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The Convex Sets in Banach Spaces and Polynomial Approximation

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ABSTRACT. A Banach space A, an open subset V of A, and an open subset U of A' are considered. Our definition introduces novel categories of topological algebras of holomorphic functions on A. We demonstrate the equality of the two sets of holomorphic functions $(\mathcal{H}_{wv}(V))$ and $(\mathcal{H}_{w^*vk}(U))$ under specific assumptions. We demonstrated that norm-dense $\mathcal{P}_{g_i}(A)$ is found in $\mathcal{P}_w(A)$ and norm-dense $\mathcal{P}_{g_i^*}(A')$ is found in $\mathcal{P}_{w^*}(A')$. Additionally, we demonstrated that $\mathcal{P}_{g_i}(A)$ is τ_k-dense in $\mathcal{H}_{wvk}(V)$ and $\mathcal{P}_{g_i^*}(A')$ is τ_{k^*}-dense in $\mathcal{H}_{w^*vk}(U)$ for a Banach space with a decreasing Schauder basis A, a polynomially convex weakly open subset V of A, and a polynomially convex weak-star open subset U of A.

1. Introduction

Consider A to be a Banach space, V and U to be open subsets of A and A', respectively. Certain categories of holomorphic functions are delineated. In this context, $\mathcal{H}_{wv}(V)$ represents the collection of holomorphic functions $g: V \to \mathbb{C}$ that exhibit weak-star uniform continuity on every weakly compact subset of V. Similarly, $\mathcal{H}_{w^*vk}(U)$ signifies the collection of holomorphic functions $f: U \to \mathbb{C}$ that demonstrate weak-star uniform continuity on every weakly compact subset of U. We begin by examining the characteristics of the algebras $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{w^*vk}(U)$. An important finding pertains to the approximation of polynomials on such algebras [4]. We demonstrate that

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in the case V is a weakly open subset of A that is polynomially convex and A is a Banach space with a shrinking Schauder basis, then $\mathcal{P}_{g_i}(A)$ is dense in $\mathcal{H}_{wvk}(V)$, assuming the topology of uniform convergence on the weakly compact subsets of V.

An equivalent outcome is obtained for the algebra $\mathcal{H}_{w^*vk}(U)$ [9]. The subsequent section provides a detailed account of the spectrum of $\mathcal{H}_{wvk}(V)$, where A represents a reflexive Banach space with a Shauder basis and V represents a weakly open, $\mathcal{P}_{wk}(A)$-convex subset of A. We demonstrate that the spectrum $\mathcal{H}_{wk}(V)$ is indeed associated with V in this instance. Additionally, we examine whether $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{wv}(V)$ coincide $\mathcal{H}_{wvk}(V) = \mathcal{H}_{wv}(V)$, for instance, if A is reflexive and V is weakly open and convex. We illustrate an additional circumstance in which $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{wv}(V)$ coincide. By utilizing these fortuitous findings, we can enhance the outcomes reported in ([11], [15]). We conclude with results on ideals of the algebra $\mathcal{H}_{wvk}(V)$ that were generated finitely and Banach–Stone theorems.

2. Banach spaces and Schauder basis

Consider the complex Banach space A ([8], [10]). V shall represent an open subset of A. We designate the distance from x to the boundary of V for each $x \in V$ as $d_V(x)$. Let V_m equal $\{x \in V: \|x\| < m$ and $d_V(x) > 2^{-m}$ for each value of $m \in \mathbb{N}$. The set of all $g \in \mathcal{H}(V)$ that are weakly continuous on each V_m is denoted by $\mathcal{H}_w(V)$, while $\mathcal{H}_{wv}(V))$ represents the set of $g \in \mathcal{H}(V)$ that are weakly uniformly continuous on each V_m. Lastly, $\mathcal{H}_b(V)$ signifies the set of $g \in \mathcal{H}(V)$ that are bounded on each V_m that $\mathcal{H}_{wv}(V) \subset \mathcal{H}_b(V)$ [3] holds for each open subset V. In the case where U is an open subset of A', let $\mathcal{H}_{w^*}(U)$ represent the collection of $f \in \mathcal{H}(U)$ elements that exhibit weak-star continuity on all U_m, and let $\mathcal{H}_{w^*v}(U)$ represent the collection of $f \in \mathcal{H}(U)$ elements that demonstrate weak-star uniform continuity on each U_m. Define $\mathcal{K}_w(V)$ as follows: $\{E \subset U: E$ is weakly compact$\}$; $\mathcal{K}_{w^*}(U)$ as follows: $\{B \subset V: B$ is weak-star compact $\}$. It is evident that V is covered by $\mathcal{K}_w(V)$ and $\mathcal{K}_{w^*}(U)$, respectively. The following lemma describes a useful property of the elements of $\mathcal{K}_w(V)$ and $\mathcal{K}_{w^*}(U)$ if V is weakly open and U is weak-star open. The set of all neighborhoods of zero in A (or A') relative to the weak topology $\sigma(A, A')$ (or weak-star topology $\sigma(A, A')$ is represented by $\mathcal{V}_w(A)$ or $\mathcal{V}_{w^*}(E')$ or (E').

Lemma 2.1. Assume the following: U is an open subset of A', V is a weakly open subset of A, and A is a Banach space. Then

(a) There exists a $W \in \mathcal{V}_w(A)$ such that $E + W \subset V$ for every $A \in \mathcal{K}_w(V)$.

(b) $W \in U_{w^*}(A')$ exists for any $B \in \mathcal{K}_{w^*}(U)$ such that $B + W \subset V$.

Proof. (a) Because V is weakly open, there exists $\mathrm{W}_x, \widetilde{W}_x \in \mathcal{V}_w(A)$ such that $\mathrm{W}_x + \mathrm{W}_x \subset \widetilde{W}_x$ and $x + \widetilde{W}_x \subset U$ for any $x \in A$. We can find $x_1, \dots, x_n \in E$ and $W_1, \dots, W_n \in \mathcal{V}_w(A)$ such that $E \subset (x_1 + W_1) \cup \dots \cup (x_m + W_m) \subset V$ because E is weakly compact. By taking $W = W_1 \cap \dots \cap W_m$, it is simple to see that $A + W \subset V$.

(b) (A)'s proof is applicable.

We write $\mathcal{P}_{g_i}(A) = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_{g_i}({}^nA)$, $\mathcal{P}_w(A) = \mathcal{P}(A) \cap \mathcal{H}_w(A)$, and $\mathcal{P}_{w\mathrm{v}}(A) = \mathcal{P}(A) \cap \mathcal{H}_{w\mathrm{v}}(A)$. In actuality, $\mathcal{P}_w(A) = \mathcal{P}_{w\mathrm{v}}(A)$ [2] corresponds to the two final sets. Assume that $\mathcal{H}_{w\mathrm{v}k}(V) = \{ g \in \mathcal{H}(V)$: g is weakly balanced continuous on every $E \in \mathcal{K}_w(V)\}$. Let V be an open subset of A. Keep in mind that if V is weakly open, then $\mathcal{H}_{w\mathrm{v}}(V) \subset \mathcal{H}_{w\mathrm{v}k}(V)$ since every weakly compact subset of V is contained in some V_m. Furthermore, $\mathcal{P}_{g_i}(A) \subset \mathcal{P}_{w\mathrm{v}}(A) \subset \mathcal{H}_{w\mathrm{v}k}(V)$ is evident. After (a), we state that if and only if P is a finite linear combination of products of weak-star continuous linear functional on A', then a polynomial $P \in \mathcal{P}_{g_i^*}(A')$.

