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Fixed Point Methodologies for ψ-Contraction Mappings in Cone Metric Spaces over Banach Algebra with Supportive Applications

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Abstract. The explicit aim of this manuscript is to obtain fixed point consequences under novel ψ-contraction mappings in a complete cone metric space over Banach algebra. We connect and relate different fixed point theorems by using the idea of ψ-contraction mappings, providing a thorough viewpoint that deepens our comprehension of this topic. Our theorems generalize and unify many results in the scientific literature. These prospective extensions offer intriguing research directions and have the potential to further advance the study of fixed point theory. The investigation of examples plays an extremely crucial role in verifying the effectiveness and validity of our theoretical results. Moreover, to support the theoretical results, some examples are investigated to emphasize these results. Ultimately, the existence and uniqueness of the solution to the Urysohn integral and nonlinear fractional differential equation are cooperated as applications to provide an authoritative basis for dealing with actual problems that include these equations.

1. Introduction

Topology and analysis both benefit from, depend on, and mutually supportive of fixed point (**FP**) theory in important ways. This theory provides fundamental tools and insights that expand our knowledge of various mathematical structures and functions, serving as an essential component for both fields. The basis for investigating the characteristics and behaviors of mappings in a variety

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of spaces is laid by the fixed point theory (**FP**), which enhances the fabric of topology and analysis by proving the presence of fixed points under particular circumstances. Mathematical modelling, mathematical physics, economics, chemistry, and biology are just a few of the disciplines where the **FP** theory is useful. In mathematical modelling, for instance, and not as a limitation, **FP** theory presents crucial resources for studying dynamic systems and identifying equilibrium states. **FP** analysis can be used to describe the stable states and long-term behaviors of these complex systems, whether researchers are simulating ecological interactions, population increase, or the spread of diseases.

Using Banach's contraction principle (**BCP**) [1], the fixed point technique may solve a variety of ordinary deferential equations (**ODEs**), partial deferential equations (**PDEs**), integral equations (**IEs**), fractional differential equations (**FDEs**) and current optimization problems (see [2–5]). The **BCP** is the most powerful and fundamental result in metric **FP** theory. By altering the axioms of the metric notion, the concept of metric has been expanded in a number of ways: quasi-metric, symmetric, dislocated, 2-metric, b-metric, D-metric, G-metric, S-metric, ultra-metric, partial metric, etc. We will concentrate on Banach-valued metric space, or more specifically, cone metric spaces (**CMSs**) over Banach algebra (**BA**).

In 1695, Leibniz introduced the idea of fractional calculus (**FC**) [6], which is one of the devolopments of ordinary calculus. Lately, the **FC** theory has an influential role in fluid mechanics, entropy, engineering and physics [7–10]. Some engineering techniques and physical models can be interpreted more practically and accurately using **FC**. As an instance, **FC**-based entropies may be used more widely than the entropy of Shannon [11]. Fractional entropy has been an extensively investigated topic because of how widely it is used [12]. In addtion, the fractional differential equations (**FDEs**) are very useful for modeling and describing a variety of phenomena [13]. This is due to the fact that a system's next state is determined by all of its previous circumstances, not simply its current form. Compared to integer-order differential equations, these equations may better reflect physical reality. It's crucial to highlight that the theory and applications of **FC** have been extensively disscused within the literature [14–18]. **FDEs** have attracted a lot of attention in recent years due to their precise explanation of complex events in viscoelastic materials, system identification, control issues, signal processing, non-Brownian motion and polymers [19]. Recent studies have concentrated on fractional functional analysis and many applications have been investigated to fractional ordinary differential systems, fractional **ODEs**, and fractional **PDEs** [20–28].

In 2007, Zhang and Huang [29] presented the idea of **CMSs** using ordered Banach space instead of the set of real numbers. They discussed some properties of the convergence sequences and showed some **FP** results of contractive mappings in such spaces. Many articles have recently used the same methodology to generalize the results from the ordinary metric space to the **CMSs**; For more details, see [30–37].

