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Some Results on the Degree of Vertices of the Power Digraph and Its Complement

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Abstract. This work is based on the ideas of L. Somer and M. Krizek, On a connection of Number theory with Graph theory. In this work, we introduce the concept of Universal directed graph $\mathbf{U_n}$ and we also define the complement of the digraph $\Gamma(n,2)$. We study some relations between the digraph $\Gamma(n,2)$ and its complement digraph $\overline{\Gamma(n,2)}$ in terms of degree of a vertex and directed arcs. A result for the number of fixed points in the digraph $\overline{\Gamma(n,2)}$ is established. We also established some results on the degree of a vertex w. r. t. a subset of the vertex set of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$.

1. Introduction

In the last few years establishing the relationship between Graph theory, Group theory, and number theory became an interesting topic, for example, see [1–4, 6, 7, 9–12, 14]. In this article, let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ denote the complete set of residues modulo n, which forms a commutative ring under addition and multiplication modulo n. For each positive integer n, a power digraph modulo n denoted by $\Gamma(n,2)$ is a digraph with vertex set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and the ordered pair (x, y) is a directed arc of $\Gamma(n,2)$ from x to y if and only if $x^2 \equiv y(mod\ n)$, where $x, y \in \mathbb{Z}_n$. In [1, 3, 5, 8, 10–12] some properties of the digraph $\Gamma(n,2)$ were investigated.

In this paper, we define universal directed graph and complement of the digraph $\Gamma(n,2)$. We study some properties of the degree of a vertex and directed arcs of the digraph $\Gamma(n,2)$ and its complement digraph $\overline{\Gamma(n,2)}$. Also, we study the degree of a vertex w. r. t. a subset of the vertex set of the digraph $\Gamma(n,2)$ and its complement digraph. We organize our paper as follows:

In section 2, we provide some definitions and basic results. In section 3, we define universal directed graph and in section 4, we define the complement of the digraph $\Gamma(n,2)$ and establish

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some relations between the digraph $\Gamma(n,2)$ and its complement digraph $\overline{\Gamma(n,2)}$ using the definition of degree of a vertex and directed arcs. A result for the number of fixed points in the digraph $\overline{\Gamma(n,2)}$ is also established. Finally, in section 5, we established some results on the degree of a vertex w. r. t. a subset of the vertex set of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$.

2. Preliminaries

For a positive integer n, we consider a directed graph $\Gamma(n,2)$ whose vertex set is $\mathbb{Z}_n$ and any two vertices $x, y \in \mathbb{Z}_n$ are connected by exactly one directed arc from x to y iff

$$x^2 \equiv y(mod\ n).$$

We denote the vertex set and arc set of the digraph $\Gamma(n,2)$ by $V(\Gamma)$ $(= \mathbb{Z}_n)$ and $A(\Gamma)$ respectively. The distinct vertices $v_1, v_2, v_3, \ldots, v_t$ in $V(\Gamma)$ will form a cycle of length t if

$$\begin{aligned} {v_1}^2 &\equiv v_2(mod\ n) \\ {v_2}^2 &\equiv v_3(mod\ n) \\ {v_3}^2 &\equiv v_4(mod\ n) \\ &\vdots \\ {v_t}^2 &\equiv v_1(mod\ n) \end{aligned}$$

We call a cycle of length t as a t- cycle and a cycle of length 1 is named as a fixed point (or a self-loop). A vertex is isolated if it is not connected to any other vertex in $\Gamma(n,2)$.

Theorem 2.1. *[13](Szalay) The number of fixed points in $\Gamma(n,2)$ is $2^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct primes dividing n.*

The in-degree of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma^-(v)$ is the number of directed arcs incident into the vertex v and the out-degree of a vertex v, denoted by $d_\Gamma^+(v)$ is the number of directed arcs incident out of the vertex v. Since the residue of a number modulo n is unique, so $d_\Gamma^+(v) = 1$ and $d_\Gamma^-(v) \geq 0$ for each vertex $v \in V(\Gamma)$. Also, for an isolated fixed point $v \in V(\Gamma)$, $d_\Gamma^+(v) = d_\Gamma^-(v) = 1$. The total degree (or simply degree) of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma(v)$ is the sum of out-degree and in-degree of v i.e., $d_\Gamma(v) = d_\Gamma^+(v) + d_\Gamma^-(v)$.

If $d_\Gamma^+(v) = d_\Gamma^-(v)$ for every vertex $v \in V(\Gamma)$, then the digraph $\Gamma(n,2)$ is said to be an isodigraph (mod n) or balanced digraph (mod n) and if $d_\Gamma^+(v) = d_\Gamma^-(v) = k$ for every vertex $v \in V(\Gamma)$, then the digraph $\Gamma(n,2)$ is said to be a regular graph of degree k (or k-regular digraph).

A component of a digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph. As the outdegree of each vertex of the digraph $\Gamma(n,2)$ is equal to 1, so the number of components of $\Gamma(n,2)$ is equal to the number of all cycles. The cycles may or may not be isolated.

From definition of $\Gamma(n,2)$, it is clear that $|A(\Gamma)| = n$. Since, the number of arcs in a directed graph is equal to the number of their tails (or their heads), we have the following theorem.

