Int. J. Anal. Appl. (2024), 22:108

International Journal of Analysis and Applications

Geometrical Analysis of Spacelike and Timelike Rectifying Curves and Their Applications

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Abstract. In the light of great importance of curves and their frames in many different branches of science, especially differential geometry as well as geometric properties and their uses in various fields, we are interested here to study a special kind of curves called rectifying curves. We consider some characterizations of a non-lightlike curve has a spacelike or timelike rectifying plane in pseudo-Euclidean space $\mathbb{E}_1^3$. Then, we demonstrate that the proportion of curvatures of any spacelike or timelike rectifying curve is a non-constant linear function of the arc length parameter s. Finally, we defray a computational example to support our main findings.

1. Introduction

The curves and their frames assume a significant part in differential geometry, and in many parts of science like mechanics and physical science, we are intrigued here with regards to concentrating on one of these curves which has numerous applications in computer-aided design, and numerical demonstrating. Additionally, these curves can be utilized in the discrete model also, as identical models are generally taken on for the design and mechanical investigation of grid structures [1–5]. In the differential geometry of a regular curve in $\mathbb{E}^3$, it is notable that one of the significant issues is the characterization of a regular curve. Space curves that have a property that their position vector generally lies in their rectifying plane are called rectifying curves [6, 7]. The notion of rectifying curves was presented by B. Y. Chen in [1]. One of the most interesting characteristics of such curves is that the ratio between the torsion and curvature of the curve is a non-constant linear function. Kinematically, rectifying curves are these curves whose instantaneous axis of rotation always passes through a fixed point. In this work, we use vector differential equations established by means of Serret-Frenet equations to give some characterizations of the above mentioned classes of curves that lie fully in $\mathbb{E}_1^3$. The paper can be organized as follows: Section 2 presents the basic

Received: May 26, 2024.

2010 *Mathematics Subject Classification.* 53C50, 53C40.

Key words and phrases. spacelike curve; timelike curve; position vector; rectifying curve.

https://doi.org/10.28924/2291-8639-22-2024-108
ISSN: 2291-8639

concepts. In Section 3, we characterize non–null (spacelike and timelike) rectifying curves lying fully in $\mathbb{E}_1^3$. The computational examples are given in Section 4. We conclude the work in Section 5.

2. Geometric Properties

To meet the prerequisites in the following segments, the essential components of the hypothesis of curves in $\mathbb{E}_1^3$ are momentarily introduced. There exists a tremendous writing regarding the matter including a few monographs (see for instance [3,5,6]).

Suppose that the metric in $\mathbb{E}_1^3$ is expressed as follows:

$$\langle da,da\rangle = da_1^2 + da_2^2 - da_3^2\,,$$

where $(a_1, a_2, a_3) \in \mathbb{E}^3$. A vector a of $\mathbb{E}_1^3$ is supposed to be spacelike if $\langle a,a\rangle > 0$ or $a = 0$, timelike if $\langle a,a\rangle < 0$ and lightlike if $\langle a,a\rangle = 0$ and $a \neq 0$. A timelike or lightlike vector in $\mathbb{E}_1^3$ is supposed to be causal. For $a \in \mathbb{E}_1^3$, the norm is characterized by $\|a\| = \sqrt{|\langle a,a\rangle|}$, then a is a spacelike if $\langle a,a\rangle = 1$ and a timelike if $\langle a,a\rangle = -1$. For $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ of $\mathbb{E}_1^3$, the scalar product is $\langle a,b\rangle = a_1b_1 + a_2b_2 - a_3b_3$, and cross product is $a \times b = ((a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), -(a_1b_2 - a_2b_1))$ [8–11].

Let $C : r = r(s)$ be a non-null curve in $\mathbb{E}_1^3$, which has a non-null rectifying plane. By $\sigma_1(s)$ and $\sigma_2(s)$, we mean the natural curvature and torsion of C, individually. Consider the Serret-Frenet frame $\{e_1(s), e_2(s), e_3(s)\}$ related with C, then the Frenet equations are:

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 & 0 \\ -\epsilon_0\epsilon_1\sigma_1 & 0 & \sigma_2 \\ 0 & -\epsilon_1\epsilon_2\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \; ' = \frac{d}{ds}, \tag{2.1}$$

where $\epsilon_0 = \langle e_1, e_1\rangle = \pm 1$, $\epsilon_1 = \langle e_2, e_2\rangle = \pm 1$, $\epsilon_2 = \langle e_3, e_3\rangle = \pm 1$, $\epsilon_0\epsilon_1\epsilon_2 = -1$.

