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Characterization of Different Prime Bi-Ideals and Its Generalization of Semirings

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Abstract. We introduce three sequences of different prime bi-ideals of semirings such that 11(12,13)-prime bi-ideal, 21(22)-prime bi-ideal and 31(32,33)-prime bi-ideal using bi-ideals. In this article, we characterize the different prime bi-ideals. We discuss that the 11-prime bi-ideal implies the 12-prime bi-ideal implies the 13-prime bi-ideal, but the reverse implication does not hold with the help of numerical examples. We investigate if a 21-prime bi-ideal implies a 22-prime bi-ideal, but the converse need not be true with the help of numerical examples. If $\Game$ is any bi-ideal of a semiring $\mathcal{S}$, then $K(\Game) = \{x \in \Game \mid x + y = z \text{ for some } y, z \in \Game\}$ is the unique largest k-bi-ideal contained in $\Game$. If Θ is a 21-prime bi-ideal of $\mathcal{S}$, then Θ is a one-sided ideal of $\mathcal{S}$. It is shown that there is a relation between $\Game$ and $K(\Game)$, in which $\Game$ is a 13-prime bi-ideal. In our communication, 11-prime bi-ideal implies 21-prime bi-ideal. An interaction between a 31-prime bi-ideal implies a 32-prime bi-ideal, and a 32-prime bi-ideal implies a 33-prime bi-ideal; however, the reverse implication is invalid by some examples. Every 13-prime bi-ideal is a 22-prime bi-ideal, but the converse need not be true with the help of examples.

1. Introduction

The concept of semirings was introduced by Vandiver, an American mathematician, in 1934, while a German mathematician, Richard Dedekind, proposed non-trivial examples of semirings in the 19th century when he studied commutative IDs for rings. In addition to their applications in the foundations of arithmetic and topological considerations, semirings occur as ideals of rings and as positive cones of partially ordered rings. It primarily derives from applied mathematics, including optimization and formal language theories. In the 20th century, non-commutative

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rings became systematically studied. A matrix is also a non-commutative entity that occurs naturally. A fundamental contribution to the science of non-commutative rings was made by Scottish mathematician Wedderburn with Wedderburn's Theorem, which states that every finite division ring is commutative. During the 18th century, commutative and non-commutative ring theories were intertwined and impacted each other. Non-commutative rings provide a natural extension of the study of prime radicals and primary ideals for commutative rings. The IDs of rings and semirings have been studied in many studies. Associative rings are the conceptual basis of algebraic number theory by Dedekind. In general, semigroups are generalizations of rings and groups. In semigroup theory, certain band decompositions are useful for studying semigroup structure. This research will open up a new field in mathematics, which aims to use semigroups of bi-ideal semirings with additively reduced semilattices. The many different ideals associated with Γ-semigroups and Γ-semiring have been described by several authors and researchers. Partially ordered relation " $\leq$ " satisfies the conditions of reflexivity, antisymmetry, and transitivity. There are different classes of semigroup and Γ-semigroup based on bi-ideals described by many researchers. An ordered semigroup is a generalization of a semigroup with a partially ordered relation constructed on a semigroup so that the relation fits with the operation. An algebraic structure such as the ordered Γ-semigroup has been studied by several authors [4–6]. Munir [11] introduced new ideals in the form of M-bi-ideals over semigroups in 2018.

Lajos studied using quasi-ideal (QI) and generalized bi-ideal (BI) with regular and intra-regular semigroups. Describe different classes of semigroups using ideals [7]. Associative rings are, in some ways, arbitrary but specified in terms of BIs. A quasi-ideal (QI) is an extension of the left ideal (LI) and right ideal (RI), which are examples of BIs. Steinfeld introduced QIs when he introduced semigroups and rings. Alarcon [1] semirings are useful for explaining prime ideals (PID). Commutative rings have been extensively studied using the PID concept. Palanikumar et al. [13], distinct prime partial BIs exist in non-commutative partial rings. Numerous studies have described various forms of IDs in algebraic structures like semirings and rings. There is no commutative requirement for semirings under either operation in an ID concept. Several authors have studied semigroups, semirings, and rings. Palanikumar et al. [12] interacted with a new type of basis for an ordered Γ-semigroup. Recently, Palanikumar et al. [8–10, 16–18] discussed some algebraic structures such as semirings and ring semigroups. The k-ideals are a class of ideals introduced by Henriksen in [3]. Sen et al. [19, 20] discussed various characterizations of k-ideals of semirings. Bhuniya et al. [2] introduced the concept of k-bi-ideals in a semiring. The k-bi-ideals on semirings were defined by Bhuniya et al. in 2011 to describe k-regular and intra-k-regular semirings. Emmy Noether was the first to introduce the concept of a prime ideal in a commutative ring. Computer science, automata, optimization, and generalized fuzzy computation extensively use semirings. Palanikumar et al. addressed semigroups, semirings, rings, and ternary semirings in their recent work [14–17].

Throughout the paper, six sections are presented. An introduction is found in Section 1. Section 2 describes semirings, relevant definitions, and results. Section 3 discusses the 11(12,13)-prime bi-ideal, providing some examples. The 21(22)-prime bi-ideals are discussed in Section 4. The 31(32,33)-prime bi-ideals are discussed in Section 5 with numerical examples. A discussion of the conclusion can be found in Section 6.

2. Preliminaries

An overview of semirings and their basic concepts is provided in this study, which will be useful for future studies.