Take note that every evaluation at a point in A is a weak-star continuous linear functional of $\mathcal{P}_{w^*}(A') = \mathcal{P}(A') \cap \mathcal{H}_{w^*}(A')$ and $\mathcal{P}_{w^*\mathrm{v}}(A') = \mathcal{P}(A') \cap \mathcal{H}_{w^*\mathrm{v}}(A')$ are also indicated, but it is evident that the final two sets coincide, that is, $\mathcal{P}_{w^*}(A') = \mathcal{P}_{w^*\mathrm{v}}(A')$. Assume that $\mathcal{H}_{w^*\mathrm{v}k}(U) = \{ f \in \mathcal{H}(V)$: f is weak-star uniformly continuous on every $B \in \mathcal{K}_{w^*}(U)\}$. Let U be an open subset of A'. Keep in mind that $\mathcal{P}_{g_i^*}(A') \subset \mathcal{P}_{w^*\mathrm{v}}(A') \subset \mathcal{H}_{w^*\mathrm{v}k}(U)$, and $\mathcal{H}_{w^*\mathrm{v}k}(A') = \mathcal{H}_{w^*\mathrm{v}}(A')$. $\mathcal{H}_{w^*uk}(V) \subset \mathcal{H}_{wuk}(V)$ if U is weak-star open. $\mathcal{H}_{w^*\mathrm{v}}(U) \subset \mathcal{H}_{w^*\mathrm{v}k}(U)$ if A is reflexive.

We confer the topology of uniform topology of uniform convergence on the elements of $\mathcal{K}_w(V)$ (respectively $\mathcal{K}_{w^*}(U)$) to $\mathcal{H}_{w\mathrm{v}k}(V)$ (respectively $(\mathcal{H}_{w^*\mathrm{v}k}(U)$, and we represent this topology by τ_k (respectively τ_{k^*}). $(\mathcal{H}_{w\mathrm{v}k}(V), \tau_k)$ (or $(\mathcal{H}_{w^*\mathrm{v}k}(U), \tau_{k^*})$) is obviously a locally m-convex algebra. We provide a coincidental finding pertaining to the algebras $\mathcal{H}_{w\mathrm{v}k}(V)$ and $\mathcal{H}_{w\mathrm{v}}(V)$ in the following example.

Example 2.2. Assume that V is a convex, weakly open subset of A and that A is a reflexive Banach space. After that, $\mathcal{H}_{w\mathrm{v}k}(V) = \mathcal{H}_{w\mathrm{v}}(V)$.

Proof. Since sine V is convex, we may infer that V_m is convex for all m in $\mathbb{N}$. Consequently, $\bar{V}_m{}^w = \bar{V}_m \subset V$. $\bar{V}_m{}^w$ is w-compact since A is reflexive, and as a result, $\bar{V}_n{}^w \in \mathcal{K}_w(V)$. Consequently, $\mathcal{H}_{w\mathrm{v}k}(V) \subset \mathcal{H}_{w\mathrm{v}}(V)$.

Given a Banach space A and a Schauder basis $(e_m)_{m\in\mathbb{N}}$, the associated linear functionals are $(\psi_m)_{m\in\mathbb{N}}$. T_m^i represents the canonical projection $T_m^i: A \to A$ for each $m \in \mathbb{N}$, where $T_m^i(x) =$

$T_m^i(\sum_{i=1}^{\infty}\psi_j(x)e_j) = \sum_{i=1}^{m}\psi_i(x)e_i$. If the associated linear functionals $(\psi_m)_{m\in\mathbb{N}}$ form a Schauder basis in A', we say that the Schauder basis is shrinking. The canonical projection $S_m: A' \to A'$ in this instance is denoted by S_m, where $S_m(\psi) = (\sum_{i=1}^{m}\psi(e_i)\psi_i$, for each $\psi \in A'$. The sequence $(T_m^i)_{m\in\mathbb{N}}$ is known to converge uniformly to the identity operator on the compact subsets of E. If we swap out compact for bounded subsets of E in the case of infinite-dimensional E, the same outcome will not hold. In fact, there would be a contradiction if it were true, as the identity operator would be a compact operator. However, we present a weaker result of this kind in the following proposition.

Proposition 2.3. Assume that A has a decreasing Schauder basis and is a Banach space. Next

(a) T_m weakly uniformly converges to the identity operator on the bound subsets of A.

(b) On the bordered subsets of A, S_m weak-star uniformly and converges to the identity operator.

Proof. (a) We have to demonstrate that for every bounded subset B of A, where $\psi \in A'$ and $\varepsilon > 0$, there exists an integer number $m_0 \in \mathbb{N}$ such that, for all $m > m_0$, $\sup_{x\in B}\left|\psi(T_m^i(x) - x)\right| < \varepsilon$. For any x in A, $\psi \in A'$, and m in $\mathbb{N}$, it is evident that $\psi(x - T_m^i(x)) = \sum_{i=m+1}^{\infty}\psi_i(x)\psi(e_i)$. A Schauder basis for A' is $(\psi_i)_{i\in\mathbb{N}}$, hence for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $\| \sum_{i=m+1}^{\infty}\psi(e_i)\psi_i \| < \varepsilon$.

For $m > m_0$, this is $\sup_{x\in B_E}|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(x)| < \varepsilon$, or equivalently,

$\sup_{x\in B}|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(x)| \le \sup_{x\in B_A}|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(rx)| < r\varepsilon$, for $m > m_0$ which is precisely for $B = B_A$. Suppose that B be the bounded set of A, and let $r > 0$ such that $B \subset rB_A$.For any $m > m_0$, the following holds true : $\sup_{x\in B}|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(x)| \le \sup_{x\in B_A}|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(rx)| < r\varepsilon$.

(b) Assume that x belongs to A, $\varepsilon > 0$, and $B \subset A'$ is a abounded subset. Assume that $B \subset B_{A'}(0,r)$ for any $r > 0$. Since $(e_m)_{m\in\mathbb{N}}$ is a Schauder basis for A, for any $m > m_0$, there exists $m_0 \in \mathbb{N}$ such that $\| \sum_{i=m+1}^{\infty}\psi_i(x)e_i \| < \frac{\varepsilon}{3}$. If we use $\psi = \sum_{i=m+1}^{\infty}\psi(e_i)\psi_i$, then

$$\sup_{\psi\in B}|S_m(\psi)(x) - \psi(x)| = \sup_{\psi\in B}\left|\sum_{i=m+1}^{\infty}\psi(e_i)\psi_i(x)\,\psi = \sum_{i=m+1}^{\infty}\psi(e_i)\psi_i\right|$$

$$\le \sup_{\psi\in B}\|\psi\|\left\|\sum_{i=m+1}^{\infty}\psi_i(x)e_i\right\| < r.\frac{\varepsilon}{r} = \varepsilon, \text{for } m \ge m_0.$$

Going forward, the lack of proof for the weak-star case in A' can be attributed to the fact that it restates the reasons presented in the proof for the weak case in A. The following corollaries apply to us.