Theorem 2.2. *[15] (Handshaking theorem) In the digraph $\Gamma(n,2)$,*

$$\sum_{v\in V(\Gamma)} d_{\Gamma}^{+}(v) = \sum_{v\in V(\Gamma)} d_{\Gamma}^{-}(v) = |A(\Gamma)|$$

A directed walk in a digraph D is an alternating sequence $v_1, e_1, v_2, e_2, v_3, \ldots, e_{n-1}, v_n$ of vertices and arcs in which each arc e_i is v_iv_{i+1}. A directed path is a walk in which all vertices are distinct. If there is a directed path from a vertex u to a vertex v, then v is said to be reachable from u.

In a digraph D, a semi-walk is an alternating sequence $v_1, e_1, v_2, e_2, v_3, \ldots, e_{n-1}, v_n$ of vertices and arcs in which each arc e_i may be either v_iv_{i+1} or $v_{i+1}v_i$. A semi-path is a semi-walk in which all vertices are distinct.

A digraph is strongly connected (or strong) if every two vertices are mutually reachable. A digraph is unilaterally connected (or unilateral) if for any two vertices at least one is reachable from the other. A digraph is weakly connected (or weak) if every two vertices are joined by a semi-path.

Every strongly connected (or strong) digraph is unilateral digraph and every unilateral digraph is weak. But the converse statements are not true.

A digraph is disconnected if it is not even weak.

Note 2.1. *From the definition of the digraph $\Gamma(n,2)$, it is clear that $\Gamma(n,2)$ is a disconnected graph, and the components of $\Gamma(n,2)$ are weakly connected.*

Definition 2.1. *[15] A simple digraph D =(V(D), A(D)) is said to be a Complete symmetric digraph (or simply complete) if both directed arcs uv and vu $\in A(D)$, for all $u, v \in V(D)$. It is denoted by K_n^*. The number of arcs in K_n^* is $n(n-1)$.*

3. Universal directed graph $\mathbf{U_n}$

Definition 3.1. *We define a Universal directed graph (or Universal digraph) as a complete symmetric digraph with self-loops at each vertex. We denote a universal directed graph having n vertices by $\mathbf{U_n}$.*

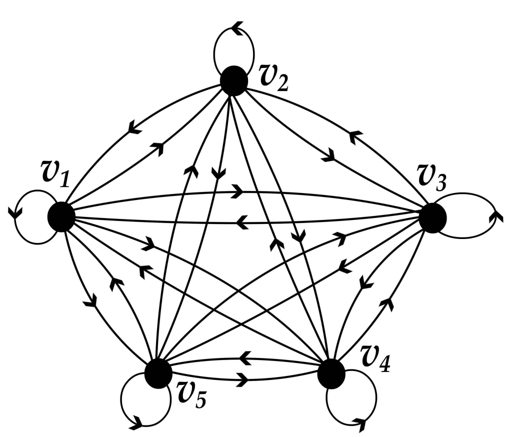

Figure 1. Universal directed graph $\mathbf{U_5}$

Some observations:

i. The number of vertices in $\mathbf{U_n}$ is n i.e. $|V(\mathbf{U_n})| = n$.
ii. The number of directed arcs in $\mathbf{U_n}$ is n^2 i.e. $|A(\mathbf{U_n})| = n^2$.
iii. The number of self-loops (or fixed points) in $\mathbf{U_n}$ is n.
iv. Indeg (v) = Outdeg $(v) = n$, for all $v \in V(\mathbf{U_n})$.
v. $\mathbf{U_n}$ is a balanced digraph.
vi. $\mathbf{U_n}$ is a n-regular digraph.
vii. $\mathbf{U_n}$ is a strongly connected digraph.

4. Complement digraph $\overline{\Gamma(n,2)}$

Definition 4.1. *We define the complement of the digraph $\Gamma(n,2)$ denoted by $\overline{\Gamma(n,2)}$ as the digraph having the same vertex set $V(\Gamma(n,2))$ as of $\Gamma(n,2)$ and there will be a directed arc from x to y in $\overline{\Gamma(n,2)}$ iff $x^2 \not\equiv y(mod\ n)$, where $x, y \in V(\Gamma(n,2))$.*

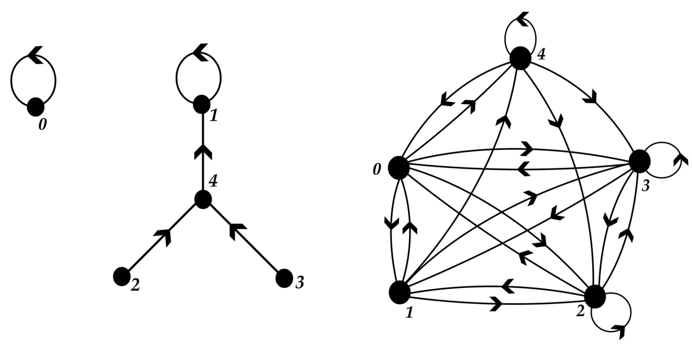

Figure 2. Digraph $\Gamma(5,2)$ and its complement digraph $\overline{\Gamma(5,2)}$

Some observations:

i. $V(\Gamma) = V(\overline{\Gamma})$
ii. $A(\overline{\Gamma}) = A(\mathbf{U_n}) - A(\Gamma)$
iii. $\Gamma(n,2) \cup \overline{\Gamma(n,2)} = \mathbf{U_n}$
iv. $|A(\Gamma)| + |A(\overline{\Gamma})| = |A(\mathbf{U_n})| = n^2$
v. $\overline{\Gamma(n,2)}$ is not necessarily a balanced digraph.