Definition 2.1. *A nonempty set $\mathcal{S}$ is said to be a semiring if*
(i) $(\mathcal{S},+)$ *is a commutative monoid,*
(ii) $(\mathcal{S},\cdot)$ *is a semigroup,*
(iii) $\epsilon(\gamma+\zeta)=\epsilon\gamma+\epsilon\zeta$ *and* $(\epsilon+\gamma)\zeta=\epsilon\zeta+\gamma\zeta$ *for all* $\epsilon,\gamma,\zeta\in\mathcal{S}$.

Definition 2.2. *[19] An additive subgroup Δ of a ring R is called an LI(RI) if $ri\in\Delta(ir\in\Delta)$ for all $i\in\Delta$ and $r\in R$. Δ is an ID if it is an RI and an LI.*

Definition 2.3. *(i) The subset Δ is a QI if Δ is a subring of a ring R and $\mathcal{S}\Delta\cap\Delta\mathcal{S}\subseteq\Delta$.*
(ii) The subset Δ is a BI if Δ is a subring of a ring R and $\Delta A\Delta\subseteq\Delta$.

Definition 2.4. *[19] A BI Δ in a semiring $\mathcal{S}$ is called a k-BI if for $i\in\Delta$ and $k\in\mathcal{S}$, $i+k\in\Delta$ imply $k\in\Delta$.*

Definition 2.5. *For a subset Δ of a semiring $\mathcal{S}$, k-closure of Δ is denoted by $\overline{\Delta}$ and is defined as $\overline{\Delta}=\{i\in\mathcal{S}\mid i+j\in\Delta \text{ for some } j\in\Delta\}$.*

Definition 2.6. *For any subsets Δ and J of a semiring $\mathcal{S}$, the product of Δ and J is defined as $\Delta J=\left\{\sum_{i=1}^{n}\epsilon_i\nu_i \mid \epsilon_i\in\Delta,\nu_i\in J \text{ and } n\in\mathbb{N}\right\}$.*

Definition 2.7. *An ID Θ of a ring R is a PID if $\Delta Z\subseteq\Theta$ implies $\Delta\subseteq\Theta$ or $Z\subseteq\Theta$ for IDs Δ and Z of R.*

Lemma 2.1. *For any nonempty subset A of a semiring $\mathcal{S}$ and $a\in A$,*
$<a>_r=\{na+a\mathcal{S}\mid n\in\mathbb{Z}^+\}$ *is an RI generated by a,*
$<a>_l=\{na+\mathcal{S}a\mid n\in\mathbb{Z}^+\}$ *is an LI generated by a,*
$<a>=\{na+\mathcal{S}a+a\mathcal{S}+\mathcal{S}a\mathcal{S}\mid n\in\mathbb{Z}^+\}$ *is an ID generated by a,*
$<a>_q=\{na\mid n\in\mathbb{Z}^+\}+(a\mathcal{S}\cap\mathcal{S}a)$ *is a QI generated by a,*
$<a>_b=\{na+ma^2+a\mathcal{S}a\mid n,m\in\mathbb{Z}^+\}$ *is a BI generated by a.*

Lemma 2.2. *[9] For any subsets Δ and Δ' of a semiring $\mathcal{S}$ which are closed under addition,*
(i) $\Delta+\Delta'\subseteq\Delta+\overline{\Delta'}\subseteq\overline{\Delta}+\overline{\Delta'}\subseteq\overline{\Delta+\Delta'}$,
(ii) $\Delta+\Delta'\subseteq\overline{\Delta}+\Delta'\subseteq\overline{\Delta}+\overline{\Delta'}\subseteq\overline{\Delta+\Delta'}$,
(iii) $\Delta\Delta'\subseteq\Delta\overline{\Delta'}\subseteq\overline{\Delta}\;\overline{\Delta'}\subseteq\overline{\Delta\Delta'}$,
(iv) $\Delta\Delta'\subseteq\overline{\Delta}\Delta'\subseteq\overline{\Delta}\;\overline{\Delta'}\subseteq\overline{\Delta\Delta'}$.

Definition 2.8. *[1] Let $2 \leq n \leq N$ and $0 \leq i < n$ and $m = n - i$. Let $B(n,i)\{0,1,2,...,n-1\}$ be the semiring with the operations defined as follows:*

$$\epsilon \ +_{B(n,i)} \ \gamma = \begin{cases} \epsilon + \gamma & \text{if } \epsilon + \gamma \leq n-1 \\ l & \text{if } \epsilon + \gamma \geq n \ \text{ with } l = (\epsilon + \gamma) \bmod m \text{ and } i \leq l \leq n-1 \end{cases}$$

$$\epsilon \ \cdot_{B(n,i)} \ \gamma = \begin{cases} \epsilon\gamma & \text{if } \epsilon\gamma \leq n-1 \\ l & \text{if } \epsilon\gamma \geq n \text{ with } l = \epsilon\gamma \bmod m \text{ and } i \leq l \leq n-1. \end{cases}$$

3. On various 1-PBIs

It is assumed that $\mathcal{S}$ denotes a semiring throughout this paper unless otherwise stated.