Let q be a fixed point in $\mathbb{E}_1^3$. Then

$$\mathbb{S}_1^2(1) = \{a \in \mathbb{E}_1^3 \mid \left\|a - q\right\|^2 = 1\}, \tag{2.2}$$

is the Lorentzian unit sphere and

$$\mathbb{H}_+^2(1) = \{a \in \mathbb{E}_1^3 \mid \left\|a - q\right\|^2 = -1\}, \tag{2.3}$$

is the hyperbolic unit sphere.

3. Geometrical analysis of non-null rectifying curves

In this section, we characterize non-null (spacelike and timelike) rectifying curves lying fully in $\mathbb{E}_1^3$. We demonstrate that the proportion of curvature and torsion of any spacelike or timelike curve is a non-constant linear function of the arc length s. Likewise, we accentuate that this property is invariant regarding the causal character of a curve and its rectifying plane.

Proposition 3.1. *Let $C : r = r(s)$ be a spacelike or timelike curve with non-null rectifying plane in $\mathbb{E}_1^3$ and $\sigma_1(s) > 0$. The next statements are equivalent:*
(1) *There is a point $q \in \mathbb{E}_1^3$ such that every non-null rectifying plane of C goes through q.*
(2) *σ_2/σ_1 has the form $\delta_1 s + \delta_2$; δ_1, δ_2 are not all constants.*
(3) *There is a point $q_0 \in \mathbb{E}_1^3$ such that $\left\|r(s) - q_0\right\|^2 = \left|\epsilon_0(s+\delta_3)^2 + \epsilon_2\delta_4^2\right|$. The constants are related by*

$$\delta_1 = \frac{\epsilon_0}{\delta_4},\ \delta_2 = \frac{\epsilon_0\delta_3}{\delta_4};\ \delta_4 \neq 0. \tag{3.1}$$

Furthermore, by the uniqueness of q, it is equivalent to q_0.

Proof. ***(1)*** Assume that every non-null rectifying plane of C passes through $q \in \mathbb{E}_1^3$. Then we have

$$\langle r(s) - q, e_2(s)\rangle = 0. \tag{3.2}$$

Differentiating Eq. (3.2) and using Eqs. (2.1), we obtain

$$\langle r(s) - q, -\epsilon_2\sigma_1 e_1 + \sigma_2 e_3\rangle = 0. \tag{3.3}$$

From Eqs. (3.2) and (3.3), it leads to, the rectifying plane is perpendicular to e_2 and $-\epsilon_2\sigma_1 e_1(s) + \sigma_2 e_3$. Thus, we find

$$r(s) - q = \eta(s)(\epsilon_0\sigma_2 e_1 - \sigma_1 e_3), \tag{3.4}$$

where $\eta = \eta(s)$ is any differentiable function.
Differentiating Eq. (3.3), we get

$$-\epsilon_1\ \sigma_1 + \langle r(s) - q, \epsilon_2\sigma_1^{'} e_1 + \sigma_2^{'} e_3\rangle = 0. \tag{3.5}$$

From Eqs. (3.4) and (3.5), we obtain

$$\eta = \frac{\sigma_1\epsilon_0}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}. \tag{3.6}$$

Substituting Eq. (3.6) in Eq. (3.4), we have

$$r(s) - q = \left(\frac{\sigma_1\sigma_2}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)e_1 - \left(\frac{\sigma_1^2\epsilon_0}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)e_3. \tag{3.7}$$

Differentiating Eq. (3.6), we get

$$\frac{dq}{ds} = (1 - \left(\frac{\sigma_1\sigma_2}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)^{'})e_1 + \left(\frac{\sigma_1^2\epsilon_0}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)^{'} e_3. \tag{3.8}$$

Therefore, the coefficients of Eq. (3.6) vanishing identically if

$$1 - \left(\frac{\sigma_1\sigma_2}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)^{'} = 0,\ \left(\frac{\sigma_1^2\epsilon_0}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2}\right)^{'} = 0,$$

whereby

$$\frac{\sigma_1\sigma_2}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2} = s + \delta_3,\quad \frac{\sigma_1^2\epsilon_0}{\sigma_2^{'}\sigma_1 - \sigma_1^{'}\sigma_2} = \delta_4,\ \ \delta_3 \in \mathbb{R}. \tag{3.9}$$