On the other hand, Xu and Liu [38] presented the idea of a **CMSs** over BA, which generalizes and extends the BCP in ordinary metric spaces. Numerous authors used **CMSs** over Banach algebras

(**BAs**) to unify the **BCP** in many directions. Without making the assumption of normality, the authors established a number of **FP** results for generalized Lipschitz mappings in the new circumstances, which have no connection to metric space in relation to the existence of the mapping's **FP**. Many studies have been written regarding **FPs** in spaces that resemble cone b-metrics over **BA** and other spaces; see [39–49] for more information.

Motivated by the previous results, the explicit purpose of this manuscript is to launch the notion of ψ-contraction mappings and examine various **FP** results in **CMSs** over **BA**. Our crucial results have been supported by many corollaries, illustrative examples and applications for the existence and uniqueness to **IEs** and nonlinear **FDEs**. our results represent a generalization, development and an extension to the scientific research in the literature.

2. Basic facts

This part is devoted to presenting basic concepts that help us for obtaining our goals.

Definition 2.1. [50] Suppose that $\mathcal{B}$ denotes a real **BA**, yield to the following properties (for every $\varpi, \varrho, \vartheta \in \mathcal{B},\ \gamma \in \mathbb{R}$):

($\mathcal{BA}_1$) $\varpi(\varrho\vartheta) = (\varpi\varrho)\vartheta$;
($\mathcal{BA}_2$) $\varpi(\varrho + \vartheta) = \varpi\varrho + \varpi\vartheta$ and $(\varpi + \varrho)\vartheta = \varpi\vartheta + \varrho\vartheta$;
($\mathcal{BA}_3$) $\gamma(\varpi\varrho) = (\gamma\varpi)\varrho = \varpi(\gamma\varrho)$;
($\mathcal{BA}_4$) $\|\varpi\varrho\| \leq \|\varpi\|\,\|\varrho\|$.

Proposition 2.1. [51] Let (Δ, d) be a complete metric space and let $d_{\mathcal{B}}(f(\varpi), f(\varrho) \leq \psi(d_{\mathcal{B}}(\varpi, \varrho))$ for all $\varpi, \varrho \in \Delta$ where $\psi : [0, \infty) \longrightarrow [0, \infty)$ is any monotone non-decresing function with $\lim\limits_{n\to\infty} \psi^n(\tau) = 0$ for any fixed $\tau > 0$. Then f has a unique **FP**.

Definition 2.2. [42] A subset $\mathcal{P}$ of $\mathcal{B}$ is called a cone if

1. $\mathcal{P}$ is closed, non-empty and $\{\theta, I\} \in \mathcal{P}$;
2. $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$ for all $\alpha, \beta \in \mathbb{R}_+$;
3. $\mathcal{P}\mathcal{P} = \mathcal{P}^2 \subset \mathcal{P}$;
4. $(-\mathcal{P}) \cap \mathcal{P} = \{\theta\}$;

where θ denotes the null of the BA $\mathcal{B}$.

For a given cone $\mathcal{P} \subset \mathcal{B}$, define a partial order $\preceq$ with respect to $\mathcal{P}$ by $\varpi \preceq \varrho$ if and only if $\varrho - \varpi \in \mathcal{P}$. $\varpi \prec \varrho$ will denote $\varpi \preceq \varrho$ and $\varpi \neq \varrho$. While $\varpi \prec \varrho$ will denote $\varrho - \varpi \in \operatorname{int}\mathcal{P}$ where $\operatorname{int}\mathcal{P}$ represents the interior of $\mathcal{P}$. If $\operatorname{int}\mathcal{P} \neq \emptyset$, then $\mathcal{P}$ is called a solid cone.