Definition 3.1. *A BI Θ in $\mathcal{S}$ is called*
(i) an 11-PBI if $\eth_1\eth_2 \subseteq \Theta$ implies $\eth_1 \subseteq \Theta$ or $\eth_2 \subseteq \Theta$ for BIs $\eth_1$ and $\eth_2$ of $\mathcal{S}$,
(ii) a 12-PBI if $\eth_1\eth_2 \subseteq \Theta$ implies $\eth_1 \subseteq \Theta$ or $\eth_2 \subseteq \Theta$ for a BI $\eth_1$ and a k-BI $\eth_2$ of $\mathcal{S}$,
(iii) a 13-PBI if $\eth_1\eth_2 \subseteq \Theta$ implies $\eth_1 \subseteq \Theta$ or $\eth_2 \subseteq \Theta$ for k-BIs $\eth_1$ and $\eth_2$ of $\mathcal{S}$.

Theorem 3.1. *Every 11-PBI is a 12-PBI.*

Proof. It is a direct result of a *k*-BI being a BI. □

Remark 3.1. *Some 12-PBIs fail to be an 11-PBI as shown in Example 3.1.*

Example 3.1. *Consider the semiring $(\mathcal{S}, +, \cdot)$ by the following table:*

$+$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f
ς_a	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f
ς_b	ς_b	ς_b	ς_c	ς_d	ς_e	ς_f
ς_c	ς_c	ς_c	ς_c	ς_f	ς_e	ς_f
ς_d	ς_d	ς_d	ς_f	ς_d	ς_e	ς_f
ς_e	ς_e	ς_e	ς_e	ς_e	ς_e	ς_e
ς_f	ς_f	ς_f	ς_f	ς_f	ς_e	ς_f

$\cdot$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f
ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_b	ς_a	ς_b	ς_c	ς_b	ς_c	ς_c
ς_c	ς_a	ς_b	ς_c	ς_b	ς_c	ς_c
ς_d	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e
ς_e	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e
ς_f	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e

Clearly, $\Theta = \{\varsigma_a, \varsigma_e\}$ is a 12-PBI of $\mathcal{S}$. But $\{\varsigma_a, \varsigma_d\}\{\varsigma_a, \varsigma_c\} = \{\varsigma_a, \varsigma_e\} \subseteq \Theta$ implies that Θ is not an 11-PBI of $\mathcal{S}$.

Theorem 3.2. *Every 12-PBI is a 13-PBI.*

Proof. It is a direct result of a *k*-BI being a BI. □

Remark 3.2. *Example 3.2 assures that 13-PBI is not a 12-PBI.*

Example 3.2. *Consider the semiring $(\mathcal{S}, +, \cdot)$ by the following table:*

$+$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_b	ς_b	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_c	ς_c	ς_c	ς_c	ς_f	ς_e	ς_f	ς_g
ς_d	ς_d	ς_d	ς_f	ς_d	ς_e	ς_f	ς_e
ς_e	ς_e	ς_e	ς_e	ς_e	ς_e	ς_e	ς_e
ς_f	ς_f	ς_f	ς_f	ς_f	ς_e	ς_f	ς_e
ς_g	ς_g	ς_g	ς_g	ς_e	ς_e	ς_e	ς_g

$\cdot$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_b	ς_a	ς_b	ς_c	ς_b	ς_c	ς_c	ς_c
ς_c	ς_a	ς_b	ς_c	ς_b	ς_c	ς_c	ς_c
ς_d	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e	ς_e
ς_e	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e	ς_e
ς_f	ς_a	ς_d	ς_e	ς_d	ς_e	ς_e	ς_e
ς_g	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g

Clearly, $\Theta = \{\varsigma_a, \varsigma_d\}$ is a 13-PBI of $\mathcal{S}$. But Θ is not a 12-PBI of $\mathcal{S}$ by $\{\varsigma_a, \varsigma_e\}\{\varsigma_a, \varsigma_b\} = \{\varsigma_a, \varsigma_d\} \subseteq \Theta$.

Theorem 3.3. *If $\Game$ is any BI of $\mathcal{S}$, then $K(\Game) = \{x \in \Game \mid x + y = z \text{ for some } y, z \in \Game\}$ is the unique largest k-BI contained in $\Game$.*

Proof. Let $\zeta_1, \zeta_2 \in K(\Game)$. Then $\zeta_1 + \xi_1 = \omega_1$ and $\zeta_2 + \xi_2 = \omega_2$ for some $\xi_1, \xi_2, \omega_1, \omega_2 \in \Game$. Now, $\zeta_1 + \xi_1 + \zeta_2 + \xi_2 = \omega_1 + \omega_2$ implies $(\zeta_1 + \zeta_2) + (\xi_1 + \xi_2) = \omega_1 + \omega_2$. Thus, $\zeta_1 + \zeta_2 \in K(\Game)$. Also, $(\zeta_1 + \xi_1)(\zeta_2 + \xi_2) = \omega_1\omega_2$ implies $\zeta_1\zeta_2 + \zeta_1\xi_2 + \xi_1\zeta_2 + \xi_1\xi_2 = \omega_1\omega_2$. Since $\zeta_1, \zeta_2, \xi_1, \xi_2 \in \Game$, we have $\zeta_1\xi_2, \xi_1\zeta_2, \xi_1\xi_2 \in \Game$ and $y' = \zeta_1\xi_2 + \xi_1\zeta_2 + \xi_1\xi_2 \in \Game$. Therefore, $\zeta_1\zeta_2 + y' = \omega_1\omega_2$. Thus, $\zeta_1\zeta_2 \in K(\Game)$. Hence, $K(\Game)$ is a subsemiring of $\mathcal{S}$. Let $a, d \in K(\Game)$ and $s \in \mathcal{S}$. Then $a + b = c$ and $d + e = f$ for some $b, c, e, f \in \Game$. Now, $(a + b)s(d + e) = csf$ implies $asd + (ase + bsd + bse) = csf$. Thus, $asd \in \Game$ for all $a, d \in \Game$ and $s \in \mathcal{S}$. Therefore, $K(\Game)$ is a BI of $\mathcal{S}$. Let $a \in K(\Game)$, $x \in \mathcal{S}$ and $x + a \in K(\Game)$. Then $a \in K(\Game) \subseteq \Game$ and $x + a \in K(\Game) \subseteq \Game$. Hence, $x \in K(\Game)$. Therefore, $K(\Game)$ is a k-BI of $\mathcal{S}$. Suppose that $\Game_1$ is any other k-BI of $\mathcal{S}$ which contained in $\Game$. Let $\nu_1 \in \Game_1 = \overline{\Game_1}$. Then $\nu_1 + b' = b''$ for some $b', b'' \in \Game_1$ and hence $\nu_1 \in K(\Game)$. Thus, $\Game_1 \subseteq K(\Game)$. Hence, $K(\Game)$ is the unique largest k-BI contained in $\Game$. □