Since $\sigma_1(s) \neq 0$, then $\delta_4 \neq 0$.
From Eqs. (3.6) and (3.8), we have

$$\left.\begin{array}{l} r(s) - q = (s + \delta_3)e_1 - \delta_4 e_3, \\ \frac{\sigma_2}{\sigma_1} = \delta_1 s + \delta_2; \ \ \delta_1 = \frac{\epsilon_0}{\delta_4}, \ \delta_2 = \frac{\epsilon_0 \delta_3}{\delta_4}, \ \delta_4 \neq 0. \end{array}\right\} \tag{3.10}$$

Therefore, we can calculate

$$\left\| r(s) - q_0 \right\|^2 = \left| \epsilon_0 (s + \delta_3)^2 + \epsilon_2 \delta_4^2 \right|. \tag{3.11}$$

From which, we obtain: ***(1)*** leads to ***(2)*** and ***(3)***. If every non-null rectifying plane passes through another point q_0, then assume that $\gamma(t)$ is the geodesic line through q_0 and q. So, for each $t \in \mathbb{R}$, we have

$$r(s) - \gamma(t) = (s + \delta_3(t))e_1(s) + \delta_4(t)e_3(s). \tag{3.12}$$

where $\delta_3(t)$ and $\delta_4(t)$ are non-zero functions.
If dot denotes the differentiation with respect to t, then from Eq. (3.11) we obtain

$$-\dot{\gamma}(t) = \dot{\delta_3}(t))e_1(s) + \dot{\delta_4}(t)e_3(s). \tag{3.13}$$

From Eqs. (3.9), we have

$$\frac{\sigma_2}{\sigma_1} = \delta_1(t)s + \delta_2(t); \ \delta_1(t) = \frac{\epsilon_0}{\delta_4(t)}, \ \delta_2(t) = \frac{\epsilon_0 \delta_3(t)}{\delta_4(t)}.$$

Therefore, by differentiating with respect to t, we get

$$\dot{\delta_1}(t)s + \dot{\delta_2}(t) = 0.$$

Thus, we have

$$\dot{\delta_1}(t) = \dot{\delta_2}(t) = 0 \Rightarrow \dot{\gamma}(t) = 0, \ q = q_0.$$

(2) Suppose that $\frac{\sigma_2}{\sigma_1} = \delta_1 s + \delta_2; \ \delta_1 \neq 0$. If we let

$$q = r(s) - \left(s + \frac{\delta_2}{\delta_1}\right)e_1 + \left(\frac{\epsilon_0}{\delta_1}\right)e_3,$$

therefore, we find $q' = 0$. Thus q is a fixed point in $\mathbb{E}_1^3$ and

$$r(s) - q = (s + \delta_3)e_1 + \delta_4 e_3, \ \delta_4 = \frac{\epsilon_0}{\delta_1}, \ \delta_3 = \frac{\delta_2}{\delta_1}.$$

This leads to ***(2)*** implies to ***(1)*** and ***(3)***.

Now assume that ***(3)*** holds, therefore

$$\langle r(s) - q, e_1 \rangle = \epsilon_0 (s + \delta_3). \tag{3.14}$$

Differentiating Eq. (3.13) and using Eqs. (2.1), we obtain

$$\sigma_1(s)\langle r(s) - q, e_2 \rangle = 0; \ \ \sigma_1(s) \neq 0 \Rightarrow \langle r(s) - q, e_2 \rangle = 0,$$

and this implies that each rectifying plane of C passes through $q \in \mathbb{E}_1^3$. Hence, this shows that ***(3)*** implies ***(1)*** and the theorem is proved. □

Now, we characterize a spacelike or timelike rectifying curve as far as its radial projection. For this purpose, suppose that C is a spacelike with a non-null principal normal in $\mathbb{E}_1^3$. Therefore, $\epsilon_0 = 1$, $\epsilon_1 = -\epsilon_2 = \epsilon = \pm 1$. For $q \in \mathbb{E}_1^3$, and by using Proposition 1, assume that

$$\zeta(s) = \frac{1}{r(s)}(r(s) - q); \quad r(s) = \left\| r(s) - q \right\| = \sqrt{\left|(s+\delta_3)^2 - \epsilon\delta_4^2\right|}, \tag{3.15}$$

is the radial projection of a spacelike curve with a spacelike or timelike principal normal of $r(s)$ into the unit sphere $S_1^2(1)$ (hyperbolic space $H_+^2(1)$). Then, we have the following

$$\left.\begin{array}{l} e_1(s) = r'(s)\zeta(s) + r(s)\zeta'(s), \\ \sigma_1(s)e_2(s) = r''(s)\zeta(s) + 2r'(s)\zeta'(s) + r(s)\zeta''(s). \end{array}\right\} \tag{3.16}$$