The cone $\mathcal{P}$ is called normal if there exist a number $W > 0$ such that for all $\varpi, \varrho \in \mathcal{B}$,

$$\theta \preceq \varpi \preceq \varrho \implies \|\varpi\| \leq W \left\|\varrho\right\|.$$

The smallest positive number verifying the above inequality is called a normal constant [52].

Definition 2.3. [29] Let Δ be a non-empty set. Suppose that the mapping $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}$ verifies, for all $\varpi, \varrho, \vartheta \in \Delta$

(i) $\theta \prec d_{\mathcal{B}}(\varpi, \varrho)$ with $\varpi \neq \varrho$ and $d_{\mathcal{B}}(\varpi, \varrho) = \theta$ if $\varpi = \varrho$;
(ii) $d_{\mathcal{B}}(\varpi, \varrho) = d_{\mathcal{B}}(\varrho, \varpi)$;
(iii) $d_{\mathcal{B}}(\varpi, \varrho) \preceq d_{\mathcal{B}}(\varpi, \vartheta) + d_{\mathcal{B}}(\vartheta, \varrho)$.
Then $d_{\mathcal{B}}$ is called a cone metric on Δ, and $(\Delta, d_{\mathcal{B}})$ is called a **CMS** over **BA**.

Example 2.1. *Assume that $\mathcal{B} = \mathbb{R}^2$, $\mathcal{P} = \{(\varpi, \varrho) \in \mathcal{B} : 0 \leq \varpi, \varrho\} \subset \mathbb{R}^2$, $\Delta = \mathbb{R}$ and $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}$ such that $d_{\mathcal{B}}(\varpi, \varrho) = (|\varpi - \varrho|, \alpha\,|\varpi - \varrho|)$, where $\alpha \geq 0$ is a constant. Then $(\Delta, d_{\mathcal{B}})$ is a* **CMS** *over* **BA**.

Definition 2.4. [38] Let $(\Delta, d_{\mathcal{B}})$ be a **CMS** over **BA** $\mathcal{B}$, $\varpi \in \Delta$ and $\{\varpi_n\}$ be a sequence in Δ. Consequently,

(i) for each $\theta \prec c$, there exists $\mathcal{N} \in \mathbb{N}$ such that $d_{\mathcal{B}}(\varpi_n, \varpi) \prec c$ for all $n \geq \mathcal{N}$.
This means that $\{\varpi_n\}$ converges to ϖ and it can be written briefly as

$$\lim_{n\to\infty} \varpi_n = \varpi \;\text{ or }\; \varpi_n \to \infty \;(n \to \infty);$$

(ii) for each $\theta \prec c$, there exists $\mathcal{N} \in \mathbb{N}$ such that $d_{\mathcal{B}}(\varpi_m, \varpi_n) \prec c$ for all $m, n \geq \mathcal{N}$.
Then, ϖ_n is a Cauchy sequence (**CS**);

(iii) if every **CS** in Δ is convergent then, $(\Delta, d_{\mathcal{B}})$ is complete.

Remark 2.1. [50] The spectral radius (**SR**) $\rho(\varpi)$ of ϖ verifies $\rho(\varpi) \leq \varpi$ for all $\varpi \in \mathcal{B}$, where $\mathcal{B}$ is a **BA** with a unit $\mathcal{I}$.

Remark 2.2. [50] The **SR** $\rho(\varpi)$ of ϖ satisfies

$$\rho(\varpi) \leq \|\varpi\|,$$

for all $\varpi \in \mathcal{B}$, where $\mathcal{B}$ is a **BA** with a unit $\mathcal{I}$.

Remark 2.3. [53] If $\rho(\varpi) < 1$ then $\|\varpi^n\| \to 0$ as $n \to \infty$.