Lemma 3.1. *If $\Game$ is a 13-PBI, then $K(\Game)$ is a 13-PBI.*

Proof. Let $\Game$ be a 13-PBI of $\mathcal{S}$. Suppose that $\Game_1\Game_2 \subseteq K(\Game)$ for k-BIs $\Game_1$ and $\Game_2$ of $\mathcal{S}$. Therefore, $\Game_1\Game_2 \subseteq K(\Game) \subseteq \Game$ implies $\Game_1 \subseteq \Game$ or $\Game_2 \subseteq \Game$. Thus, $\Game_1 \subseteq K(\Game)$ or $\Game_2 \subseteq K(\Game)$. Hence, $K(\Game)$ is a 13-PBI of $\mathcal{S}$. □

Theorem 3.4. *If $\Game$ is a 13-PBI, then $K(\Game)$ is an 11-PBI.*

Proof. Let $\Game$ be a 13-PBI of $\mathcal{S}$. Suppose that $\Game_1\Game_2 \subseteq K(\Game)$ for BIs $\Game_1$ and $\Game_2$ of $\mathcal{S}$. Now, $\Game_1\Game_2 \subseteq (\overline{\Game}_1)(\overline{\Game}_2) \subseteq \overline{\Game_1\Game_2} \subseteq \overline{K(\Game)} = K(\Game)$ implies $\overline{\Game}_1 \subseteq K(\Game)$ or $\overline{\Game}_2 \subseteq K(\Game)$. Hence, $K(\Game)$ is an 11-PBI of $\mathcal{S}$. □

Remark 3.3. *A 13-PBI $\Game$ is not sufficient for $K(\Game)$ to be 11-PBI.*

Example 3.3. *By Example 3.1, for $\Game = \{\varsigma_a, \varsigma_b\}$, $K(\Game) = \{\varsigma_a\}$ is an 11-PBI of $\mathcal{S}$. But $\Game$ is not a 13-PBI of $\mathcal{S}$ by $\{\varsigma_a, \varsigma_b, \varsigma_c\}\{\varsigma_a, \varsigma_b, \varsigma_d\} \subseteq \Game$, but $\{\varsigma_a, \varsigma_b, \varsigma_c\} \nsubseteq \Game$ and $\{\varsigma_a, \varsigma_b, \varsigma_d\} \nsubseteq \Game$.*

4. On various 2-PBIs

The two types of 2-PBIs, namely 21-PBI and 22-PBI, are introduced.

Definition 4.1. *A BI Θ in $\mathcal{S}$ is called*
(i) a 21-PBI if $a\mathcal{S}b \subseteq \Theta$ implies $a \in \Theta$ or $b \in \Theta$ for $a, b \in \mathcal{S}$,
(ii) a 22-PBI if $\overline{a\mathcal{S}b} \subseteq \Theta$ implies $a \in \Theta$ or $b \in \Theta$ for $a, b \in \mathcal{S}$.

Remark 4.1. *Example 4.1 guarantees that a 13-PBI differs from a 21-PBI.*

Example 4.1. *(i) $\Theta = \{\varsigma_a, \varsigma_c\}$ is a 13-PBI as in Example 3.2. But Θ is not a 21-PBI by $\varsigma_a\mathcal{S}\varsigma_e = \{\varsigma_a, \varsigma_c\} \subseteq \Theta$.*
(ii) Consider the semiring $\mathcal{S} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in B(2,1) \right\}$, where $B(2,1)$ is defined as in [1]. Clearly, $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a 21-PBI of $\mathcal{S}$.
Now, $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq \Theta$, hence Θ is not a 13-PBI of $\mathcal{S}$.