Theorem 3.1. *Let $C : r = r(s)$ be a spacelike curve with a spacelike or timelike principal normal in $\mathbb{E}_1^3$ and $\sigma_1(s) > 0$. If $q \in \mathbb{E}_1^3$ is a fixed point, then*

(1) *$(r(s_\zeta) - q)$ is a spacelike position vector lying in a spacelike rectifying plane if and only if, up to a parametrization, $(r(s_\zeta) - q)$ is obtained as*

$$(r(s_\zeta) - q) = \left(\frac{\delta_4}{\cos s_\zeta}\right)\zeta(s_\zeta), \tag{3.17}$$

where $\zeta(s_\zeta)$ is a spacelike curve lying in $\mathbb{S}_1^2(1)$ and

$$(\sigma_1)_\zeta^2 = \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2 - 1,$$

holds.

(2) *$(r(s_\zeta) - q)$ is a spacelike lying in a timelike rectifying plane if and only if, up to a parametrization, $(r(s_\zeta) - q)$ is found as*

$$(r(s_\zeta) - q) = \left(\frac{\delta_4}{\sinh s_\zeta}\right)\zeta(s_\zeta), \tag{3.18}$$

where $\zeta(s_\zeta)$ is a timelike curve lying in $\mathbb{S}_1^2(1)$ and

$$(\sigma_1)_\zeta^2 = \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2 + 1,$$

holds.

(3) *$(r(s_\zeta) - q)$ is a timelike lying fully in a timelike rectifying plane if and only if, up to a parametrization, $(r(s_\zeta) - q)$ is obtained as*

$$(r(s_\zeta) - q) = \left(\frac{\delta_4}{\cosh_\zeta}\right)\zeta(s_\zeta), \tag{3.19}$$

where $\zeta(s_\zeta)$ is a spacelike curve lying in $\mathbb{H}_+^2(1)$;

$$(\sigma_1)_\zeta^2 = \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2 - 1,$$

holds.

Here $\sigma_{1\zeta}$ denotes the geodesic curvature of the curve $\zeta = \zeta(s)$.

Proof. Assume that $\zeta(s)$ is lying fully in $\mathbb{S}_1^2(1)$. Then, we can write

$$\langle\zeta(s),\zeta(s)\rangle = 1,\ \langle\zeta(s),\zeta'(s)\rangle = 0,\ \ r^2(s)= (s+\delta_3)^2 - \epsilon\delta_4^2. \tag{3.20}$$

From Eqs. (19) and (23), we get

$$\left\|\zeta'(s)\right\|^2 = -\frac{\epsilon\delta_4^2}{r^4}, \tag{3.21}$$

which leads to, $\zeta(s)$ is a non-null curve if $\epsilon = -1$ or $\epsilon = 1$, respectively. Thus, without loss of generality, we may assume that $\delta_4 > 0$ and apply a translation with respect to s, such that the arc length of $\zeta(s)$ is

$$s_\zeta := \int \left\|\zeta'(s)\right\| ds = \int \left(\frac{\delta_4}{s^2 - \epsilon\delta_4^2}\right) ds. \tag{3.22}$$

(i) On account of $\epsilon = -1$, the position vector $(r(s_\zeta) - q)$ is lying in a spacelike rectifying plane. From Eq. (3.21), since we have $s_\zeta = \delta_4 \tan s_\zeta$, we obtain $r = \delta_4 \sec s_\zeta$. Substituting in (17), we obtain Eq. (3.16).

Conversely, assume that $(r(s_\zeta) - q)$ is given by Eq. (3.16), where $\zeta(s_\zeta)$ is unit speed spacelike curve lying on $S_1^2(1)$. If we calculate the derivative of Eq. (3.16), we have

$$(r(s_\zeta) - q)' = \zeta(s_\zeta)\sin s_\zeta + \frac{d\zeta(s_\zeta)}{ds_\zeta}\cos\zeta(s_\zeta). \tag{3.23}$$

