$ *and* $(\mathcal{V}, \mathcal{T})$ *satisfies b - (E.A.) Property.*

(3) *Clearly, the subspace* $\mathcal{T}(\Delta)$ *is b - closed in* $\Delta$. *Moreover,* $0$ *is the coincidence point of the pairs* $(\mathcal{U}, \mathcal{S})$ *and* $(\mathcal{V}, \mathcal{T})$.

(4) *Further, we have*

$$\mathcal{U}\mathcal{S}\xi = \mathcal{U}[\mathcal{S}(\xi)] = \mathcal{U}\left[\xi^2\right] = \frac{\xi^2}{2}; \quad \mathcal{S}\mathcal{U}\xi = \mathcal{S}[\mathcal{U}(\xi)] = \mathcal{S}\left[\frac{\xi}{2}\right] = \left[\frac{\xi}{2}\right]^2$$

*and*

$$\mathcal{V}\mathcal{T}\xi = \mathcal{V}[\mathcal{T}(\xi)] = \mathcal{V}[\xi] = \frac{\xi^2}{2} \quad \mathcal{T}\mathcal{V}\xi = \mathcal{T}[\mathcal{V}(\xi)] = \mathcal{T}\left[\frac{\xi^2}{2}\right] = \frac{\xi^2}{2}$$

*Hence the pair* $(\mathcal{U}, \mathcal{S})$ *and* $(\mathcal{V}, \mathcal{T})$ *are weakly compatible.*

(5) *We will show that our defined example verify the inequality (3.9).*
*Consider*

$$LHS = s^{\epsilon}\rho(\mathcal{U}\xi, \mathcal{V}\eta) = (2)^{1.2}\left[\frac{\xi}{2} + \frac{\eta^2}{2}\right]^2 = 0.5743\left[\xi + \eta^2\right]^2.$$

*On simplifying, we get*

$$RHS = \psi(\mathcal{A}(\xi, \eta)) = \psi(\left(\xi^2 + \eta\right)^2).$$

*Here further three cases arise :*

*Case I: If* $\xi = 0$ *and* $\eta \in [0, 1)$, *then*

$$\begin{aligned} LHS &= 0.5743 \times \eta^4 \\ &\leq 0.6666\eta^2 = \psi\left(\eta^2\right) = RHS \end{aligned}$$

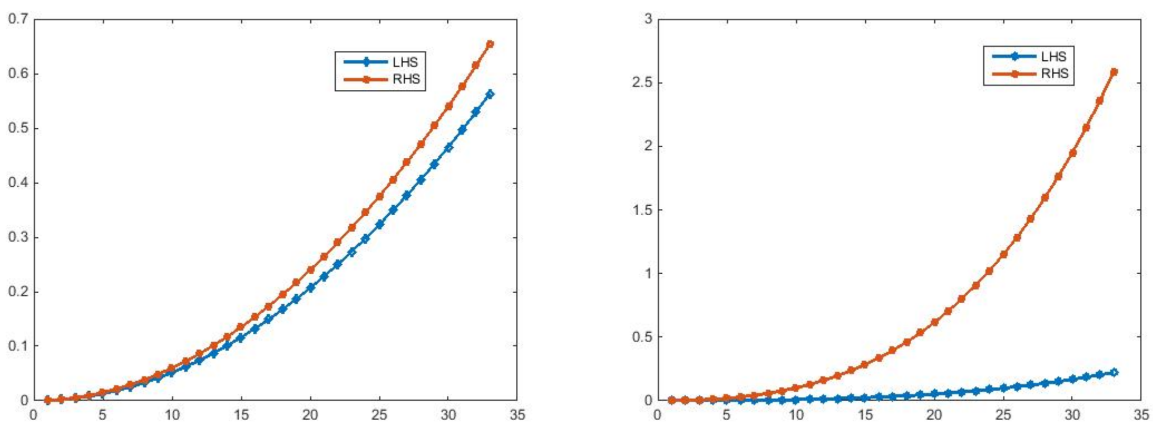
Table 1. Behavior of inequality (3.9) of the Example 4.2

| $\eta$ | *L.H.S.* | *R.H.S.* |
|---|---|---|
| 0.1 | 0.000 | 0.0067 |
| 0.2 | 0.001 | 0.0268 |
| 0.3 | 0.005 | 0.0603 |
| 0.4 | 0.015 | 0.1072 |
| 0.5 | 0.036 | 0.1675 |
| 0.6 | 0.074 | 0.2412 |
| 0.7 | 0.138 | 0.3283 |
| 0.8 | 0.235 | 0.4288 |
| 0.9 | 0.377 | 0.5427 |
| 1 | 0.574 | 0.67 |

| $\xi = \eta$ | LHS | RHS |
|---|---|---|
| 0.1 | 0.0069 | 0.0081 |
| 0.2 | 0.0331 | 0.0384 |
| 0.3 | 0.0874 | 0.1014 |
| 0.4 | 0.1801 | 0.2090 |
| 0.5 | 0.3230 | 0.3750 |
| 0.6 | 0.5293 | 0.6143 |
| 0.7 | 0.8133 | 0.9440 |
| 0.8 | 1.1909 | 1.3823 |
| 0.9 | 1.6793 | 1.9492 |
| 1 | 2.2972 | 2.6664 |

*(a) If $\xi = \eta$ and $\xi, \eta > 0$, then*

$$\begin{aligned} LHS &= 0.05743(\eta + \eta^2)^2 \\ &\leq 0.6666(\eta + \eta^2)^2 = \psi(\mathcal{A}(\xi, \eta)) = RHS . \end{aligned}$$




(a) for the case when $\xi = 0$ and $\eta \in [0, 1)$

(b) for the case when $\xi = \eta$ for all $\xi, \eta > 0$.

Figure 2. Graphical behavior of inequality (3.9) of the Example 4.2

*Therefore, all the conditions of Theorem 3.2 have been fulfilled. The inequality (3.9) is shown in a tabular and graphical format in Table 1, and in Figure 2 respectively. Furthermore, the number "0" is the only fixed point that is common to all of the maps.*

## 5. Conclusion

Within this study, we have obtained two fixed point results for a given pair of self maps satisfying compatible and $b$ - (E.A.) property in frame work of $b$- metric spaces and partially ordered $b$-metric spaces. Furthermore, to substantiate the validity of our findings, we have presented a few corollaries and two examples that demonstrates the effectiveness of the obtained results. Our results extended some of the existing results of the literature such as the results of Jungck [4] and Ozturk and Radenovic [21].

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