In the digraph $\overline{\Gamma(n,2)}$, we denote the in-degree, the out-degree, and the total degree (or degree) of a vertex $v \in V(\overline{\Gamma})$ by $d^-_{\overline{\Gamma}}(v)$, $d^+_{\overline{\Gamma}}(v)$ and $d_{\overline{\Gamma}}(v)$ respectively.

We now try to establish some results between the degree of a vertex of the digraph $\Gamma(n,2)$ and its complement digraph $\overline{\Gamma(n,2)}$. Also, we try to establish some results on the directed arcs of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$.

Theorem 4.1. $d^-_{\Gamma}(v) + d^-_{\overline{\Gamma}}(v) = n$, *for every vertex* $v \in V(\Gamma)$.

Proof. Let $d^-_\Gamma(v) = k$, then there is k number of directed arcs coming into v in Γ and so by definition of $\overline{\Gamma(n,2)}$, there will be $(n-k)$ number of arcs coming into v in $\overline{\Gamma(n,2)}$ giving $d^-_{\overline{\Gamma}}(v) = n-k$ Thus, $d^-_\Gamma(v) + d^-_{\overline{\Gamma}}(v) = k + (n-k) = n$. □

Theorem 4.2. *$d^+_\Gamma(v) + d^+_{\overline{\Gamma}}(v) = n$, for every vertex $v \in V(\Gamma)$.*

Proof. The proof is straightforward using the definition. □

Theorem 4.3. *$d_\Gamma(v) + d_{\overline{\Gamma}}(v) = 2n$, for every vertex $v \in V(\Gamma)$.*

Proof. we have,

$$\begin{aligned} d_\Gamma(v) + d_{\overline{\Gamma}}(v) &= (d^+_\Gamma(v) + d^-_\Gamma(v)) + (d^+_{\overline{\Gamma}}(v) + d^-_{\overline{\Gamma}}(v)) \\ &= (d^+_\Gamma(v) + d^+_{\overline{\Gamma}}(v)) + (d^-_\Gamma(v) + d^-_{\overline{\Gamma}}(v)) \\ &= n + n \text{ [By Theorem 4.1 and Theorem 4.2]} \\ &= 2n. \end{aligned}$$

□

Theorem 4.4. *In $\overline{\Gamma(n,2)}$, the outdegree of each vertex is $(n-1)$ i.e. $d^+_{\overline{\Gamma}}(v) = n-1$, for any $v \in V(\overline{\Gamma})$.*

Proof. In $\Gamma(n,2)$, we have $d^+_\Gamma(v) = 1$, for any $v \in V(\Gamma)$. Also, by using Theorem 4.2 we get

$$\begin{aligned} & d^+_\Gamma(v) + d^+_{\overline{\Gamma}}(v) = n \\ & \Rightarrow d^+_{\overline{\Gamma}}(v) = n - d^+_\Gamma(v) \\ & \Rightarrow d^+_{\overline{\Gamma}}(v) = n-1. \end{aligned}$$

□

Theorem 4.5. *The degree of a vertex of the graph $\overline{\Gamma(n,2)}$ can not exceed $(2n-1)$.*

Proof. For any vertex $v \in \Gamma(n,2)$, $d^+_\Gamma(v) = 1$ and $d^-_\Gamma(v) \geq 0$.
So, $d^+_{\overline{\Gamma}}(v) = n-1$ and $d^-_{\overline{\Gamma}}(v) \leq n$ and
we have,

$$\begin{aligned} d_{\overline{\Gamma}}(v) &= d^+_{\overline{\Gamma}}(v) + d^-_{\overline{\Gamma}}(v) \\ &\leq (n-1) + n \\ &= 2n-1 \end{aligned}$$

Thus, $d^+_\Gamma(v) \leq (2n-1)$. □

Theorem 4.6. *In the digraph $\Gamma(n,2)$, $\sum_{i=1}^n d_\Gamma(v_i) = 2n$; $v_i \in V(\Gamma)$.*

Proof. By the Handshaking theorem, we have

$$\sum_{i=1}^{n} d_{\Gamma}^{+}(v_i) = \sum_{i=1}^{n} d_{\Gamma}^{-}(v_i) = |A(\Gamma)|$$
$$\Rightarrow \sum_{i=1}^{n} d_{\Gamma}^{+}(v_i) + \sum_{i=1}^{n} d_{\Gamma}^{-}(v_i) = 2n, \text{where } |A(\Gamma)| = n$$
$$\Rightarrow \sum_{i=1}^{n} (d_{\Gamma}^{+}(v_i) + d_{\Gamma}^{-}(v_i)) = 2n$$
$$\Rightarrow \sum_{i=1}^{n} d_{\Gamma}(v_i) = 2n$$

□

Theorem 4.7. *The number of directed arcs in the digraph* $\overline{\Gamma(n,2)}$ *is* $n^2 - n$ *.*