Lemma 2.1. If $\mathcal{B}$ is a unital **BA** with unit $\mathcal{I}$, $\psi : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$ and $\rho(\psi(\varpi)) < 1$. Then $(\mathcal{I}_{\mathcal{B}} - \psi)$ is invertible and

$$(\mathcal{I}_{\mathcal{B}} - \psi)^{-1}(\varpi) = \sum_{n=0}^{\infty} \psi^n(\varpi).$$

Lemma 2.2. [50] Let $\mathcal{B}$ be a **BA** with a unit $\mathcal{I}$ and $\varpi, \varrho \in \mathcal{B}$. If ϖ commutes with ϱ, then

$$\rho(\varpi + \varrho) \leq \rho(\varpi) + \rho(\varrho), \qquad \rho(\varpi\varrho) \leq \rho(\varpi)\,\rho(\varrho).$$

Definition 2.4. [54] Let $\Psi_{\mathcal{B}}$ be the set of all positive functions $\psi_{\mathcal{B}} : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$ satisfying the following conditions:

(i) $\psi(\varpi + \varrho) = \psi(\varpi) + \psi(\varrho)$,

(ii) $\psi(\varpi\varrho) = \psi(\varpi)\,\psi(\varrho)$,

(iii) $\lim_{n\to+\infty} \psi^n(\varpi) = \theta$ for all $\varpi > 0$,

(iv) $\psi(\varpi) = \theta$ iff $\varpi = 0$.

Let us display some major definitions, constructions and basic concepts of **FC** (see [55,56]). If $h : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function, then the Caputo fractional derivative (**CFD**) of order α is defined by

$$^{C}D^{\alpha}(h(t)) \;=\; \int_0^t \frac{h^{(n)}(s)}{\Gamma(n-\alpha)\,(t-s)^{1+\alpha-n}}\;ds,$$

$(n-1<\alpha<n,\ n=[\alpha]+1)$, where Γ is a gamma function and $[\alpha]$ stands for the integer part of the positive real number α. Also, $^{R-L}J^{\alpha}$ represents the Riemann-Liouville fractional integral (**R-LFI**) of order α, and defined as follows:

$$^{R-L}J^{\alpha}(h(t)) \;=\; \int_0^t \frac{h(s)}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\;ds.$$

3. FP results in CMS over BA

We begin this part with proving the following lemmas:

Lemma 3.1. If $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is any monotone non-decreasing function with $\psi(\varpi) < \varpi$ for all $\varpi > 0$, $\rho\,(\varpi) < 1$ and $\|\psi^n(\varpi)\| \le \|\psi\|^n\,.\,\|\varpi\|^n$ then $\rho\,(\psi^n(\varpi)) < 1$ where ρ is the spectral radius.

Proof.

$$\begin{aligned}
\rho\Big(\psi^n(\varpi)\big) &= \rho\Big(\psi(\varpi)\,.\,\psi(\varpi)\,...\,\psi(\varpi)\Big)\\
&= \rho\,(\psi(\varpi))\,.\,\rho\,(\psi(\varpi))\,...\,\rho(\psi(\varpi))\\
&= \Big(\rho\,(\psi(\varpi))\Big)^n < \Big(\rho\,(\varpi)\Big)^n < 1.
\end{aligned}$$

Therefore, $\rho\,(\psi^n(\varpi)) < 1$.

Lemma 3.2. If $\psi \prec \mathcal{I}_{\mathcal{B}}$, then $(\mathcal{I}_{\mathcal{B}} - \psi)$ has an inverse where

$$(\mathcal{I}_{\mathcal{B}} - \psi)^{-1} = \sum_{n=0}^{\infty} \psi^n.$$

Proof.

$$\begin{aligned}
(\mathcal{I}_{\mathcal{B}} - \psi)\,(\mathcal{I}_{\mathcal{B}} - \psi)^{-1} &= (\mathcal{I}_{\mathcal{B}} - \psi)\Big(\sum_{n=0}^{\infty} \psi^n\Big)\\
&= (\mathcal{I}_{\mathcal{B}} - \psi)\,(\mathcal{I}_{\mathcal{B}} + \psi + \psi^2 + ...)\\
&= \mathcal{I}_{\mathcal{B}}.
\end{aligned}$$

Thus, $(\mathcal{I}_{\mathcal{B}} - \psi)$ has an inverse.