Theorem 4.1. *Every 11-PBI is a 21-PBI.*

Proof. Let $a\mathcal{S}b \subseteq \Theta$ for $a, b \in \mathcal{S}$. Now, $(a\mathcal{S})(\mathcal{S}b) \subseteq a\mathcal{S}b \subseteq \Theta$ implies $a\mathcal{S} \subseteq \Theta$ or $\mathcal{S}a \subseteq \Theta$. Suppose that $a\mathcal{S} \subseteq \Theta$. Then $< a >_b < b >_b \subseteq a\mathcal{S} \subseteq \Theta$ implies $a \in \Theta$. Similarly, $\mathcal{S}b \subseteq \Theta$ implies $b \in \Theta$. Therefore, Θ is a 21-PBI of $\mathcal{S}$. □

Remark 4.2. *Example 4.2 shows that there is a 21-PBI, which is not an 11-PBI.*

Example 4.2. *By Example 4.1 (ii), $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a 21-PBI of $\mathcal{S}$. Now, $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \Theta$, hence Θ is not an 11-PBI of $\mathcal{S}$.*

Theorem 4.2. *Every 21-PBI is a 22-PBI.*

Proof. Let $\overline{a\mathcal{S}b} \subseteq \Theta$ for $a, b \in \mathcal{S}$. Now, $a\mathcal{S}b \subseteq \overline{a\mathcal{S}b} \subseteq \Theta$ implies $a \subseteq \Theta$ or $b \subseteq \Theta$. Therefore, Θ is a 22-PBI of $\mathcal{S}$. □

Remark 4.3. *A 22-PBI is not a 21-PBI by Example 4.3.*

Example 4.3. *In Example 3.1, $\Theta = \{\varsigma_a, \varsigma_d\}$ is a 22-PBI of $\mathcal{S}$, but not a 21-PBI of $\mathcal{S}$ by $\varsigma_e\mathcal{S}\varsigma_b = \{\varsigma_a, \varsigma_d\} \subseteq \Theta$ with $\varsigma_e \notin \Theta$ and $\varsigma_b \notin \Theta$.*

Theorem 4.3. *Every 13-PBI is a 22-PBI.*

Proof. If there exists $\overline{a\mathcal{S}b} \subseteq \Theta$, but $a \notin \Theta$ and $b \notin \Theta$, then $a\mathcal{S}a \nsubseteq \Theta$ and $b\mathcal{S}b \nsubseteq \Theta$. Now, $(a\mathcal{S}a)(b\mathcal{S}b) \subseteq a\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}b \subseteq a\mathcal{S}b$ implies $(\overline{a\mathcal{S}a})(\overline{b\mathcal{S}b}) \subseteq \overline{a\mathcal{S}b} \subseteq \Theta$, contradicts $a\mathcal{S}a \nsubseteq \Theta$ and $b\mathcal{S}b \nsubseteq \Theta$. Hence, Θ is a 22-PBI of $\mathcal{S}$. □

Remark 4.4. *Disprove the converse of Theorem 4.3 by Example 4.4.*

Example 4.4. *By Example 4.1 (ii),* $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ *is a 22-PBI of* $\mathcal{S}$.
Now, $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \subseteq \Theta$, *hence* Θ *is not a 13-PBI of* $\mathcal{S}$.

Theorem 4.4. *For a BI* Θ, *the following statements are equivalent:*
(i) Θ *is a 21-PBI,*
(ii) $Q_1Q_2 \subseteq \Theta$ *implies* $Q_1 \subseteq \Theta$ *or* $Q_2 \subseteq \Theta$,
(iii) $QL \subseteq \Theta$ *implies* $Q \subseteq \Theta$ *or* $L \subseteq \Theta$,
(iv) $RQ \subseteq \Theta$ *implies* $R \subseteq \Theta$ *or* $Q \subseteq \Theta$,
(v) $RL \subseteq \Theta$ *implies* $R \subseteq \Theta$ *or* $L \subseteq \Theta$, *where* Q, Q_1, *and* Q_2 *are QIs,* R *is an RI, and* L *is an LI.*

Proof. To prove the equivalence, we are going to prove that
$(i) \implies (ii) \implies (iii) \implies (v) \implies (i)$ and $(ii) \implies (iv) \implies (v)$.
$(i) \implies (ii)$: Let $a\mathcal{S}b \subseteq \Theta$ for $a, b \in \mathcal{S}$. If there exist QIs Q_1 and Q_2 of $\mathcal{S}$ such that $Q_1Q_2 \subseteq \Theta$, but $Q_1 \nsubseteq \Theta$, then $a \in Q_1 \setminus \Theta$. For any $b \in Q_2$, $a\mathcal{S}b \subseteq < a >_q < b >_q \subseteq Q_1Q_2 \subseteq \Theta$ implies $b \in \Theta$. Thus, $Q_2 \subseteq \Theta$.
$(ii) \implies (iii), (iii) \implies (v), (ii) \implies (iv)$ and $(iv) \implies (v)$ are straightforward.
$(v) \implies (i)$: Suppose that $a\mathcal{S}b \subseteq \Theta$ for $a, b \in \mathcal{S}$. Now, $(a\mathcal{S})(\mathcal{S}b) \subseteq a\mathcal{S}b \subseteq \Theta$ implies $a\mathcal{S} \subseteq \Theta$ or $\mathcal{S}b \subseteq \Theta$. If $a\mathcal{S} \subseteq \Theta$, then

$$\begin{aligned} < a >_r < b >_l &= [\{na \mid n \in \mathbb{Z}^+\} + a\mathcal{S}][\{mb \mid m \in \mathbb{Z}^+\} + \mathcal{S}b] \\ &= namb + na\mathcal{S}b + a\mathcal{S}mb + a\mathcal{S}\mathcal{S}b \\ &\subseteq \{n'ab \mid n' \in \mathbb{Z}^+\} + a\mathcal{S}b \\ &\subseteq a\mathcal{S} \subseteq \Theta. \end{aligned}$$

Thus, $a \in \Theta$ or $b \in \Theta$. Similarly, if $\mathcal{S}b \subseteq \Theta$, then $< a >_r < b >_l \subseteq \mathcal{S}b \subseteq \Theta$. Thus, $a \in \Theta$ or $b \in \Theta$. □

Definition 4.2. *(i) An RI* Θ *in* $\mathcal{S}$ *is called a PRI if* $R_1R_2 \subseteq \Theta$ *implies* $R_1 \subseteq \Theta$ *or* $R_2 \subseteq \Theta$ *for RIs* R_1 *and* R_2 *in* $\mathcal{S}$.
(ii) An LI Θ *in* $\mathcal{S}$ *is called a PLI if* $L_1L_2 \subseteq \Theta$ *implies* $L_1 \subseteq \Theta$ *or* $L_2 \subseteq \Theta$ *for LIs* L_1 *and* L_2 *in* $\mathcal{S}$.