By the assumption, $\left\|\zeta(s)\right\| = \left\|\frac{d\zeta(s_\zeta)}{ds_\zeta}\right\| = 1$ and from Eqs. (3.16) and (3.22), we obtain

$$\left.\begin{aligned} &\left\|(r(s_\zeta) - q)'\right\|^2 = \sin^2 s_\zeta + \cos^2 s_\zeta = 1, \\ &\langle (r(s_\zeta) - q)', r(s_\zeta) - q\rangle = \delta_4 \tan s_\zeta. \end{aligned}\right\} \tag{3.24}$$

Let us write

$$(r(s_\zeta) - q) = \mu(s)(r(s_\zeta) - q)' + (r(s_\zeta) - q)^\perp,$$

for any function $\mu(s)$, where $(r(s_\zeta) - q)^\perp$ is the normal component of $(r(s_\zeta) - q)$. Thus, in the light of Eqs. (3.22) and (3.23), we easily find that

$$\mu(s) = \frac{\langle (r(s_\zeta) - q)', r(s_\zeta) - q\rangle}{\left\|(r(s_\zeta) - q)'\right\|^2} = \delta_4 \tan s_\zeta.$$

Therefore, we have

$$\left\|(r(s_\zeta) - q)^\perp\right\|^2 = \left\|(r(s_\zeta) - q)\right\|^2 - \mu^2(s)\left\|(r(s_\zeta) - q)'\right\|^2 = \delta_4^2 = \delta onst,$$

which means that $(r(s_\zeta) - q)$ is lying fully in a spacelike rectifying plane.

We now calculate the curvature of $\zeta(s)$. By the assumption and using Eqs. (3.19) and (3.20), we have

$$\langle\zeta'(s),\zeta''(s)\rangle = -\frac{2\delta_4^2 r'}{r^5},\ \langle\zeta(s),\zeta''(s)\rangle = -\frac{\delta_4^2}{r^4}. \tag{3.25}$$

Since

$$\zeta'(s) = \left\|\zeta'(s)\right\| e_{1\zeta}(s),\ \left\|\zeta'(s)\right\| = \frac{\delta_4}{r^2},$$

we also have

$$\zeta''(s) = \left(\frac{\delta_4^2}{r^4}\right)\sigma_{1\zeta}(s)e_{2\zeta}(s) - 2\left(\frac{r'\delta_4}{r^3}\right)e_{1\zeta}(s) \Rightarrow \left\|\zeta''(s)\right\|^2 = \left(\frac{4\delta_4^2 r'^2}{r^6}\right) - \left(\frac{\delta_4^4}{r^8}\right)(\sigma_1)_\zeta^2. \tag{3.26}$$

Therefore, by using Eqs. (3.24) and (3.25) in Eqs. (3.12), we can calculate

$$\sigma_1^2 := \left\|r''(s)\right\|^2 = \frac{\delta_4^4}{r^6}(1 + (\sigma_1)_\zeta^2),$$

as it is claimed.

(2) The confirmation is practically equivalent to the proof of the statement ***(1)***. Moreover, that's what a comparable to contentions show that

$$\langle\zeta'(s), \zeta''(s)\rangle = \left(\frac{2\delta_4^2 r'}{r^5}\right),\ \langle\zeta(s), \zeta''(s)\rangle = \frac{\delta_4^2}{r^4},\ \left\|\zeta''(s)\right\|^2 = -\left(\frac{4\delta_4^2 r'}{r^6}\right) + \left(\frac{\delta_4^4}{r^8}\right)(\sigma_1)_\zeta^2, \tag{3.27}$$

therefore, we get

$$(\sigma_1)_\zeta^2 = \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2 + 1.$$

Because $\zeta(s_\zeta)$ is lying fully in $\mathbb{H}_+^2(1)$. Then, we can write

$$\langle\zeta(s), \zeta(s)\rangle = -1,\ \langle\zeta(s), \zeta'(s)\rangle = 0,\ \ r^2(s) = -(s+\delta_3)^2 + \epsilon\delta_4^2. \tag{3.28}$$

Also, we get

$$\left\|\zeta'(s)\right\|^2 = \frac{\epsilon\delta_4^2}{r^4}, \tag{3.29}$$

which leads to $\zeta(s)$ is a non-null curve if $\epsilon = 1 (= -1)$.

(3) On account of $\epsilon = 1$, $(\ r(s_\zeta) - q)$ is lying in a timelike rectifying plane. By a similar arguments as in Case ***(1)***, we have

$$s_\zeta := \int \left(\frac{\delta_4}{\delta_4^2 - s^2}\right) ds;\ |s| < \delta_4, \tag{3.30}$$

and therefore

$$r = \frac{\delta_4}{\cosh_\zeta}. \tag{3.31}$$

Substituting in Eq. (3.14), we obtain Eq. (3.18).