Proof. Let, $v_1, v_2, v_3, \ldots, v_n \in V(\overline{\Gamma})$. By Handshaking theorem, we have

$$\sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) = \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = a, \text{ where} |A(\overline{\Gamma})| = a$$
$$\Rightarrow \sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) + \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = 2a$$
$$\Rightarrow \sum_{i=1}^{n} (d_{\overline{\Gamma}}^{+}(v_i) + d_{\overline{\Gamma}}^{-}(v_i)) = 2a$$
$$\Rightarrow \sum_{i=1}^{n} d_{\overline{\Gamma}}(v_i) = 2a$$
$$\Rightarrow \sum_{i=1}^{n} (2n - d_{\Gamma}(v_i)) = 2a \text{ [By Theorem 4.3]}$$
$$\Rightarrow n \cdot 2n - \sum_{i=1}^{n} d_{\Gamma}(v_i) = 2a$$
$$\Rightarrow 2n^2 - 2n = 2a \text{ [By Theorem 4.6]}$$
$$\Rightarrow a = n^2 - n$$
$$i.e.\ |A(\overline{\Gamma})| = n^2 - n.$$

□

Corollary 4.1. *In* $\overline{\Gamma(n,2)}, \sum_{i=1}^{n} d_{\overline{\Gamma}}(v_i) = 2(n^2 - n)$.

Corollary 4.2. $|A(\Gamma)| + |A(\overline{\Gamma})| = n^2$.

Theorem 4.8. *The number of fixed points in* $\overline{\Gamma(n,2)}$ *is* $n - 2^{\omega(n)}$.

Proof. By Theorem 2.1, the number of fixed points in $\Gamma(n,2)$ is $2^{\omega(n)}$. So, there are $n-2^{\omega(n)}$ number of points in $\Gamma(n,2)$ which are not fixed points. By definition of $\overline{\Gamma(n,2)}$, the points which are not fixed points in $\Gamma(n,2)$ are fixed points in $\overline{\Gamma(n,2)}$. Therefore, number of fixed points in $\overline{\Gamma(n,2)}$ is $n-2^{\omega(n)}$. □

Theorem 4.9. *The digraph $\overline{\Gamma(n,2)}$ is strongly connected.*

Proof. From the definition of the digraph $\Gamma(n,2)$, it is clear that the digraph $\Gamma(n,2)$ is disconnected and $V(\Gamma) = V(\overline{\Gamma})$. Let u and v be any two distinct vertices in $V(\overline{\Gamma})$. Then $u, v \in V(\Gamma)$. As the digraph $\Gamma(n,2)$ is disconnected so there must exist at least two components C_1 and C_2 (say) with the following two cases:

Case I: Suppose, u and v are in different components and let $u \in C_1$ & $v \in C_2$. Then arc $uv \notin A(\Gamma)$ and arc $vu \notin A(\Gamma)$. By definition of $\overline{\Gamma}$, we get arc $uv \in A(\overline{\Gamma})$ and arc $vu \in A(\overline{\Gamma})$, which means v is reachable from u and u is reachable from v in $\overline{\Gamma}$.

Case II: Suppose, u and v are in the same component and let $u, v \in C_1$. As $\Gamma(n,2)$ is disconnected so there must exist at least one vertex $w \in C_2$ such that arc uw & arc $wu \notin A(\Gamma)$ and arc vw & arc $wv \notin A(\Gamma)$. By definition of $\overline{\Gamma}$, we get arc uw & arc $wu \in A(\overline{\Gamma})$ and arc vw & arc $wv \in A(\overline{\Gamma})$. As arc uw & arc $wv \in A(\overline{\Gamma})$, so v is reachable from u in $\overline{\Gamma}$. Also, arc vw & arc $wu \in A(\overline{\Gamma})$, so u is reachable from v in $\overline{\Gamma}$.

Thus, any two vertices $u, v \in V(\overline{\Gamma})$ are reachable from one another, and hence the digraph $\overline{\Gamma(n,2)}$ is strongly connected. □

Corollary 4.3. *The digraph $\overline{\Gamma(n,2)}$ is unilaterally connected as well as weakly connected.*

Corollary 4.4. *There is no isolated fixed point in $\overline{\Gamma(n,2)}$.*

5. Degree with respect to a subset of the vertex set of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$

Let $S \subseteq V(\Gamma)$. The out-degree of any vertex $v \in V(\Gamma)$ of the digraph $\Gamma(n,2)$ with respect to S (denoted by $d^+_{\Gamma_S}(v)$) is the number of directed arcs coming from the vertex v into a vertex of S and the in-degree of any vertex $v \in V(\Gamma)$ of the digraph $\Gamma(n,2)$ with respect to S (denoted by $d^-_{\Gamma_S}(v)$) is the number of directed arcs coming from a vertex of S into the vertex v. The degree of any vertex $v \in V(\Gamma)$ of the digraph $\Gamma(n,2)$ with respect to S (denoted by $d_{\Gamma_S}(v)$) is the sum of the out-degree and in-degree of the vertex v w. r. t. S i.e., $d_{\Gamma_S}(v) = d^+_{\Gamma_S}(v) + d^-_{\Gamma_S}(v)$.