Lemma 4.1. *If* Θ *is a 21-PBI, then* Θ *is a one-sided ID.*

Proof. Let Θ be a 21-PBI of $\mathcal{S}$. Then $(\Theta\mathcal{S})(\mathcal{S}\Theta) \subseteq \Theta\mathcal{S}\Theta \subseteq \Theta$. Thus, $\Theta\mathcal{S} \subseteq \Theta$ or $\mathcal{S}\Theta \subseteq \Theta$. Hence, Θ is a one-sided ID of $\mathcal{S}$. □

Theorem 4.5. *If* Θ *is a 21-PBI, then* Θ *is a prime one-sided ID.*

Proof. By Lemma 4.1, we have Θ is a one-sided ID of $\mathcal{S}$. Suppose Θ is an RI of $\mathcal{S}$. If there exist RIs R_1 and R_2 of $\mathcal{S}$ such that $R_1R_2 \subseteq \Theta$, but $R_1 \nsubseteq \Theta$ then $a \in R_1 \setminus \Theta$. For any $b \in R_2$ and by Theorem 4.4, we get

$$\begin{aligned} < a >_r < b >_l &\subseteq R_1[mb + \mathcal{S}b] \\ &\subseteq R_1R_2 + R_1\mathcal{S}R_2 \\ &\subseteq R_1R_2 \\ &\subseteq \Theta \end{aligned}$$

implies $b \in \Theta$. Thus, $R_2 \subseteq \Theta$. Therefore, Θ is a PRI of $\mathcal{S}$. Similarly, if Θ is an LI of $\mathcal{S}$, then Θ is a PLI of $\mathcal{S}$. □

Remark 4.5. *(i) Theorem 4.5 and Example 4.5 contrast a 12-PBI from a 21-PBI of $\mathcal{S}$.*
(ii) There is a 22-PBI which differs from a 21-PBI of $\mathcal{S}$.

Example 4.5. *Example 3.1, $\{\varsigma_a, \varsigma_c\}$ is a 12-PBI of $\mathcal{S}$, but neither an RI nor an LI of $\mathcal{S}$.*
In Example 3.2, $\{\varsigma_a, \varsigma_b\}$ is a 22-PBI of $\mathcal{S}$, but neither an RI nor an LI of $\mathcal{S}$.

5. On various 3-PBIs

In this section, we introduce 31(32,33)-prime bi-ideals of semirings.

Definition 5.1. *A BI Θ in $\mathcal{S}$ is called*
(i) a 31-PBI if $\Lambda_1\Lambda_2 \subseteq \Theta$ implies $\Lambda_1 \subseteq \Theta$ or $\Lambda_2 \subseteq \Theta$ for IDs Λ_1 and Λ_2 in $\mathcal{S}$,
(ii) a 32-PBI if $\Lambda_1\Lambda_2 \subseteq \Theta$ implies $\Lambda_1 \subseteq \Theta$ or $\Lambda_2 \subseteq \Theta$ for an ID Λ_1 and a k-ID Λ_2 in $\mathcal{S}$,
(iii) a 33-PBI if $\Lambda_1\Lambda_2 \subseteq \Theta$ implies $\Lambda_1 \subseteq \Theta$ or $\Lambda_2 \subseteq \Theta$ for k-IDs Λ_1 and Λ_2 in $\mathcal{S}$.

Theorem 5.1. *Every 21-PBI is a 31-PBI.*

Proof. If there exist IDs Λ_1 and Λ_2 of $\mathcal{S}$ such that $\Lambda_1\Lambda_2 \subseteq \Theta$, but $\Lambda_1 \nsubseteq \Theta$. Then $a \in \Lambda_1 \setminus \Theta$. For any $b \in \Lambda_2$, $a\mathcal{S}b \subseteq < a >< b > \subseteq \Lambda_1\Lambda_2 \subseteq \Theta$ implies $b \in \Theta$. Thus, $\Lambda_2 \subseteq \Theta$. Therefore, Θ is a 31-PBI of $\mathcal{S}$. □

Remark 5.1. *Disprove the converse of Theorem 5.1 by Example 5.1.*

Example 5.1. *In Example 3.1, $\Theta = \{\varsigma_1, \varsigma_2\}$ is a 31-PBI of $\mathcal{S}$. But Θ is not a 21-PBI of $\mathcal{S}$ by $\varsigma_3\mathcal{S}\varsigma_4 = \{\varsigma_1, \varsigma_2\} \subseteq \Theta$.*