Conversely, assume that $(r(s_\zeta) - q)$ is given by Eq. (22) where $\zeta(s_\zeta)$ is a spacelike curve lying on $H_+^2(1)$. If we calculate the derivative of Eq. (3.18), we get

$$(r(s_\zeta) - q)' = -\zeta(s_\zeta)\sinh s_\zeta + \frac{d\zeta(s_\zeta)}{ds_\zeta}\cosh\zeta(s_\zeta). \tag{3.32}$$

By the assumption $\left\|\zeta(s)\right\| = -1$, $\left\|\frac{d\zeta(s_\zeta)}{ds_\zeta}\right\| = 1$ and from Eqs. (3.18) and (3.31), it follows that

$$\left.\begin{array}{l} \left\|(r(s_\zeta) - q)'\right\|^2 = -\sinh^2 s_\zeta + \cosh^2 s_\zeta = 1, \\ \langle(r(s_\zeta) - q)', r(s_\zeta) - q\rangle = \delta_4 \tanh s_\zeta. \end{array}\right\} \tag{3.33}$$

By a similar procedure as in Case ***(1)***, we obtain

$$\left\|(r(s_\zeta) - q)^\perp\right\| = |\delta_4| = const,$$

which means that $(r(s_\zeta)-q)$ is lying in a timelike rectifying plane.
Finally, if we repeat the above discussion as in Case ***(1)***, we obtain $(\sigma_1)^2_\zeta = \left(\frac{r^6}{\delta_4^4}\sigma_1^2 - 1\right)$ as well. Thus, the proof of the theorem is completed. □

According to the Case ***(1)*** in Theorem 2, we have the following proposition:

Proposition 3.2. *The arc length of the unit speed spacelike curve $\zeta(s)$ lying fully in $\mathbb{S}^2_1(1)$ is less than π.*

Proof. If (s_1, s_2) be the domain of $r(s)$, then the pseudo arc length of $\zeta(s)$ satisfies the following:

$$s_\zeta = \int_{s_1}^{s_2} \left\|\zeta'(s)\right\| ds = \tan^{-1}\left(\frac{s_2+\delta_3}{|\delta_4|}\right) - \tan^{-1}\left(\frac{s_1+\delta_3}{|\delta_4|}\right) < \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Hence, the result is clear. □

Proposition 3.3. *Let $r=r(s)$ be a timelike curve in $\mathbb{E}^3_1$ and $\sigma_1(s)>0$. Then, for a fixed point $q \in \mathbb{E}^3_1$, we have the following:*
(1) *$(r(s_\zeta)-q)$ is a spacelike lying in a timelike rectifying plane if and only if, up to a parametrization, $(r(s_\zeta)-q)$ is found as*

$$r(s_\zeta) - q = \left(\frac{\delta_4}{\sinh s_\zeta}\right)\zeta(s_\zeta), \tag{3.34}$$

where $\zeta(s_\zeta)$ is a timelike curve lying on $\mathbb{S}^2_1(1)$ and

$$(\sigma_1)^2_\zeta = 1 - \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2,$$

holds.
(2) *$(r(s_\zeta)-q)$ is a timelike lying in a timelike rectifying plane if and only if, up to a parametrization, $(r(s_\zeta)-q)$ is obtained as*

$$(r(s_\zeta) - q) = \left(\frac{\delta_4}{\cosh s_\zeta}\right)\zeta(s_\zeta), \tag{3.35}$$

where $\zeta(s_\zeta)$ is a spacelike curve lying in $\mathbb{H}^2_+(1)$ and

$$(\sigma_1)^2_\zeta = 1 - \left(\frac{r^6}{\delta_4^4}\right)\sigma_1^2,$$

holds.