Note 5.1. *$d^+_{\Gamma_{\overline{S}}}(v), d^-_{\Gamma_{\overline{S}}}(v)$ and $d_{\Gamma_{\overline{S}}}(v)$ denotes the out-degree, in-degree and degree of the vertex $v \in V(\Gamma)$ w. r. t. the set $\overline{S} \subseteq V(\Gamma)$ in the digraph $\Gamma(n,2)$, where $\overline{S}$ is the complement of the set S. Similarly, $d^+_{\overline{\Gamma}_S}(v), d^-_{\overline{\Gamma}_S}(v)$ and $d_{\overline{\Gamma}_S}(v)$ denotes the out-degree, in-degree and degree of the vertex $v \in V(\overline{\Gamma})$ w. r. t. the set $S \subseteq V(\overline{\Gamma})$ in the complement digraph $\overline{\Gamma(n,2)}$ and $d^+_{\overline{\Gamma}_{\overline{S}}}(v), d^-_{\overline{\Gamma}_{\overline{S}}}(v)$ and $d_{\overline{\Gamma}_{\overline{S}}}(v)$ denotes the out-degree, in-degree and degree of the vertex $v \in V(\overline{\Gamma})$ w. r. t. the set $\overline{S} \subseteq V(\overline{\Gamma})$ in the digraph $\overline{\Gamma(n,2)}$.*

Remark 5.1. *If $S = \phi$, then $d^+_{\Gamma_S}(v) = 0, d^-_{\Gamma_S}(v) = 0, \forall v \in V(\Gamma)$.*

Remark 5.2. *If $S = V(\Gamma)$, then $d^+_{\Gamma_S}(v) = d^+_{\Gamma}(v), d^-_{\Gamma_S}(v) = d^-_{\Gamma}(v)$ and $d_{\Gamma_S}(v) = d_{\Gamma}(v), \forall v \in V(\Gamma)$.*

Example 5.1. *Consider the digraph $\Gamma(6,2)$:*

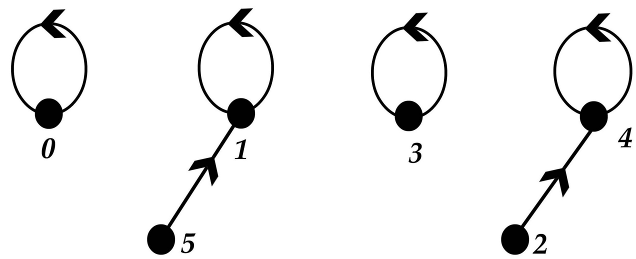

Figure 3. Digraph $\Gamma(6,2)$

Here, $V(\Gamma) = \{0,1,2,3,4,5\}$. Let $S = \{0,2,3\}$, then $S \subseteq V(\Gamma)$ .We have,

$$d^+_{\Gamma_S}(0) = 1, d^+_{\Gamma_S}(1) = 0, d^+_{\Gamma_S}(2) = 0, d^+_{\Gamma_S}(3) = 1, d^+_{\Gamma_S}(4) = 0, d^+_{\Gamma_S}(5) = 0$$
$$d^-_{\Gamma_S}(0) = 1, d^-_{\Gamma_S}(1) = 0, d^-_{\Gamma_S}(2) = 0, d^-_{\Gamma_S}(3) = 1, d^-_{\Gamma_S}(4) = 1, d^-_{\Gamma_S}(5) = 0$$
$$d_{\Gamma_S}(0) = 2, d_{\Gamma_S}(1) = 0, d_{\Gamma_S}(2) = 0, d_{\Gamma_S}(3) = 2, d_{\Gamma_S}(4) = 1, d_{\Gamma_S}(5) = 0$$

Also, $\overline{S} = \{1,4,5\}$, then $\overline{S} \subseteq V(\Gamma)$. We have,

$$d^+_{\Gamma_{\overline{S}}}(0) = 0, d^+_{\Gamma_{\overline{S}}}(1) = 1, d^+_{\Gamma_{\overline{S}}}(2) = 1, d^+_{\Gamma_{\overline{S}}}(3) = 0, d^+_{\Gamma_{\overline{S}}}(4) = 1, d^+_{\Gamma_{\overline{S}}}(5) = 1$$
$$d^-_{\Gamma_{\overline{S}}}(0) = 0, d^-_{\Gamma_{\overline{S}}}(1) = 2, d^-_{\Gamma_{\overline{S}}}(2) = 0, d^-_{\Gamma_{\overline{S}}}(3) = 0, d^-_{\Gamma_{\overline{S}}}(4) = 1, d^-_{\Gamma_{\overline{S}}}(5) = 0$$
$$d_{\Gamma_{\overline{S}}}(0) = 0, d_{\Gamma_{\overline{S}}}(1) = 3, d_{\Gamma_{\overline{S}}}(2) = 1, d_{\Gamma_{\overline{S}}}(3) = 0, d_{\Gamma_{\overline{S}}}(4) = 2, d_{\Gamma_{\overline{S}}}(5) = 1.$$