Corollary 2.4. Assume that A has a decreasing Schauder basis and is a Banach space. In $\mathcal{P}_w(A)$, $\mathcal{P}_{g_i}(A)$ is norm-dense, and in $\mathcal{P}_{w^*}(A')$, $\mathcal{P}_{g_i^*}(A')$ is norm-dense.

Proof. For every m in $\mathbb{N}$, let $c > 1$ be such that $\|T_m^i\| \le c$. Suppose that $B = B(0, r)$, and let $C = B(0, cr)$. Let x, y are in C, $x - y$ is in W, $W \in \mathcal{V}_w(A)$ and $\varepsilon > 0$ then

$$|P(x) - P(y)| < \varepsilon.$$

According to Proposition (1.3), for any x in B and $m > m_0$, there exists $m_0 \in \mathbb{N}$ such that $T_m^i(x) - x \in W$. As a result, for all x in B where $m > m_0$,$|P \circ T_m^i(x) - P(x)| < \varepsilon$. Now note that, for every n in $\mathbb{N}$, $P \circ T_m^i \in \mathcal{P}_{g_i}(A)$.

Assume that A denote a subset of the Banach space E, and $\mathcal{G} \subset \mathcal{P}(A)$. Then for all $g \in \mathcal{G}$, the $\mathcal{G}$-hull of E is defined as the set

$$\widehat{E}_{\mathcal{G}} = \{x \in A: |f(x)| \le \sup_E |g|\}$$

Corollary 2.5. Suppose that A represents a Banach space characterized by a diminishing Schauder basis. Define E and B as abounded and bounded subsets, respectively, of A and A'. Then $\widehat{E}_{\mathcal{P}_{g_i}(A)} = \widehat{E}_{\mathcal{P}_w(A)}$, and $\widehat{B}_{\mathcal{P}_{g_i^*}(A')} = \widehat{B}_{\mathcal{P}_{w^*}(A')}$.

Corollary 2.6. Permit A to represent a Banach space characterized by a diminishing Schauder basis. Define V as a weakly open subset of A, and U as a weak-star open subset of A'.

(a) Given $m > m_0$ and $E \in \mathcal{K}_\omega(V)$, there are $W \in \mathcal{V}_\omega(A)$ and $m_0 \in \mathbb{N}$ in which $E + W \subset V$ and $T_m^i(E) + W \subset V$ are both true. Specifically, $T_m^i(E) \in \mathcal{K}_\omega(V)$ holds true for all values of $m \ge m$.

(b) There exists a $W \in \mathcal{V}_{\omega^*}(A')$ and $m_0 \in \mathbb{N}$ pairwise compatible such that $B + W \subset U$ and $S_m(B) + W \subset U$, for all $m > m_0$, for each $B \in \mathcal{K}_{\omega^*}(U)$. More specifically, $S_m(B) \in \mathcal{K}_{\omega^*}(U)$ as m approaches to zero.

(c) The set $\mathcal{K}_\omega(V)$ contains the elements $C = E \cup \{T_m^i(E): m \ge m_0$

(d) $\mathcal{K}_{\omega^*}(U)$ contains the set $D = B \cup \{S_m(B): m \ge m_0\}$.

Proof. (a) Assume that $E \in \mathcal{K}_\omega(V)$ is present. We can determine $W, \widetilde{W} \in \mathcal{V}_\omega(A)$ by Lemma 2.1, given that $W + W \subset \widetilde{W}$ and $E + \widetilde{W} \subset V$. According to Proposition 2.3, for all $x \in E$ and $m \ge m_0$, there exists $m_0 \in \mathbb{N}$ such that $T_m(x) - x \in W$. Therefore, $T_m^i(E) \subset E + W \subset V$ holds true for all $m \ge m_0$ as well as hence $T_m^i(E) \subset E + W \subset V$, where $m < m_0$.

(c) By (a), specifically, we obtain $C \subset V$. To demonstrate the weak compactness of C, consider $(W_\alpha)_{\alpha \in E}$ as a weakly open cover for C, such that $C \subset \bigcup_{\alpha \in E} W_\alpha$. Given that $E \subset C$ is weakly compact, $\alpha_1, \dots, \alpha_k \in E$ must be present for $A \subset \bigcup_{j=1}^k W_{\alpha_j}$. Consider $W \in \mathcal{V}_w(A)$ to be such that $E +$

$W \subset \bigcup_{j=1}^{k} W_{\alpha_j}$ according to Lemma 2.1. Proposition 2.3 states that for all $x \in A$ and $m \geq m_1$, there exists $m_1 \geq m_0$ in which $T_m(x) - x \in W$. This implies that $T_m^i(x) \in \bigcup_{j=1}^{k} W_{\alpha_j}$, for all $x \in A$ and $m \geq m_1$. It is now evident that $T_m^i(E)$, where $m = m_0, \dots, m_1$, belongs to a finite subfamily of $(W_\alpha)_{\alpha \in E}$.

Corollary 2.7. Denoted as A, this space follows a diminishing Schauder basis. Subsequently, $\mathcal{P}_{g_i}(A)$ and $\mathcal{P}_{g_i^*}(A')$ both exhibit norm-dense characteristics.

Proof. Whenever $\|T_m^i\| \leq 1 + \delta_i$, and $m \in \mathbb{N}$. $B = B(0, r)$, $C = B(0, (1 + \delta_i)\, r)$ and $P^i \in \mathcal{P}_w(A) = \mathcal{P}_{wv}(A)$.. There exists $W \in \mathcal{V}_w(A)$ for which $\varepsilon > 0$, such that if $x, y \in C$ and $x - y \in W$, then

$$\sum_{i=1}^{m} \left|P^i(x) - P^i(y)\right| < \varepsilon.$$

There exists $m_0 \in \mathbb{N}$ in accordance with Proposition 2.3 such that $T_m^i(x) - x \in W$, where $x \in B$ and $m \geq m_0$. This is why

$$\sum_{i=1}^{m} \left|P^i \circ T_m^i(x) - P^i(x)\right| < \epsilon,$$

in the given $x \in B$ and $m \geq m_0$. For all $m \in \mathbb{N}$, observe that $P^i \circ T_m^i(x) \in \mathcal{P}_{g_i}(A)$**.**

Proposition 2.8. Define V as a weakly open subset of A, U as a weak-star open subset of A', and A as a Banach space with a contracting Schauder basis. Assign g to $g \in \mathcal{H}_{wvk}(V)$and f to f $\in \mathcal{H}_{w^*vk}(U)$). Then

(a) There is a value of $m_0 \in \mathbb{N}$ such that $\sup_{x \in E} \left|g\left(T_m^i(x)\right) - g(x)\right| < \varepsilon$, for all $m \geq m_0$, for each $E \in \mathcal{K}_w(V)$ and $\varepsilon > 0$.