Theorem 3.1. Suppose that $(\Delta, \mathcal{B}, d_{\mathcal{B}})$ be a complete **CMS** over **BA** $\mathcal{B}$. Let $\mathcal{S} : \Delta \longrightarrow \Delta$ and $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \;\preceq\; \psi\Big(d_{\mathcal{B}}(\varpi, \varrho)\Big), \qquad \text{for all } \varpi, \varrho \in \Delta,$$

where $\psi : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Proof. Let an arbitrary element ϖ_0 in Δ. Define a sequence $\{\varpi_n\}$ by $\mathcal{S}^n\varpi_0 = \mathcal{S}\varpi_{n-1} = \varpi_n,\ n \geq 1$. From our contraction condition, we get

$$\begin{aligned} d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) &= d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi_{n-1}) \\ &\preceq \psi\Big(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\Big) \\ &= \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi_{n-1}, \mathcal{S}\varpi_{n-2})\Big) \\ &\preceq \psi^2\Big(d_{\mathcal{B}}(\varpi_{n-1}, \varpi_{n-2})\Big) \\ &\vdots \\ &\preceq \psi^n\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big). \end{aligned}$$

Then, for $n > m$, we find

$$\begin{aligned} d_{\mathcal{B}}(\varpi_n, \varpi_m) &\preceq d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + d_{\mathcal{B}}(\varpi_{n-1}, \varpi_{n-2}) + \dots + d_{\mathcal{B}}(\varpi_{m+1}, \varpi_m) \\ &\preceq \psi^{n-1}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) + \psi^{n-2}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) + \dots + \psi^{m}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \\ &\preceq \sum_{k=m}^{n-1} \psi^k\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \longrightarrow 0_{\mathcal{B}}, \qquad \text{for all } d_{\mathcal{B}}(\varpi_1, \varpi_0) > 0_{\mathcal{B}}, \quad n, m \longrightarrow \infty. \end{aligned}$$

Hence $\{\varpi_n\}$ is a **CS** in Δ. Based on the completeness of Δ, there exists $\varpi^* \in \Delta$ such that $\lim_{n\to\infty} \varpi_n = \varpi^*$. Therefore, one has

$$\begin{aligned} 0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi_n) + d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi^*) \\ &\preceq d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\varpi_n, \varpi^*)\Big). \end{aligned}$$

It follows that

$$\begin{aligned} 0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \varpi^*) + \psi\Big(d_{\mathcal{B}}(\varpi^*, \varpi^*)\Big) \\ &\preceq 0_{\mathcal{B}} + \psi(0_{\mathcal{B}}). \end{aligned}$$

Thus, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) \preceq 0_{\mathcal{B}}$, which is a contradiction. Then, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) = 0_{\mathcal{B}}$, i.e., $\varpi^* = \mathcal{S}\varpi^*$ is a **FP** of $\mathcal{S}$.

Now, if $(\varrho^* \neq 0) \neq \varpi^*$ is another **FP** of the mapping $\mathcal{S}$, then

$$\begin{aligned} 0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \varrho^*) &= d_{\mathcal{B}}(\mathcal{S}\varpi^*, \mathcal{S}\varrho^*) \\ &\preceq \psi\Big(d_{\mathcal{B}}(\varpi^*, \varrho^*)\Big), \end{aligned}$$

that is,

$$(\mathcal{I}_{\mathcal{B}}-\psi)\Big(d_{\mathcal{B}}(\varpi^{*},\varrho^{*})\Big) \ \preceq \ 0_{\mathcal{B}}.$$