Example 5.2. *Consider the digraph $\overline{\Gamma(6,2)}$:*

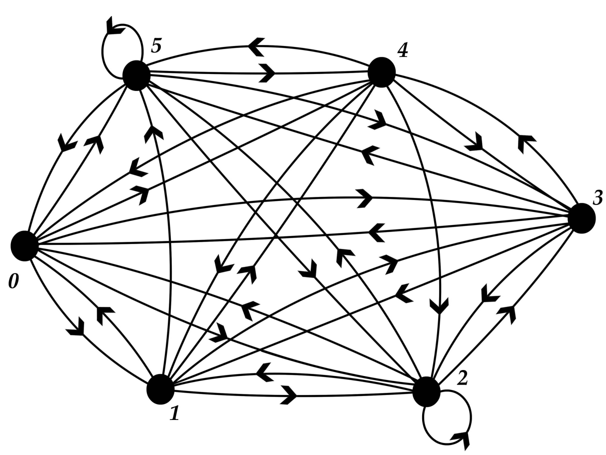

Figure 4. Complement digraph $\overline{\Gamma(6,2)}$

Here, $V(\overline{\Gamma}) = \{0, 1, 2, 3, 4, 5\}$. Let $S = \{0, 2, 3\}$, then $S \subseteq V(\overline{\Gamma})$. We have,

$$d^+_{\overline{\Gamma}_S}(0) = 2, d^+_{\overline{\Gamma}_S}(1) = 3, d^+_{\overline{\Gamma}_S}(2) = 3, d^+_{\overline{\Gamma}_S}(3) = 2, d^+_{\overline{\Gamma}_S}(4) = 3, d^+_{\overline{\Gamma}_S}(5) = 3$$
$$d^-_{\overline{\Gamma}_S}(0) = 2, d^-_{\overline{\Gamma}_S}(1) = 3, d^-_{\overline{\Gamma}_S}(2) = 3, d^-_{\overline{\Gamma}_S}(3) = 2, d^-_{\overline{\Gamma}_S}(4) = 2, d^-_{\overline{\Gamma}_S}(5) = 3$$
$$d_{\overline{\Gamma}_S}(0) = 4, d_{\overline{\Gamma}_S}(1) = 6, d_{\overline{\Gamma}_S}(2) = 6, d_{\overline{\Gamma}_S}(3) = 4, d_{\overline{\Gamma}_S}(4) = 5, d_{\overline{\Gamma}_S}(5) = 6.$$

Also, $\overline{S} = \{1, 4, 5\}$, then $\overline{S} \subseteq V(\overline{\Gamma})$. We have,

$$d^+_{\overline{\Gamma}_{\overline{S}}}(0) = 3, d^+_{\overline{\Gamma}_{\overline{S}}}(1) = 2, d^+_{\overline{\Gamma}_{\overline{S}}}(2) = 2, d^+_{\overline{\Gamma}_{\overline{S}}}(3) = 3, d^+_{\overline{\Gamma}_{\overline{S}}}(4) = 2, d^+_{\overline{\Gamma}_{\overline{S}}}(5) = 2$$
$$d^-_{\overline{\Gamma}_{\overline{S}}}(0) = 3, d^-_{\overline{\Gamma}_{\overline{S}}}(1) = 1, d^-_{\overline{\Gamma}_{\overline{S}}}(2) = 3, d^-_{\overline{\Gamma}_{\overline{S}}}(3) = 3, d^-_{\overline{\Gamma}_{\overline{S}}}(4) = 2, d^-_{\overline{\Gamma}_{\overline{S}}}(5) = 3$$
$$d_{\overline{\Gamma}_{\overline{S}}}(0) = 6, d_{\overline{\Gamma}_{\overline{S}}}(1) = 3, d_{\overline{\Gamma}_{\overline{S}}}(2) = 5, d_{\overline{\Gamma}_{\overline{S}}}(3) = 6, d_{\overline{\Gamma}_{\overline{S}}}(4) = 4, d_{\overline{\Gamma}_{\overline{S}}}(5) = 5$$

The following results on the degree of a vertex w. r. t. a subset of the vertex set $V(\Gamma)$ can be established easily using the definition.

Theorem 5.1. *For any vertex set $S \subseteq V(\Gamma)$,*

(i) $d^+_{\Gamma_S}(v) \leq d^+_{\Gamma}(v)$
(ii) $d^-_{\Gamma_S}(v) \leq d^-_{\Gamma}(v)$
(iii) $d_{\Gamma_S}(v) \leq d_{\Gamma}(v)$, $\forall v \in V$

Theorem 5.2. *For any vertex set $S \subseteq V(\Gamma)$,*

(i) $d^+_{\Gamma}(v) = d^+_{\Gamma_S}(v) + d^+_{\Gamma_{\overline{S}}}(v)$
(ii) $d^-_{\Gamma}(v) = d^-_{\Gamma_S}(v) + d^-_{\Gamma_{\overline{S}}}(v)$
(iii) $d_{\Gamma}(v) = d_{\Gamma_S}(v) + d_{\Gamma_{\overline{S}}}(v)$, $\forall v \in V$
where $\overline{S}$ is the complement of the set S.