(b) There exists a value of $m_0 \in \mathbb{N}$ such that $\sup_{y' \in B} \left|f\left(S_m^i(y')\right) - f(y')\right| < \varepsilon$, for all $m \geq m_0$, where $\varepsilon > 0$ and $\in \mathcal{K}_{w^*}(U)$.

Proof. Assume that $E \in \mathcal{K}_w(U)$. Using Corollary 2.6, there exists an integer number $m_1 \in \mathbb{N}$ such that $E \cup \{T_m^i(E) : m \geq m_1\} = C \in \mathcal{K}_w(V)$. Since $g \in \mathcal{H}_{wvk}(V)$, there is $W \in \mathcal{V}_w(A)$ such that if $x, y \in C$ and $x - y \in W$ then

$$|g(x) - g(y)| < \varepsilon.$$

There exists a W for which $m_2 \in \mathbb{N}$ guarantees that $T_m(x) - x \in W$, given that $x \in C$ and $m \geq m_2$. Define m_0 as the maximum of m_1, m_2 given that $x \in E$ and $m \geq m_0$. Following this, $x, T_m(x) \in C$, $T_m(x) - x \in W$, and thus $\left|g(T_m^i(x)) - g(x)\right|$ is less than ε.

In essence, proposition 2.8 states that $g \circ T_m^i$ converges uniformly to g across the elements of $w(V)$. However, this would be a linguistic distortion, as not all compositions $g \circ T_m^i$ are precisely defined for each value of $m \in \mathbb{N}$. Our first significant finding regarding the two algebras $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{w^*\text{v}k}(U)$ is the Next theorem.

Theorem 2.9. Consider A a Banach space with a diminishing Schauder basis, V a weakly open subset of A that is polynomially convex, and U a weak-star open subset of A' that is also polynomially convex. $\mathcal{P}_{g^*}(A')$ is τ_{k^*} -dense in $\mathcal{H}_{w^*\text{v}k}(U)$, whereas $\mathcal{P}_g(A)$ is τ_k-dense in $\mathcal{H}_{wvk}(V)$.

Proof. Assume that Let $E \in \mathcal{K}_w(V)$, $g \in \mathcal{H}_{wvk}(V)$ and $\varepsilon > 0$. By applying Proposition 2.8 and Corollary 2.6, we can identify an integer number $m_0 \in \mathbb{N}$ such that

$$T_{m_0}^i(E) \in \mathcal{K}_w(V) \text{ and } \left|g \circ T_{m_0}^i(x) - g(x)\right| < \frac{\varepsilon}{2}, \text{for all } x \in E.$$

$V \cap T_{m_0}^i(A)$ is polynomially convex in $T_{m_0}^i(A)$, which follows from the fact that V is polynomially convex [10]. Conversely, it is evident that $T_{m_0}^i(A)$ constitutes a compact subset of $V \cap T_{m_0}^i(A)$. Subsequently, it can be deduced from [10] that $P \in \mathcal{P}(T_{m_0}^i(A))$ exists in such a way that ensures the discrepancy between $|P(y) - g(y)|$ and $\frac{1}{2}$ is present uniformly on $y \in T_{m_0}^i(E)$ alternatively stated,

$$\sup_{x \in E}\left|p \circ T_{m_0}^i(x) - g \circ T_{m_0}^i(x)\right| < \frac{\varepsilon}{2}.$$

The conclusion is now presented in (a) and (b).

The initial assertion in Corollary 2.4 becomes evident when A' possesses the property of approximation [3]. Knowing the second assertion in Theorem 2.9 requires that A possesses the approximation property [1] and $U = A'$. But the proof presented here is considerably simpler when A has a diminishing Schauder basis.

Corollary 2.10. Define V as a weakly open subset of A, U as a weak-star open subset of A', and A as a Banach space with a contracting Schauder basis. Allow $g_i \in \mathcal{H}_{wvk}(V)$ and $f_i \in \mathcal{H}_{w^*\text{v}k}(U)$. Thus,

(a) Given $\varepsilon > 0$ and $E \in \mathcal{K}_w(V)$, there is a $m_0 \in \mathbb{N}$ value in which

$$\sup_{x \in E}\sum_{i=1}^{m}\left|g_i\left(T_m^i(x)\right) - g_i(x)\right| < \varepsilon, \text{for all } m \geq m_0.$$

(b) In the case where $B \in \mathcal{K}_{w^*}(U)$ and $\varepsilon > 0$, $m_0 \in \mathbb{N}$ is a valid value such that

$$\sup_{y' \in B}\sum_{i=1}^{m}\left|f_i\left(S_m^i(y')\right) - f_i(y')\right| < \varepsilon, \text{for all } m \geq m_0.$$

Proof. Assume $E \in \mathcal{K}_w(V)$. It is implied by corollary 2.6 that for an integer number $m_1 \in \mathbb{N}$, there exists $E \cup \{T_m^i(E): m \geq m_1\} = C \in \mathcal{K}_w(V)$ condition. Given that $g_i \in \mathcal{H}_{wvk}(V)$, there is an element $W \in \mathcal{V}_w(A)$ in which $x, y \in C$ and $x - y \in W$, then

$$\sum_{i=1}^{m} |g_i(x) - g_i(y)| < \varepsilon.$$

There exists a value of $\mathbb{N}$ such that $T_m^i(x) - x \in W$ for this W, given that $x \in C$ and $m \geq m_2$. Define m_0 as the maximum of m_1, m_2 given that $x \in E$ and $m \geq m_0$. Consequently, $x, T_m^i(x) \in C$ and $T_m^i(x) - x \in W$, and hence

$$\sum_{i=1}^{m} \left|g_i(T_m^i(x)) - g_i(x)\right| < \varepsilon.$$

Corollary 2.11. Consider A a Banach space with a diminishing Schauder basis, V a weakly open subset of A that is polynomially convex, and U a weak-star open subset of A' that is also polynomially convex. Subsequently, $\mathcal{P}_{g_i}(A)$ becomes τ_k-dense in $\mathcal{H}_{w\mathrm{v}k}(V)$, while $\mathcal{P}_{g_i^*}(A')$ is τ_{k^*}-dense in $\mathcal{H}_{w^*\mathrm{v}k}(U)$ [16].

Proof. Let $g_i \in \mathcal{H}_{w\mathrm{v}k}(V)$ and $E \in \mathcal{K}_w(V)$ both have $\varepsilon > 0$. In the case where an integer m_0 is such that $T_{m_0}^i(E) \in \mathcal{K}_w(V)$ and $\sum_{i=1}^{m} \left|g_i \circ T_{m_0}^i(x) - g_i(x)\right| < \frac{\varepsilon}{2}$, for all $x \in E$.

$V \cap T_{m_0}^i(A)$ is polynomially convex in $T_{m_0}^i(A)$, given that V is polynomially convex [10]. The compact subset of $V \cap T_{m_0}^i(A)$ is denoted as $T_{m_0}^i(E)$. [1] demonstrates that $P^i \in \mathcal{P}(T_{m_0}^i(A)$ exists, such that

$$\sum_{i=1}^{m} \left|P^i(y) - g_i(y)\right| < \frac{\varepsilon}{2},$$

concerning $y \in T_{m_0}^i(E)$, or

$$\sup_{x \in E} \sum_{i=1}^{m} \left|P^i \circ T_{m_0}^i(x) - g_i \circ T_{m_0}^i(x)\right| < \frac{\varepsilon}{2}.$$

This is the consequence.