Since $(\mathcal{I}_{\mathcal{B}}-\psi)$ is invertible, then

$$\begin{aligned}(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,(\mathcal{I}_{\mathcal{B}}-\psi)\Big(d_{\mathcal{B}}(\varpi^{*},\varrho^{*})\Big) \quad &\preceq \quad (\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,(0_{\mathcal{B}}) \\ \Longleftrightarrow \quad \mathcal{I}_{\mathcal{B}}\Big(d_{\mathcal{B}}(\varpi^{*},\varrho^{*})\Big) \quad &\preceq \quad \sum_{n=0}^{\infty}\psi^{n}(0_{\mathcal{B}}).\end{aligned}$$

Again, we get a contradiction. Therefore,

$$d_{\mathcal{B}}(\varpi^{*},\varrho^{*})=0_{\mathcal{B}} \quad \Longrightarrow \quad \varpi^{*}=\varrho^{*}.$$

This implies that the **FP** is unique.

The example below support Theorem 3.1:

Example 3.1. Let $\mathcal{B}_{+}=\mathbb{R}^{2}$ and $\big\|(\varpi_1,\varpi_2)\big\|=|\varpi_1|+|\varpi_2|$ for all $(\varpi_1,\varpi_2)\in\mathcal{B}$. Define the operation of multiplication by

$$\varpi\varrho=(\varpi_1,\varpi_2).(\varrho_1,\varrho_2)=(\varpi_1\varrho_1\,,\,\varpi_1\varrho_2+\varpi_2\varrho_1).$$

Then $\mathcal{B}$ is a **BA** with unit $\mathcal{I}=(1,0)$.
Let $\mathcal{P}=\{\,(\varpi_1,\varpi_2)\in\mathbb{R}^{2}\mid \varpi_1,\varpi_2\geq 0\,\}$.
Let $\Delta=\mathbb{R}$ and the metric $d_{\mathcal{B}}:\Delta\times\Delta\longrightarrow\mathbb{R}^{2}_{+}$ be defined by

$$d_{\mathcal{B}}(\varpi,\varrho)=\Big(\big|\varpi-\varrho\big|,\beta\big|\varpi-\varrho\big|\Big)\in\mathcal{P}, \quad \text{with} \quad \beta\geq 0.$$

Then, $(\Delta,\mathcal{B},d)$ is a complete **CMS** over **BA**.
Now, define the mapping $\mathcal{S}:\Delta\longrightarrow\Delta$ by

$$\mathcal{S}\varpi=\frac{\varpi}{4},$$

and $\psi:\mathcal{B}_{+}\longrightarrow\mathcal{B}_{+}$ by

$$\psi(t)=\frac{t}{3}.$$

where $\mathcal{B}_{+}=\mathbb{R}^{2}_{+}$. Then, we obtain

$$\begin{aligned}d_{\mathcal{B}}(\mathcal{S}\varpi,\mathcal{S}\varrho) \quad &= \quad \Big(\big|\mathcal{S}\varpi-\mathcal{S}\varrho\big|\ ,\ \beta\big|\mathcal{S}\varpi-\mathcal{S}\varrho\big|\Big) \\ &= \quad \frac{1}{4}\Big(\big|\varpi-\varrho\big|\ ,\ \beta\big|\varpi-\varrho\big|\Big) \\ &= \quad \frac{1}{4}\ d_{\mathcal{B}}(\varpi,\varrho) \\ &= \quad \frac{3}{4}\left(\frac{d_{\mathcal{B}}(\varpi,\varrho)}{3}\right)\end{aligned}$$

$$\begin{aligned} &\le \frac{d_{\mathcal{B}}(\varpi,\varrho)}{3} \\ &= \psi\Big(d_{\mathcal{B}}(\varpi,\varrho)\Big). \end{aligned}$$

Therefore all hypotheses of Theorem 3.1 are satisfied and 0 is a unique **FP** of $\mathcal{S}$.