4. Application

In this section, we give an example as an application of spacelike and timelike slant helices in Minkowski 3-space E^3_1 and illustrate their pictures by using Mathematica program.
Let us first consider the following parametric representation of a spacelike slant helix of E^3_1:

$$r_1(u) = \begin{pmatrix} \frac{15}{136}\sin(17u), \\ \frac{25}{144}\sin(9u) + \frac{9}{400}\sin(25u), \\ \frac{25}{144}\cos(9u) - \frac{9}{400}\cos(25u) \end{pmatrix}.$$

The tangent, normal, and binormal vector fields of $r_1(u)$ are computed as follows:

$$\begin{aligned} e_1 &= \frac{1}{\mu_1}\left(30\cos(17u), 25\cos(9u) + 9\cos(25u), -25\sin(9u) + 9\sin(25u)\right); \\ \mu_1 &= \sqrt{450\cos(16u) + 625\cos(18u) + 9(50 + 50\cos(34u) + 9\cos(50u))}, \end{aligned}$$

$$\begin{aligned} e_2 &= \frac{1}{\mu_2}\left(625\cos(18u) - 81\cos(50u), 480\cos(8u), 30(25\sin(26u) + 9\sin(42u))\right); \\ \mu_2 &= \left(\begin{array}{c} 450\cos(16u) - 625\cos(18u) - 9(-50 + 50\cos(34u) + 9\cos(50u)) \\ 450\cos(16u) + 625\cos(18u) + 9(50 + 50\cos(34u) + 9\cos(50u)) \end{array}\right)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} e_3 &= \frac{1}{2\mu_3}\left(\begin{array}{c} -60\sin(17u), \\ (34\cos(8u) - 25\cos(26u) - 9\cos(42u))\csc(17u), \\ \csc(17u)(-34\sin(8u) - 25\sin(26u) + 9\sin(42u)) \end{array}\right); \\ \mu_3 &= \sqrt{450\cos(16u) - 625\cos(18u) - 9(-50 + 50\cos(34u) + 9\cos(50u))}, \end{aligned}$$

and its curvatures are obtained as

$$\begin{aligned} \sigma_1 &= \frac{3840\sqrt{450\cos(16u) - 625\cos(18u) - 9(-50 + 50\cos(34u) + 9\cos(50u))}\sin(17u)}{(450\cos(16u) + 625\cos(18u) + 9(50 + 50\cos(34u) + 9\cos(50u)))^{3/2}}, \\ \sigma_2 &= \frac{3840\cos(17u)}{-450\cos(16u) + 625\cos(18u) + 9(-50 + 50\cos(34u) + 9\cos(50u))}. \end{aligned}$$

The position vector of this curve is a spacelike vector, and it lies on a pseudo-sphere of E_1^3 (see Fig. 1).

Now, we consider a timelike slant helix parameterized by:

$$r_2(u) = \left(\begin{array}{c} \frac{25}{144}\sinh(9u) + \frac{9}{400}\sinh(25u), \\ -\frac{25}{144}\cosh(9u) + \frac{9}{400}\cosh(25u), \\ \frac{15}{136}\sinh(17u) \end{array}\right).$$

The trihedron vectors of $r_2(u)$ are computed as follows:

$$\begin{aligned} e_1 &= \frac{1}{v_1}\left(25\cosh(9u) + 9\cosh(25u), -25\sinh(9u) + 9\sinh(25u), 30\cosh(17u)\right); \\ v_1 &= \sqrt{450\cosh(16u) - 900\cosh(17u)^2 + 625\cosh(18u) + 81\cosh(50u)} \end{aligned}$$

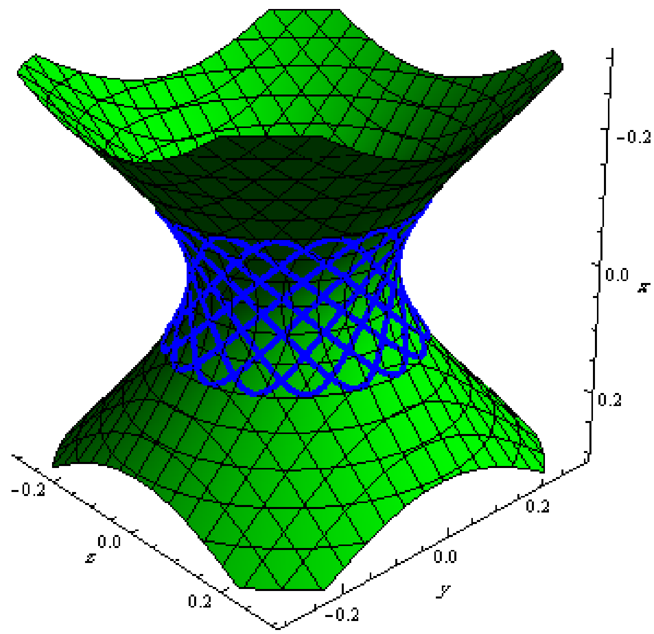

FIGURE 1. The spacelike slant helix lies on a pseudo-sphere of E_1^3.