We will now discuss a number of applications of the prior research. The findings pertain to novel categories of open subsets of Banach spaces. The definition of 2.1 was derived from [15].

3. The convex sets and the compactness

Definition 3.1. Consider A to be a Banach space, V and U to be open subsets of A and A', respectively. We assert that:

(a) For all $E \in \mathcal{K}_w(V)$, V is $\mathcal{P}_{wk}(A)$ -convex if $\widehat{E}_{\mathcal{P}_\omega(A)} \cap V \in \mathcal{K}_w(V)$.

(b) U is convex with respect to $\mathcal{P}_{w^*k}(A')$- if $\hat{B}_{\mathcal{P}_w(A')} \cap U \in \mathcal{K}_{w^*}(U)$, for all $B \in \mathcal{K}_{w^*}(U)$.

(c) If $\widehat{E}_{\mathcal{P}_w(A)} \subset V$, and $\widehat{E}_{\mathcal{P}_w(A)} \in \mathcal{K}_w(V)$ for all $E \in \mathcal{K}_w(V)$, then V is strongly $\mathcal{P}_{wk}(A)$ -convex.

(d) U is considered to be strongly $\mathcal{P}_{w^*k}(E')$ -convex if $\widehat{B}_{\mathcal{P}_w(A')} \subset U$ and $\widehat{B}_{\mathcal{P}_w(A')} \in \mathcal{K}_{w^*}(U)$, for all $B \in \mathcal{K}_w(U)$.

We have demonstrated in the following lemma that the final conditions of Definitions 3.1 (c) and (d) are superfluous.

Lemma 3.2. Consider A to be a Banach space, V and U to be open subsets of A and A', respectively. Suppose that $E \in \mathcal{K}_\omega(V)$ and $B \in \mathcal{K}_{\omega^*}(U)$. If $\widehat{E}_{\mathcal{P}_\omega(A)} \subset V$, and $\widehat{E}_{\mathcal{P}_{\omega^*}(A')} \subset U$ then $\widehat{E}_{\mathcal{P}_w(A)} \in \mathcal{K}_w(V)$, and $\hat{B}_{\mathcal{P}_{\omega^*}(A')} \in \mathcal{K}_{\omega^*}(U)$ respectly.

Proof. Given that $\mathbb{C} \oplus A' \subset \mathcal{P}_w(A)$, it can be deduced that $\hat{E}_{\mathcal{P}_w(A)} \subset \hat{E}_{\mathbb{C}\oplus A'} = co^{-w}(E)$, with the final equality being derived from [8]. Given the weak compactness of $co^{-w}(E)$ and the weak closure of $\widehat{E}_{\mathcal{P}_w(A)}$, it can be deduced that $\widehat{E}_{\mathcal{P}_\omega(A)} \subset V$ is also weakly compact, and thus $\widehat{E}_{\mathcal{P}_w(A)} \in \mathcal{K}_w(V)$. Since $\widehat{B}_{\mathcal{P}_{\omega^*}(A')}$ is weak-star closed and bounded, and thus weak-star compact, the second assertion is superfluous.

Lemma 3.3. Define A as a Banach space, and E as a subset of A' that is abounded. Subsequently, $\widehat{E}_{\mathbb{C}\oplus A'} = co^{-w^*}(E)$, where $\mathbb{C} \oplus A$ represents the set $\{e + \delta_x : e \in \mathbb{C}, x \in A\} \subset A''$.

Proof. The proof is continued by applying the Hahn Banach Theorem to the space $(A', \sigma(A', A))$ that is locally convexymorphic [10].

Example 3.4. Suppose A represents a Banach space, with P and Q ranging over $\mathcal{P}_{g_i}(A)$ and $\mathcal{P}_{g_i}(A')$ Consequently, then:

(a) each weakly open convex subset of A is strongly $\mathcal{P}_{wk}(A)$-convex

(b) Each convex weak-star open subset of A' possesses the property of $\mathcal{P}_{w^*k}(A')$-convexity.

(c) $V = \{x \in A : |P(x)| < 1\}$ is a weakly open set that is strongly $\mathcal{P}_{wk}(A)$-convex.

(d) $U = \{x \in A' : |Q(x)| < 1\}$ is an open set that is strongly $\mathcal{P}_{w^*k}(A')$ -convex weak-star.

Proof. Let $E \in \mathcal{K}_w(V)$ in (a). To begin, we shall demonstrate that $\overline{co}^w(E) \in \mathcal{K}_w(V)$. Assume, by Lemma 2.1, that $\widetilde{W} \in \mathcal{V}_w(A)$ is such that $E + \widetilde{W} \subset V$. Given that V is convex, it is evident that $co(E) + \widetilde{W} \subset co(E + \widetilde{W}) \subset V$. Based on the equation $\overline{co}^w(E) = \bigcap_{W \in \mathcal{V}_w(A)} co((E) + W)$, it is evident that $\overline{co}^w(E) \subset co(E) + \widetilde{W} \subset V$. Consequently, $\overline{co}^w(E) \in \mathcal{K}_w(V)$.

Currently, $\widehat{E}_{\mathbb{C}\oplus A'} \subset \widehat{E}_{\mathcal{P}_w(A)} = \overline{co}^w(E) \in \mathcal{K}_w(V)$, with the final equality being deduced from [10]. Consequently, V is strongly $\mathcal{P}_{wk}(A)$-convex.

(b) We apply the identical reasoning as in (a), substituting Lemma 3.2 for [10].

(c) It is evident that V has a feeble opening. When $E \in \mathcal{K}_w(V)$, we will demonstrate that $\sup_E |P| < 1$.

Consider the case where $\sup_E |P| = 1$. There is a sequence $(x_m) \in E$ such that $|P(x_m)|$approaches to 1. Given that E is compact in the w-direction, a subsequence of (x_m) called (x_{m_k}) exists in which $x_{m_k} \overset{w}{\to} x \in E \subset V$. Therefore, $|P(x_{m_k})| \to |P(x)| = 1$.

precisely, $x \notin V$, which is inherently contradictory. At this time, let $y \in \widehat{E}_{\mathcal{P}_w(A)}$. Then $|P(y)| \leq \sup_E |P| < 1$, which establishes that $\widehat{E}_{\mathcal{P}_w(A)} \subset V$. Now V strongly $\mathcal{P}_{wk}(A)$ convex according to Lemma 3.2.