Theorem 5.3. *For any vertex set $S \subseteq V(\Gamma)$,*

(i) $\sum_{v\in\ V(\Gamma)} d^+_{\Gamma_S}(v) = \sum_{v\in S} d^+_{\Gamma_S}(v) + \sum_{v\in\overline{S}} d^+_{\Gamma_S}(v)$
(ii) $\sum_{v\in\ V(\Gamma)} d^-_{\Gamma_S}(v) = \sum_{v\in S} d^-_{\Gamma_S}(v) + \sum_{v\in\overline{S}} d^-_{\Gamma_S}(v)$
(iii) $\sum_{v\in\ V(\Gamma)} d_{\Gamma_S}(v) = \sum_{v\in S} d_{\Gamma_S}(v) + \sum_{v\in\overline{S}} d_{\Gamma_S}(v)$

Theorem 5.4. *For any vertex set $S \subseteq V(\Gamma)$,*

(i) $\sum_{v\in S} d^+_{\Gamma_{\overline{S}}}(v) = \sum_{v\in\overline{S}} d^-_{\Gamma_S}(v)$
(ii) $\sum_{v\in S} d^-_{\Gamma_{\overline{S}}}(v) = \sum_{v\in\overline{S}} d^+_{\Gamma_S}(v)$
(iii) $\sum_{v\in S} d_{\Gamma_{\overline{S}}}(v) = \sum_{v\in\overline{S}} d_{\Gamma_S}(v)$

We now try to establish some results related to the definition of the degree of a vertex w. r. t. a subset of the vertex set $V(\Gamma)$.

Theorem 5.5. *In the digraph $\Gamma(n, 2)$, for any vertex set $S \subseteq V(\Gamma)$*

$$\sum_{v\in\ V(\Gamma)} d_{\Gamma_S}(v) = \sum_{v\in S} d_{\Gamma}(v)$$

Proof. We have,

$$\begin{aligned}
\sum_{v\in V(\Gamma)} d_{\Gamma_S}(v) &= \sum_{v\in V(\Gamma)} (d^+_{\Gamma_S}(v) + d^-_{\Gamma_S}(v)) \ [\because d_{\Gamma_S}(v) = d^+_{\Gamma_S}(v) + d^-_{\Gamma_S}(v)] \\
&= \sum_{v\in V(\Gamma)} d^+_{\Gamma_S}(v) + \sum_{v\in V(\Gamma)} d^-_{\Gamma_S}(v) \\
&= \Big(\sum_{v\in S} d^+_{\Gamma_S}(v) + \sum_{v\in \overline{S}} d^+_{\Gamma_S}(v)\Big) + \Big(\sum_{v\in S} d^-_{\Gamma_S}(v) + \sum_{v\in \overline{S}} d^-_{\Gamma_S}(v)\Big) \ \text{[By Theorem 5.3]} \\
&= \Big(\sum_{v\in S} d^+_{\Gamma_S}(v) + \sum_{v\in S} d^+_{\Gamma_{\overline{S}}}(v)\Big) + \Big(\sum_{v\in S} d^-_{\Gamma_S}(v) + \sum_{v\in S} d^-_{\Gamma_{\overline{S}}}(v)\Big) \ \text{[By Theorem 5.4]} \\
&= \Big(\sum_{v\in S} d^+_{\Gamma_S}(v) + \sum_{v\in S} d^-_{\Gamma_S}(v)\Big) + \Big(\sum_{v\in S} d^+_{\Gamma_{\overline{S}}}(v) + \sum_{v\in S} d^-_{\Gamma_{\overline{S}}}(v)\Big) \\
&= \sum_{v\in S} \Big(d^+_{\Gamma_S}(v) + d^-_{\Gamma_S}(v)\Big) + \sum_{v\in S} \Big(d^+_{\Gamma_{\overline{S}}}(v) + d^-_{\Gamma_{\overline{S}}}(v)\Big) \\
&= \sum_{v\in S} d_{\Gamma_S}(v) + \sum_{v\in S} d_{\Gamma_{\overline{S}}}(v) \\
&= \sum_{v\in S} \Big(d_{\Gamma_S}(v) + d_{\Gamma_{\overline{S}}}(v)\Big) \\
&= \sum_{v\in S} d_\Gamma(v) \ \text{[By Theorem 5.2]}
\end{aligned}$$

□

Theorem 5.6. *In the complement digraph $\overline{\Gamma(n,2)}$, for any vertex set $S \subseteq V(\overline{\Gamma})$*

$$\sum_{v\in V(\overline{\Gamma})} d_{\overline{\Gamma}_S}(v) = \sum_{v\in S} d_{\overline{\Gamma}}(v)$$

Proof. This theorem can be proved in the same way as we have proved Theorem 5.5. □

Theorem 5.7. *In the digraph $\Gamma(n,2)$, for any two sets $S, T \subseteq V(\Gamma)$*

$$\sum_{v\in V(\Gamma)} d_{\Gamma_{S\cup T}}(v) = \sum_{v\in V(\Gamma)} d_{\Gamma_S}(v) + \sum_{v\in V(\Gamma)} d_{\Gamma_T}(v) - \sum_{v\in V(\Gamma)} d_{\Gamma_{S\cap T}}(v)$$

Proof. We have,

$$\begin{aligned}
\sum_{v\in V(\Gamma)} d_{\Gamma_{S\cup T}}(v) &= \sum_{v\in S\cup T} d_\Gamma(v) \ \text{[By Theorem 5.5]} \\
&= \sum_{v\in S} d_\Gamma(v) + \sum_{v\in T} d_\Gamma(v) - \sum_{v\in S\cap T} d_\Gamma(v) \ [\because |S\cup T| = |S| + |T| - |S\cap T|] \\
&= \sum_{v\in V(\Gamma)} d_{\Gamma_S}(v) + \sum_{v\in V(\Gamma)} d_{\Gamma_T}(v) - \sum_{v\in V(\Gamma)} d_{\Gamma_{S\cap T}}(v) \ \text{[By Theorem 5.5]}
\end{aligned}$$