Theorem 5.2. *Every 31-PBI is a 32-PBI.*

Proof. Let I be a 31-PBI of $\mathcal{S}$. Suppose that $\Lambda_1\overline{\Lambda_2} \subseteq I$ for an ID Λ_1 and a k-ID Λ_2 of $\mathcal{S}$. Now, $\Lambda_1\Lambda_2 \subseteq \Lambda_1\overline{\Lambda_2} \subseteq I$ implies $\Lambda_1 \subseteq I$ or $\Lambda_2 \subseteq I$. Hence, I is a 32-PBI of $\mathcal{S}$. □

Remark 5.2. *A 32-PBI fails to be a 31-PBI by Example 5.2.*

Example 5.2. *Consider the semiring $(\mathcal{S}, +, \cdot)$ by the following table:*

$+$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_b	ς_b	ς_e	ς_f	ς_g	ς_e	ς_f	ς_g
ς_c	ς_c	ς_f	ς_g	ς_g	ς_f	ς_g	ς_g
ς_d	ς_d	ς_g	ς_g	ς_g	ς_g	ς_g	ς_g
ς_e	ς_e	ς_e	ς_f	ς_g	ς_e	ς_f	ς_g
ς_f	ς_f	ς_f	ς_g	ς_g	ς_f	ς_g	ς_g
ς_g	ς_g	ς_g	ς_g	ς_g	ς_g	ς_g	ς_g

$\cdot$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_b	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_c	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_d	ς_a	ς_e	ς_g	ς_g	ς_e	ς_g	ς_g
ς_e	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_f	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_g	ς_a	ς_e	ς_g	ς_g	ς_e	ς_g	ς_g

Clearly, $\Theta = \{\varsigma_a, \varsigma_b, \varsigma_e, \varsigma_f, \varsigma_g\}$ *is a 32-PBI of* $\mathcal{S}$*, but not a 31-PBI of* $\mathcal{S}$ *by* $\{\varsigma_a, \varsigma_b, \varsigma_d, \varsigma_e, \varsigma_g\}\{\varsigma_a, \varsigma_d, \varsigma_e, \varsigma_g\} = \{\varsigma_a, \varsigma_e, \varsigma_g\} \subseteq \Theta$ *with* $\{\varsigma_a, \varsigma_b, \varsigma_d, \varsigma_e, \varsigma_g\} \nsubseteq \Theta$ *and* $\{\varsigma_a, \varsigma_d, \varsigma_e, \varsigma_g\} \nsubseteq \Theta$*.*

Theorem 5.3. *Every 32-PBI is a 33-PBI.*

Proof. It is a direct result of a k-BI being a BI. □

Remark 5.3. *A 33-PBI fails to be a 32-PBI by Example 5.3.*

Example 5.3. *Consider the semiring* $(\mathcal{S}, +, \cdot)$ *by the following table:*

$+$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_b	ς_b	ς_b	ς_d	ς_d	ς_g	ς_g	ς_g
ς_c	ς_c	ς_d	ς_e	ς_g	ς_e	ς_f	ς_g
ς_d	ς_d	ς_d	ς_g	ς_g	ς_g	ς_g	ς_g
ς_e	ς_e	ς_g	ς_e	ς_g	ς_e	ς_f	ς_g
ς_f	ς_f	ς_g	ς_f	ς_g	ς_f	ς_g	ς_g
ς_g	ς_g	ς_g	ς_g	ς_g	ς_g	ς_g	ς_g

$\cdot$	ς_a	ς_b	ς_c	ς_d	ς_e	ς_f	ς_g
ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a	ς_a
ς_b	ς_a	ς_b	ς_a	ς_b	ς_a	ς_b	ς_b
ς_c	ς_a	ς_a	ς_c	ς_c	ς_e	ς_e	ς_e
ς_d	ς_a	ς_b	ς_c	ς_d	ς_e	ς_g	ς_g
ς_e	ς_a	ς_a	ς_e	ς_e	ς_e	ς_e	ς_e
ς_f	ς_a	ς_b	ς_e	ς_g	ς_e	ς_f	ς_g
ς_g	ς_a	ς_b	ς_e	ς_g	ς_e	ς_g	ς_g

Clearly, $\Theta = \{\varsigma_a, \varsigma_b, \varsigma_e, \varsigma_g\}$ *is a 33-PBI of* $\mathcal{S}$*. Now,* $\{\varsigma_a, \varsigma_b, \varsigma_e, \varsigma_f, \varsigma_g\}\{\varsigma_a, \varsigma_c, \varsigma_e\} = \{\varsigma_a, \varsigma_e\} \subseteq \Theta$*, but* $\{\varsigma_a, \varsigma_b, \varsigma_e, \varsigma_f, \varsigma_g\} \nsubseteq \Theta$ *and* $\{\varsigma_a, \varsigma_c, \varsigma_e\} \nsubseteq \Theta$*. Hence,* Θ *is not a 32-PBI of* $\mathcal{S}$*.*

Remark 5.4. *Example 5.4 guarantees that a 22-PBI differs from a 31-PBI.*

Example 5.4. *By Example 5.2,* $\Theta = \{\varsigma_a, \varsigma_b, \varsigma_e, \varsigma_f, \varsigma_g\}$ *is a 22-PBI of* $\mathcal{S}$*, but not a 31-PBI of* $\mathcal{S}$ *by* $\{\varsigma_a, \varsigma_b, \varsigma_d, \varsigma_e, \varsigma_g\}\{\varsigma_a, \varsigma_d, \varsigma_e, \varsigma_g\} = \{\varsigma_a, \varsigma_e, \varsigma_g\} \subseteq \Theta$ *with* $\{\varsigma_a, \varsigma_b, \varsigma_d, \varsigma_e, \varsigma_g\} \nsubseteq \Theta$ *and* $\{\varsigma_a, \varsigma_d, \varsigma_e, \varsigma_g\} \nsubseteq \Theta$*. By Example 4.1 (ii),* $\Theta = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ *is a 31-PBI of* $\mathcal{S}$*, but not a 22-PBI of* $\mathcal{S}$ *by* $\overline{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\mathcal{S}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \subseteq \Theta$ *and* $\overline{\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}} \subseteq \Theta$ *with* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin \Theta$ *and* $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin \Theta$*.*