$\mathcal{P}_{wk}(A)$-convexity and weak openness both indicate that V is polynomially convex. Indeed, $K \in \mathcal{K}_w(V)$ if K is a compact subset of V. Given that $\mathcal{B}(V)$ is in a state of $\mathcal{P}_w(V) \subset \mathcal{P}(A)$, it follows that $\widehat{K}_{\mathcal{P}(A)} \subset \widehat{K}_{\mathcal{P}_w(A)}$. Consequently, $\widehat{K}_{\mathcal{P}(A)} \cap V \subset \widehat{K}_{\mathcal{P}_w(A)} \cap V \in \mathcal{K}_w(V) \subset \mathcal{B}(V)$. It is worth noting that according to [15], an open subset V of a Banach space A is considered $\mathcal{P}_b(A)$-convex if $\widehat{E}_{\mathcal{P}(A)} \cap V \in \mathcal{B}(V)$ for every $E \in \mathcal{B}(V)$. Furthermore, V is considered strongly $\mathcal{P}_b(A)$-convex if $\widehat{E}_{\mathcal{P}(A)} \subset V$ and $\widehat{E}_{\mathcal{P}(A)} \in \mathcal{B}(V)$ for every $E \in \mathcal{B}(V)$. In contrast, we demonstrate in [15] that the final condition $\widehat{E}_{\mathcal{P}(A)} \in \mathcal{B}(V)$ is unnecessary. When the value of V is balanced, both concepts are concurrent [15]. The outcome is analogous when $\mathcal{P}_{wk}(A)$-convex sets are considered; this is demonstrated in Theorem 3.6. To illustrate this theorem, the subsequent result is required.

Theorem 3.5. Consider the Banach space A.

(a) Consider a weakly compact subset of A denoted by $E \subset A$ and a weakly open subset of $A \subset E$ denoted by V, such that $\widehat{E}_{\mathcal{P}(A)} \subset V$. Subsequently, a weakly open set $\tilde{V}$ exists that is $\mathcal{P}_{wk}(A)$-convex and such that $\widehat{E}_{\mathcal{P}_{g_i}(A)} \subset \tilde{V} \subset V$.

(b) Denote a weak-star compact subset of A' denoted as $B \subset A'$ and a weak-star open subset of A' referred to as U, such that $\widehat{B}_{\mathcal{P}_{g_i^*}(A')} \subset U$. Subsequently, a weak-star open set $\widetilde{U}$ is generated, which is strongly $\mathcal{P}_{w^*k}(A')$-convex. This implies that the set $\widehat{B}_{\mathcal{P}_{g_i^*}(A')} \subset \widetilde{U} \subset U$.

Proof. (a) Our strategies are motivated by the concepts put forth in [10]. It can be deduced that $C = \overline{co}^w(E)$ is weakly compact, given that E is weakly compact. In the event that $C \subset V$, $\tilde{V} = C +$

W is obtained, given that $W \in \mathcal{V}_w(V)$ is convex and such that $C + W \subset V$ (Lemma 2.1). Given that $\mathbb{C} \oplus A' \subset \mathcal{P}_{g_i}(A)$, it can be deduced that $\widehat{E}_{\mathcal{P}_{g_i}(A)} \subset \widehat{E}_{\mathbb{C} \oplus A'} = C$. This last equality is supported by reference [10]. Example 3.4 demonstrates that $\tilde{V}$ is now strongly $\mathcal{P}_{wk}(A)$-convex; therefore, $\tilde{V}$ is the intended set. In the absence of C being contained in V, there exists a $P \in \mathcal{P}_{g_i}(A)$ such that $\sup_E |P| < 1 < |P(y)|$, for every $y \in C \setminus V$. Given that $C \setminus V$ is weakly compact, it is possible to identify polynomials $P_1, P_2, \dots P_k \in \mathcal{P}_{g_i}(A)$ that satisfy the following conditions:

$$C \setminus V \subset \bigcup_{i=1}^{k} \{ x \in A : |P_i(x)| > 1$$

This is why

$$C \cap \{ x \in A : |P_i(x)| \leq 1, \text{for } i = 1,2, \dots \} \subset V.$$

We assert that $W \in \mathcal{V}_w(A)$ exists in such a way that

$$(C + W) \cap \{x \in A : |P_i(x)| < 1, \text{for } i = 1, \dots, k\} \subset V.$$

In the event that this condition is not met, there exists a set $z_W = x_W + y_W$ for each $W \in \mathcal{V}_w(V)$, where $x_W \in C, y_W \in W$, and $|P_i(z_W)| < 1$ for $i = 1,2, \dots, k$; such that $z_W \notin V$. Without sacrificing generality, since C is weakly compact, there exists $x \in C$ such that $x_W \xrightarrow{w} x \in C$, and thus $z_W \xrightarrow{w} x \in C$.

It follows that since $P_i(z_W) \to P_i(x)$ for $i = 1,2, \dots, k$, $|P_i(z_W)| \leq 1$, $i = 1, \dots, k$, which indicates that $x \in V$, by (c). Define W as such that $x + \widetilde{W} \subset V$. There exists a $W_0 \in \mathcal{V}_w(V)$ for which $z_{W_0} \in x + \widetilde{W} \subset V$, which is in contradiction with the given $\widetilde{W}$. Consequently, $\tilde{V} = (C + W) \cap \{x \in A : |P_i(x)| < 1, \text{ for } i = 1, \dots, k\}$ is strongly $\mathcal{P}_{wk}(V)$-convex by nature, as it is a finite intersection of sets that are $\mathcal{P}_{wk}(A)$-convex (Example 3.4) at this point. Ultimately, it is evident that $\widehat{E}_{\mathcal{P}_{g_i}(A)} \subset \tilde{V} \subset V$.

(b) We adopt the identical methodology as in (a), substituting Lemma 3.3 for [10].

Theorem 3.6. The space A, which has a shriking Schauder basis, V is a weakly open subset of A. U on the other hand, is a weak-star open subset of A'. Then

(a) V is $\mathcal{P}_{wk}(A)$-convex if and only if V is strongly $\mathcal{P}_{wk}(A)$-convex.

(b) U is $\mathcal{P}_{w^*k}(A')$-convex if and only if U is strongly $\mathcal{P}_{w^*k}(A')$-convex.

Proof. To illustrate the nontrivial consequence, let $E \in \mathcal{K}_w(V)$. It is sufficient to demonstrate, by Lemma 2.10, that $\widehat{E}_{\mathcal{P}_w(A)} \subset V$. We consider that $\widehat{E}_{\mathcal{P}_w(A)} = \left(\widehat{E}_{\mathcal{P}_w(A)} \cap V\right) \cup (\widehat{E}_{\mathcal{P}_w(A)} \setminus V)$. Since V is $\mathcal{P}_{wk}(A)$-convex, we have that $\widehat{E}_{\mathcal{P}_w(A)} \cap V \in \mathcal{K}_w(V)$ and then by Lemma 2.1 there is a $\widetilde{W} \in \mathcal{V}_w(V)$

in which $\widehat{E}_{\mathcal{P}_w(A)} \cap V + \widetilde{W} \subset V$, which implies that $(\widehat{E}_{\mathcal{P}(A)} \cap V + \widetilde{W}) \cap (\widehat{E}_{\mathcal{P}(A)} \setminus V) = \emptyset$. Determine $W \in \mathcal{V}_w(V)$ in which $W + W \subset \widetilde{W}$. $(E_0 + W) \cap (E_1 + W) = \emptyset$, where $E_0 = (\widehat{E}_{\mathcal{P}_w(A)}) \cap V$ and $E_1 = \widehat{E}_{\mathcal{P}_w(A)} \setminus V$, as deduced from [15].