Theorem 3.2. Let $(\Delta,\mathcal{B},d_{\mathcal{B}})$ be a complete **CMS** with **BA** $\mathcal{B}$. Let $\mathcal{S}:\Delta\longrightarrow\Delta$ and $d_{\mathcal{B}}:\Delta\times\Delta\longrightarrow\mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi,\mathcal{S}\varrho) \le \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi,\varpi)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varrho)\Big), \qquad \text{for all } \varpi,\varrho\in\Delta,$$

where $\psi:\mathcal{B}_+\longrightarrow\mathcal{B}_+$ and $\rho\left(\psi(\varpi)\right)<\dfrac{1}{3}$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Proof. Let an arbitrary element ϖ_0 in Δ. Define a sequence $\{\varpi_n\}$ by $\mathcal{S}^n\varpi_0=\mathcal{S}\varpi_{n-1}=\varpi_n,\ n\ge 1$. From our contraction condition, we obtain

$$\begin{aligned} d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) &= d_{\mathcal{B}}(\mathcal{S}\varpi_n,\mathcal{S}\varpi_{n-1}) \\ &\le \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi_n,\varpi_n)+d_{\mathcal{B}}(\mathcal{S}\varpi_{n-1},\varpi_{n-1})\Big) \\ &= \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi_n)+d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big) \\ &\le \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi_n)\Big)+\psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big). \end{aligned}$$

Then

$$(\mathcal{I}_{\mathcal{B}}-\psi)\,d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) \le \psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big).$$

By multiplying $(\mathcal{I}-\psi)^{-1}$

$$\begin{aligned} d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) &\le (\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big) \\ &\ \vdots \\ &\le \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^n\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big). \end{aligned}$$

Now, we shall prove that

$$\rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right)<1, \qquad \text{if} \quad \rho\left(\psi(\varpi)\right)<\frac{1}{3}.$$

Therefore,

$$\begin{aligned} \rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right) &\le \rho\Big((\mathcal{I}_{\mathcal{B}}-\psi)^{-1}(\varpi)\Big).\,\rho\left(\psi(\varpi)\right) \\ &= \rho\Big(\sum_{n=0}^{\infty}\psi^n(\varpi)\Big).\,\rho\left(\psi(\varpi)\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{n=0}^{\infty} \rho\left(\psi^{n}(\varpi)\right)\right). \rho\left(\psi(\varpi)\right) \\
&= \left(\sum_{n=0}^{\infty} \left[\rho\left(\psi(\varpi)\right)\right]^{n}\right). \rho\left(\psi(\varpi)\right) \\
&\leq \frac{\rho\left(\mathcal{I}_{\mathcal{B}}\right)}{\rho\left(\mathcal{I}_{\mathcal{B}}\right)-\rho\left(\psi(\varpi)\right)} . \rho\left(\psi(\varpi)\right) \\
&= \frac{1}{1-\frac{1}{3}} . \frac{1}{3} = \frac{1}{2} < 1.
\end{aligned}
$$

Thus, for $n > m$, we get

$$
\begin{aligned}
& d_{\mathcal{B}}(\varpi_n, \varpi_m) \preceq d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + d_{\mathcal{B}}(\varpi_{n-1}, \varpi_{n-2}) + ... + d_{\mathcal{B}}(\varpi_{m+1}, \varpi_m) \\
\preceq\ & \left[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\psi\right]^{n-1}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) + \left[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\psi\right]^{n-2}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \\
& + ... + \left[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\psi\right]^{m}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \\
\preceq\ & \sum_{l=m}^{n-1}\left[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\psi\right]^{l}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \longrightarrow 0_{\mathcal{B}}, \quad \text{for all } d_{\mathcal{B}}(\varpi_1, \varpi_0) \succ 0_{\mathcal{B}}, \quad n, m \longrightarrow \infty.
\end{aligned}
$$