$$e_2 = \frac{1}{v_2}\begin{pmatrix} (30(25\sinh(9u)-9\sinh(25u)-25\sinh(43u)+9\sinh(59u))) \\ (480(-\cosh(9u)+\cosh(25u))) \\ (625\sinh(u)-81\sinh(33u)-625\sinh(35u)+81\sinh(67u)) \end{pmatrix};$$

$$v_2 = \left(\frac{\sqrt{450\cosh(16u)-900\cosh(17u)^2+625\cosh(18u)+81\cosh(50u)}}{\sqrt{\begin{array}{c}(-34\cosh(8u)+25\cosh(26u)+9\cosh(42u))^2-3600\mathrm{Sinh}[17u]^4 \\ +(34\sinh(8u)+25\sinh(26u)-9\sinh(42u))^2\end{array}}}\right),$$

$$e_3 = \frac{1}{v_3}\begin{pmatrix} 34\cosh(8u)-25\cosh(26u)-9\cosh(42u), \\ -34\sinh(8u)-25\sinh(26u)+9\sinh(42u), \\ 60\sinh(17u)^2 \end{pmatrix};$$

$$v_3 = \sqrt{\begin{array}{c}(-34\cosh(8u)+25\cosh(26u)+9\cosh(42u))^2-3600\sinh(17u)^4 \\ +(34\sinh(8u)+25\sinh(26u)-9\sinh(42u))^2\end{array}},$$

and its curvatures are calculated as follows:

$$\sigma_1 = \frac{1920\sqrt{\begin{array}{c}(-34\cosh(8u)+25\cosh(26u)+9\cosh(42u))^2-3600\sinh(17u)^4 \\ +(34\sinh(8u)+25\sinh(26u)-9\sinh(42u))^2\end{array}}}{(450\cosh(16u)-900\cosh(17u)^2+625\cosh(18u)+81\cosh(50u))^{3/2}},$$

$$\sigma_2 = -\frac{15360\cosh(17u)\sinh(17u)^2}{\begin{array}{c}(-34\cosh(8u)+25\cosh(26u)+9\cosh(42u))^2-3600\sinh(17u)^4\\+(34\sinh(8u)+25\sinh(26u)-9\sinh(42u))^2\end{array}}.$$

In this case, the position vector is a timelike vector, and it lies on a hyperboloid of one sheet in E_1^3 (see Fig. 2).

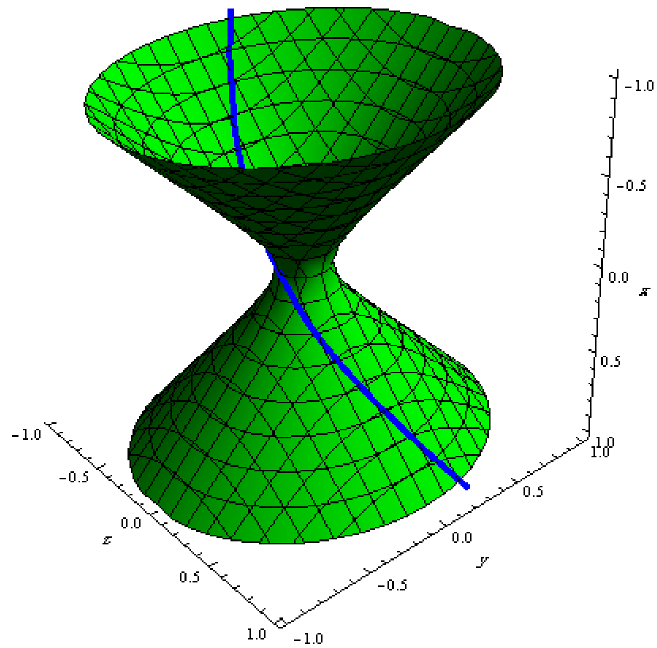

FIGURE 2. The timelike slant helix lies on a hyperboloid of one sheet in E_1^3.

5. CONCLUSION

In the three-dimensional Minkowski space, some characterizations of spacelike and timelike slant helices have a non-null axis are presented. By using vector differential equations established by means of Serret-Frenet equations in Lorentz-Minkowski space E_1^3, the differential geometric properties of these curves are investigated. Finally, by using *Wolfram Mathematica 0.7*, an example for spacelike and timelike slant helices is given and plotted. In future works, we plan to study the rectifying curves in Galilean and pseudo-Galilean spaces for different queries and further improve the results in this paper, combined with the techniques and results in our latest publications [12–16].

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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