By representing $V' = (E_0 + W) \cup (E_1 + W)$, it becomes evident that $V' = \widehat{E}_{\mathcal{P}_w(A)} + W = \widehat{E}_{\mathcal{P}_g(A)} + W$, with the final equality being derived from Corollary 2.5. Define $g \in \mathcal{H}_{wvk}(V')$ as the condition that $g = 0$ in $E_0 + U$ and $g = 1$ in $E_1 + U$. Let V' signify a weakly open subset of A consisting of $\widehat{E}_{\mathcal{P}_g(A)}$. There exists a weakly open set $\tilde{V}$ that is strongly $\mathcal{P}_{wk}(A)$-convex, as stated in Theorem 3.5, such that $\widehat{E}_{\mathcal{P}_{g_i}(A)} \subset \tilde{V} \subset V'$. We have that $\widehat{E}_{\mathcal{P}_{g_i}(A)} \in \mathcal{K}_w(\tilde{V})$ due to the weak compactness of $\widehat{E}_{\mathcal{P}_{g_i}(A)}$. Given that V is $\mathcal{P}_{wk}(A)$-strongly convex and $g|_{\tilde{V}} \in \mathcal{H}_{wvk}(\tilde{V})$, Theorem 2.9 can be utilised to identify a polynomial $P \in \mathcal{P}_g(A)$ such that $\sup_{\widehat{A}_{\mathcal{P}_{g_i}(A)}} |g|_{\tilde{V}} - P| < 1/2$. Given that $E \subset E_0$, it follows that $\sup_E |P| < 1/2$ and hence $\sup_{\hat{E}_{\mathcal{P}_{g_i}(A)}} |P| < 1/2$.

Currently, let $y \in E_1 \subset \tilde{V}$. Then we have

$$\frac{1}{2} > |p(y) - g|_{\tilde{V}}(y)| = |P(y) - 1| = |1 - P(y)| \geq 1 - |P(y)|.$$

It follows that $||P(y)| > 1/2$ is greater than $1/2$, which is a contradiction.

4. Banach stone theorems and holomorphic mappings

Following this, the spectral efficiencies of $\mathcal{H}_{wvk}(V)$ when A is reflexive will be examined . Given that the two algebras $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{w^*vk}(U)$ are of the same type, it is adequate to deal with $\mathcal{H}_{wvk}(V)$. Let V be an open subset of A and denote A as Banach space. $S_{wvk}(V)$ represents the spectrum of $\mathcal{H}_{wvk}(V)$, which consists of every continuous complex homomorphism $T: \mathcal{H}_{wvk}(V) \to \mathbb{C}$. Consider $z \in V$. Then $\delta_z: \mathcal{H}_{wvk}(V) \to \mathbb{C}$ is referred to as evaluation at z. It is defined by $\delta_z(g) = g(z)$ for all $g \in \mathcal{H}_{wvk}(V)$. It is evident that $\delta_z \in S_{wvk}(V)$ for each $z \in V$; therefore, we can say that $S_{wvk}(V)$ contains V. Subsequently, we demonstrate that, under specific conditions on A and V, every element of $S_{wvk}(V)$ consists of an evaluation at some point of V; thus, we say that $S_{wvk}(V)$ is identified with V [16].

Theorem 4.1. For A to be a reflexive Banach space with a Schauder basis, consider V to be a weakly open subset of A that is a $\mathcal{P}_{wk}(A)$-convex. Following this, the spectrum of $\mathcal{H}_{wvk}(V)$ is correlated with V.

Proof. We adopt the concepts put forth in [8]. Denote $T \in S_{wvk}(V)$. and $c > 0$ are both necessary conditions for T to be continuous, ensuring that $||T(g)| \leq c\sup_E|g|$ for all $g \in \mathcal{H}_{wvk}(V)$. We may infer that c equals 1 based on the classical argument that T is multiplicative. Consider $r > 0$ in the sense that $E \subset B(0,r)$. Specifically, for all $g \in A'$, we have that $|T(g)| \leq \sup_E|g| \leq \sup_{A(0,r)}|g|$. Therefore, given that $T \in A'' = A$ and $a \in A$ is unique such that $T(g) = g(a)$ for all $g \in A', T(P) = P(a)$ for all $P \in \mathcal{P}_{g_i}(A)$, we conclude that $T(P) = P(a)$. Subsequently, for all $P \in \mathcal{P}_{g_i}(A)$, it can be deduced that $|P(a)| = |T(P)| \leq \sup_E|P|$. This implies that $a \in \widehat{E}_{\mathcal{P}_{g_i}(A)} = \widehat{E}_{\mathcal{P}_w(A)}$, with the final equality being deduced from Corollary 2.5.

We now have, by Theorem 3.6 that V is strongly $\mathcal{P}_{wk}(A)$ convex; therefore, $a \in V$. $T(g) = g(a)$ is then obtained by applying Theorem 2.9 to all $g \in \mathcal{H}_{wvk}(V)$ values.

Example 4.2. Consider A to be a Banach space that is reflexive, and V to be a convex and weakly open subset of A. Example 2.2 demonstrates that $\mathcal{H}_{wvk}(V)$ equals $\mathcal{H}_{wv}(V)$. V is strongly $\mathcal{P}_{wk}(A)$-convex, as demonstrated by Example 3.4, given that V is convex. Assuming A possesses a Shauder basis, it follows that $\mathcal{P}_{g_i}(A)$ is dense in $\mathcal{H}_{wv}(V)$. according to Theorem 2.9. Additionally, Theorem 4.1 dictates that $S_{wv}(V)$ equals V. $V \subset A$ is a convex and balanced open set, and if A is a Banach space such that A' possesses the approximation property, then $S_{wv}(V) = \mathrm{int}(\bar{V}^{w^*})$, where the interior is taken in the norm A'', as demonstrated in ([6], [7]). $S_{wv}(V)$ equals V specifically if A is reflexive with a Shauder basis. Therefore, in the reflexive case, the hypothesis that V is balanced can be disregarded; however, it is necessary to presume that V is only weakly open.

Example 4.3. Denote A a Banach space that is reflexive, such that $\mathcal{P}(A) = \mathcal{P}_w(A)$. Consider V to be a weakly open subset of A that is $\mathcal{P}_{wk}(A)$-convex due to its strong $\mathcal{P}_b(A)$-convexity. It can be deduced that $\bar{V}_m^w \subset (\hat{V}_m)_{\mathcal{P}_w(A)} = (\hat{V}_m)_{\mathcal{P}(A)} \subset V$. Given that A is reflexive, it follows that $\bar{V}_m^w$is weakly compact; therefore, $\bar{V}_m^w \in \mathcal{K}_w(V)$. As a result, $\mathcal{H}_{wvk}(V)$ equals $\mathcal{H}_{wv}(V)$. Furthermore, under the assumption that A possesses a Schauder basis, it can be deduced from Example 4.2 that $\mathcal{P}_w(A)$ is dense in $\mathcal{H}_{wv}(V)$ and $S_{wv}(V)$equals V. An instance of a Banach space that possesses every one of the necessary properties is Tsirelson's space [13]. It is demonstrated in reference [15] $S_{wv}(V) = V$ if A is a reflexive Banach space in which $\mathcal{P}(A) = \mathcal{P}_w(A)$, $V \subset A$ is balanced, and the $\mathcal{P}_b(A)$-convex open set is strongly $\mathcal{P}_b(A)$-convex. As previously noted, each balanced $\mathcal{P}_b(A)$-convex open set possesses the strongly $\mathcal{P}_b(A)$ -convex property. In the specific instance where A represents Tsireson's space, we further enhance the outcomes reported in reference [15].