□

Note 5.2. *If $S \cap T = \phi$, then*

$$\sum_{v \in V(\Gamma)} d_{\Gamma_{S\cup T}}(v) = \sum_{v \in V(\Gamma)} d_{\Gamma_S}(v) + \sum_{v \in V(\Gamma)} d_{\Gamma_T}(v)$$

Theorem 5.8. *In the complement digraph $\overline{\Gamma(n,2)}$, for any two sets $S, T \subseteq V(\overline{\Gamma})$*

$$\sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S\cup T}}(v) = \sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_S}(v) + \sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_T}(v) - \sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S\cap T}}(v)$$

Proof. This theorem can be proved in the same way as we have proved Theorem 5.7. □

Theorem 5.9. *In the digraph $\Gamma(n,2)$, for any two sets $S, T \subseteq V(\Gamma)$*

$$\sum_{v \in V(\Gamma)} d_{\Gamma_{S-T}}(v) = \sum_{v \in V(\Gamma)} d_{\Gamma_S}(v) - \sum_{v \in V(\Gamma)} d_{\Gamma_{S\cap T}}(v)$$

Proof. We have,

$$\begin{aligned}
\sum_{v \in V(\Gamma)} d_{\Gamma_{S-T}}(v) &= \sum_{v \in S-T} d_{\Gamma}(v) \text{ [By Theorem 5.5]} \\
&= \sum_{v \in S} d_{\Gamma}(v) - \sum_{v \in S\cap T} d_{\Gamma}(v) \ (\because |S-T| = |S| - |S\cap T|) \\
&= \sum_{v \in V(\Gamma)} d_{\Gamma_S}(v) - \sum_{v \in V(\Gamma)} d_{\Gamma_{S\cap T}}(v) \text{ [By Theorem 5.5]}
\end{aligned}$$

□

Theorem 5.10. *In the complement digraph $\overline{\Gamma(n,2)}$, for any two sets $S, T \subseteq V(\overline{\Gamma})$*

$$\sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S-T}}(v) = \sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_S}(v) - \sum_{v \in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S\cap T}}(v)$$

Proof. This theorem can be proved in the same way as we have proved Theorem 5.9. □

Theorem 5.11. *In the digraph $\Gamma(n,2)$, for any two sets $S, T \subseteq V(\Gamma)$*

$$\sum_{v \in V(\Gamma)} d_{\Gamma_{S\Delta T}}(v) = \sum_{v \in V(\Gamma)} d_{\Gamma_S}(v) + \sum_{v \in V(\Gamma)} d_{\Gamma_T}(v) - 2\sum_{v \in V(\Gamma)} d_{\Gamma_{S\cap T}}(v)$$

Proof. We have,

$$\begin{aligned}
\sum_{v \in V(\Gamma)} d_{\Gamma_{S\Delta T}}(v) &= \sum_{v \in S\Delta T} d_{\Gamma}(v) \text{ [By Theorem 5.5]} \\
&= \sum_{v \in S-T} d_{\Gamma}(v) + \sum_{v \in T-S} d_{\Gamma}(v) \ [\because |S\Delta T| = |S-T| + |T-S|] \\
&= (\sum_{v \in S} d_{\Gamma}(v) - \sum_{v \in S\cap T} d_{\Gamma}(v)) + (\sum_{v \in T} d_{\Gamma}(v) - \sum_{v \in S\cap T} d_{\Gamma}(v)) \text{ [By Theorem 5.9]} \\
&= \sum_{v \in S} d_{\Gamma}(v) + \sum_{v \in T} d_{\Gamma}(v) - 2\sum_{v \in S\cap T} d_{\Gamma}(v) \\
&= \sum_{v \in V(\Gamma)} d_{\Gamma_S}(v) + \sum_{v \in V(\Gamma)} d_{\Gamma_T}(v) - 2\sum_{v \in V(\Gamma)} d_{\Gamma_{S\cap T}}(v) \text{ [By Theorem 5.5]}
\end{aligned}$$

□

Theorem 5.12. *In the digraph $\overline{\Gamma(n,2)}$, for any two sets $S, T \subseteq V(\overline{\Gamma})$*

$$\sum_{v\in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S\Delta T}}(v) = \sum_{v\in V(\overline{\Gamma})} d_{\overline{\Gamma}_S}(v) + \sum_{v\in V(\overline{\Gamma})} d_{\overline{\Gamma}_T}(v) - 2\sum_{v\in V(\overline{\Gamma})} d_{\overline{\Gamma}_{S\cap T}}(v)$$

Proof. This theorem can be proved in the same way as we have proved Theorem 5.11. □

6. Conclusions

In this paper, we have defined the Universal directed graph $\mathbf{U_n}$, and the complement digraph $\overline{\Gamma(n,2)}$ of the digraph $\Gamma(n,2)$. We have studied the structure of $\overline{\Gamma(n,2)}$ and established some results on the degree of a vertex and directed arcs of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$. Additionally, we have established a formula for the number of fixed points in the digraph $\overline{\Gamma(n,2)}$ and proved that the digraph $\overline{\Gamma(n,2)}$ is strongly connected. Moreover, we have obtained some results on the degree of a vertex w. r. t. a subset of the vertex set of the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$.

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