Theorem 5.4. *For a BI* Θ*, then following statements are equivalent:*
(i) Θ *is a 31-PBI,*

(ii) $IR \subseteq \Theta$ *implies* $I \subseteq \Theta$ *or* $R \subseteq \Theta$,
(iii)$LI \subseteq \Theta$ *implies* $L \subseteq \Theta$ *or* $I \subseteq \Theta$, *where R is an RI, L is an LI, and I is an ID.*

Proof. $(i) \implies (ii)$: If there exist an ID I and an RI R of $\mathcal{S}$ such that $IR \subseteq \Theta$, but $R \nsubseteq \Theta$, then $b \in R \setminus \Theta$. For any $a \in I$, $< a >< b >\subseteq I[R + \mathcal{S}b + \mathcal{S}b\mathcal{S}] \subseteq IR \subseteq \Theta$ implies $a \in \Theta$. Thus, $I \subseteq \Theta$. Therefore, (ii) holds.
$(ii) \implies (iii)$: If there exist an LI L and an ID I of $\mathcal{S}$ such that $LI \subseteq \Theta$, but $L \nsubseteq \Theta$, then $a \in L \setminus \Theta$. For any $b \in I$, $< a >< b >_r\subseteq [L + a\mathcal{S} + \mathcal{S}a\mathcal{S}]I \subseteq LI \subseteq \Theta$ implies $b \in \Theta$. Thus, $I \subseteq \Theta$. Therefore, (iii) holds.
$(iii) \implies (i)$: If there exist IDs Λ_1 and Λ_2 of $\mathcal{S}$ such that $\Lambda_1\Lambda_2 \subseteq \Theta$, but $\Lambda_1 \nsubseteq \Theta$, then $a \in \Lambda_1 \setminus \Theta$. For any $b \in \Lambda_2$, $< a >_l< b >\subseteq \Lambda_1\Lambda_2 \subseteq \Theta$ implies $b \in \Theta$. Thus, $\Lambda_2 \subseteq \Theta$. Therefore, Θ is a 31-PBI of $\mathcal{S}$. □

Remark 5.5. *A 31-PBI is neither an RI nor an LI.*

Example 5.5. *By Example 4.1 (ii),* $\Theta = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$ *is a 31-PBI of* $\mathcal{S}$*, but neither an RI nor an LI of* $\mathcal{S}$.

Definition 5.2. *For any BI* $\eth$ *of* $\mathcal{S}$*, we define* $L(\eth) = \{x \in \eth \mid \mathcal{S}x \subseteq \eth\}$ *and* $H(\eth) = \{y \in L(\eth) \mid y\mathcal{S} \subseteq L(\eth)\}$. *Then* $H(\eth)$ *is the unique largest two-sided ID of* $\mathcal{S}$ *contained in* $\eth$.

Theorem 5.5. *A BI* $\eth$ *is a 31-PBI of* $\mathcal{S}$ *if and only if* $H(\eth)$ *is a PID of* $\mathcal{S}$.

Proof. Let a BI $\eth$ be a 31-PBI of $\mathcal{S}$. Suppose that $\Lambda_1\Lambda_2 \subseteq H(\eth)$ for IDs Λ_1 and Λ_2 and a 31-PBI $\eth$ of $\mathcal{S}$. Thus, $\Lambda_1\Lambda_2 \subseteq H(\eth) \subseteq \eth$ implies $\Lambda_1 \subseteq \eth$ or $\Lambda_2 \subseteq \eth$. Therefore, $\Lambda_1 \subseteq H(\eth)$ or $\Lambda_2 \subseteq H(\eth)$. Hence, $H(\eth)$ is a PID of $\mathcal{S}$.

Conversely, let $H(\eth)$ be a PID of $\mathcal{S}$. Suppose that $\Lambda_1\Lambda_2 \subseteq \eth$ for IDs Λ_1 and Λ_2 of $\mathcal{S}$. Then $\Lambda_1\Lambda_2 \subseteq H(\eth)$ implies $\Lambda_1 \subseteq H(\eth) \subseteq \eth$ or $\Lambda_2 \subseteq H(\eth) \subseteq \eth$. Hence, $\eth$ is a 31-PBI of $\mathcal{S}$. □

6. Conclusion

Many different PBIs of semirings are introduced in this paper. We introduced three sequences of different PBIs based on different BIs. These sequences include 11(12,13)-PBIs, 21(22)-PBIs, and 31(32,33)-PBIs. For example, an 11-PBI implied a 12-PBI implied a 13-PBI, but the reverse implication did not hold. A numerical example does not support the opposite implication that a 21-PBI implies a 22-PBI. Future research will focus on b-semirings, ternary semirings, and hyper semirings based on BIs, QIs and bi-quasi ideals. Future research will focus on b-semirings, ternary semirings and hyper semirings using various prime ideals and tri-ideals. We will develop semirings to b-semirings using various prime ideals and prime bi-ideals.

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