As a result of Theorem 4.1, the Next Theorem follows. It states that, according to the same Theorem 4.1 hypotheses, each proper finitely generated ideal of $\mathcal{H}_{wv}k(V)$. shares a zero. The substantiation shall be omitted in accordance with the tenets of [11].

Theorem 4.4. Assume that V is a a $\mathcal{P}_{wk}(A)$-convex and weakly open subset of A, where A is a reflexive Banach space with a Schauder basis. Consequently, if $g_1, g_2, \dots, g_m \in \mathcal{H}_{\omega vk}(V)$ and none of them have any common zeros, there is exists $f_{1,}, f_2, \dots, f_m \in \mathcal{H}_{\omega v}(V)$ in which $\sum_{i=1}^{m} g_i f_i = 1$. .

With respect to the algebra $\mathcal{H}_{wv}(V)$, the subsequent corollary follows in the spirit of Example 4.2.

Corollary 4.5. Consider A to be a Schauder-basis reflexive Banach space, and V to be a convex and weakly open subset of A. Subsequently, if $g_{1,}, \dots, g_m \in \mathcal{H}_{wv}(V)$ and none of the elements contain common zeros, there is a $f_{1,}, f_2, \dots, f_m \in \mathcal{H}_{\omega v}(V)$ in which $\sum_{i=1}^{m} g_i f_i = 1$.

Denoted as Banach spaces A and G, let $V \subset A$ and $U \subset G$ represent open subsets. The set of holomorphic mappings $\psi: U \to V$ is represented by $\mathcal{H}_{wvk}(V, U)$ in which $\psi: \big(V, \sigma(G, G')\big) \to (V, \sigma(G, G'))$ remains uniformly continuous when limited to each $B \in \mathcal{K}_w(U)$. Suppose that $\psi \in \mathcal{H}_{wuvk}(U, V)$. It can be readily observed that the continuous algebra-homomorphism $C_\psi: \mathcal{H}_{wvk}(V) \to \mathcal{H}_{wvk}(U)$, where $C_\psi(g) = g \circ \psi$, holds true for all $g \in \mathcal{H}_{wvk}(V)$. Such a homomorphism is referred to as a composition operator. Subsequently, we demonstrate that every continuous algebra-homomorphism from $\mathcal{H}_{wvk}(V)$ to $\mathcal{H}_{wvk}(U)$ is a composition operator, under the same conditions as Theorem 4.1.

Theorem 4.6. Consider the two Banach spaces A and G, where A is reflexive and has a Shauder basis. Consider $V \subset A$ to be weakly open and $\mathcal{P}_{wk}(A)$-convex, while $U \subset G$ represents an open subset. Consequently, all continuous algebra-homomorphisms $T: \mathcal{H}_{wvk}(V) \to \mathcal{H}_{wvk}(U)$ can be classified as composition operators.

Proof. Our principles are derived from [14]. It is necessary to identify a mapping $\psi \in \mathcal{H}_{wvk}(U, V)$ that guarantees $T = C_\psi$. It is observed that $\delta_w \circ T \in S_{wvk}(V)$ and let $\omega \in U$. A unique $z \in V$ exists such that $\delta_w \circ T = \delta_z$, as stated in Theorem 4.1. By establishing $\psi(w) = z$, we can deduce that $T(g) = g \circ \psi$, for all $g \in \mathcal{H}_{wvk}(V)$. Specifically, $g \circ \psi$ is holomorphic for all $g \in A'$; therefore, ψ is a holomorphic mapping according to [10]. To demonstrate that the set $\psi: (U, \sigma(G, G')) \to (V, \sigma(A, A'))$ remains uniformly continuous while being limited to a single $B \in \mathcal{K}_w(U)$. Therefore, let $B \in \mathcal{K}_w(U)$, $g \in A'$ and $\omega > 0$. Given that $g \circ \psi \in \mathcal{H}_{wvk}(U)$, there is $W \in \mathcal{V}_w(G)$ in which $|g \circ \psi(x) - g \circ \psi(y)| < \varepsilon$, and $x, y \in W$, then $x - y \in W$. This demonstrates $\psi \in \mathcal{H}_{wvk}(U, V)$ [16].

Corollary 4.7. Consider Banach spaces A and G, where A is reflexive and has a Shauder basis. Denote $V \subset A$ as a weakly open and convex set, while denoting $U \subset G$ as an open subset. Then all continuous algebra homomorphisms $T: \mathcal{H}_{wv}(V) \to \mathcal{H}_{wv}(U)$ can be classified as composition operators.

Corollary 4.7 presents comparable findings to those presented in [7] regarding absolutely convex open subsets of Banach spaces whose dual possesses the property of approximation.
Two compact metric spaces X and Y are homeomorphic if and only if the Banach algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isometrically isomorphic, as demonstrated in [5]. The well-known Banach-Stone theorem was extended to arbitrary compact Hausdorff topological spaces by M.H. Stone in [12]. Comparable outcomes are established for the algebras $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{wvk}(U)$ in the following theorem.

Theorem 4.8. Consider the reflexive Banach spaces A and G to be Shauder bases. Assume that $V \subset A$ and $U \subset G$ are weakly open sets, with V and U being $\mathcal{P}_{wk}(A)$-convex and $\mathcal{P}_{wk}(G)$-convex respectively. Subsequently, the subsequent conditions are equivalent.

(a) A bijective mapping $\psi: U \to V$ is present, in which $\psi \in \mathcal{H}_{wvk}(U, V)$ and $\psi^{-1} \in \mathcal{H}_{wvk}(V, U)$.

(b) $\mathcal{H}_{wvk}(V)$ and $\mathcal{H}_{wvk}(U)$ are topologically isomorphic algebras.

Proof. Our principles are derived from [14].

(a)⇒(b) The composition operator $C_\psi: \mathcal{H}_{wvk}(V) \to \mathcal{H}_{wvk}(U)$ shall be examined. It is then evident that C_ψ is bijective, and $(C_\psi)^{-1} = C_{\psi^{-1}}$.

$(b) \Rightarrow (a)$ Consider an example of a topological isomorphism $T: \mathcal{H}_{wvk}(V) \to \mathcal{H}_{wvk}(U)$. There exist $\psi \in \mathcal{H}_{wvk}(U, V)$ and $\phi \in \mathcal{H}_{wvk}(U, V)$ in which $T = C_\psi$ and $T^{-1} = C_\phi$, respectively, according to Theorem 4.6. It is subsequently uncomplicated to observe that$\phi = \psi^{-1}$; this concludes the proof [16].

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