Hence $\{\varpi_n\}$ is a **CS** in Δ. Based on the completeness of Δ, there exists $\varpi^* \in \Delta$ such that $\lim_{n\to\infty} \varpi_n = \varpi^*$. Therefore, one has

$$
\begin{aligned}
0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi_n) + d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi^*) \\
&\preceq d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi_n, \varpi_n) + d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varpi^*)\Big) \\
&\preceq d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n)\Big) + \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varpi^*)\Big).
\end{aligned}
$$

Taking $\lim_{n\to\infty}$, we get

$$
\begin{aligned}
d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \varpi^*) + \psi\Big(d_{\mathcal{B}}(\varpi^*, \varpi^*)\Big) + \psi\Big(d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*)\Big) \\
&\preceq 0_{\mathcal{B}} + 0_{\mathcal{B}} + \psi\Big(d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*)\Big),
\end{aligned}
$$

that is,

$$
(\mathcal{I}_{\mathcal{B}}-\psi)\Big(d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*)\Big) \preceq 0_{\mathcal{B}}.
$$

Since $(\mathcal{I}_{\mathcal{B}} - \psi)$ is invertible, then

$$(\mathcal{I}_{\mathcal{B}} - \psi)^{-1} (\mathcal{I}_{\mathcal{B}} - \psi)\Big(d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*)\Big) \preceq (\mathcal{I}_{\mathcal{B}} - \psi)^{-1} (0_{\mathcal{B}}).$$

Thus, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) \preceq 0_{\mathcal{B}}$, which is a contradiction. Then, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) = 0_{\mathcal{B}}$, i.e., $\varpi^* = \mathcal{S}\varpi^*$ is a **FP** of $\mathcal{S}$.
Now, if $(\varrho^* \neq 0) \neq \varpi^*$ is another **FP** of the mapping $\mathcal{S}$, then

$$\begin{aligned}
0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \varrho^*) &= d_{\mathcal{B}}(\mathcal{S}\varpi^*, \mathcal{S}\varrho^*) \\
&\preceq \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varpi^*) + d_{\mathcal{B}}(\mathcal{S}\varrho^*, \varrho^*)\Big) \\
&= \psi\Big(d_{\mathcal{B}}(\varpi^*, \varpi^*)\Big) + \psi\Big(d_{\mathcal{B}}(\varrho^*, \varrho^*)\Big),
\end{aligned}$$

that is,

$$d_{\mathcal{B}}(\varpi^*, \varrho^*) \preceq 0_{\mathcal{B}},$$

which is a contradiction. Therefore,

$$d_{\mathcal{B}}(\varpi^*, \varrho^*) = 0_{\mathcal{B}} \implies \varpi^* = \varrho^*.$$

This implies that the **FP** is unique.

The example below support Theorem 3.2:

Example 3.2. Assume the same hypotheses of Example 3.1 then, we get

$$\begin{aligned}
d_{\mathcal{B}}(\mathcal{S}\varpi, \varpi) + d_{\mathcal{B}}(\mathcal{S}\varrho, \varrho) &= \Big(|\mathcal{S}\varpi - \varpi| \ , \ \beta |\mathcal{S}\varpi - \varpi|\Big) + \Big(\left|\mathcal{S}\varrho - \varrho\right| \ , \ \beta \left|\mathcal{S}\varrho - \varrho\right|\Big) \\
&= \frac{3}{4}\Big(|\varpi| \ , \ \beta |\varpi|\Big) + \frac{3}{4}\Big(\left|\varrho\right| \ , \ \beta \left|\varrho\right|\Big) \\
&= \frac{3}{4}\Big(|\varpi| + \left|\varrho\right| \ , \ \beta\Big(|\varpi| + \left|\varrho\right|\Big)\Big) \\
&\succeq \frac{3}{4}\Big(\left|\varpi - \varrho\right| \ , \ \beta \left|\varpi - \varrho\right|\Big) \\
&= 3\Big(\left|\frac{\varpi}{4} - \frac{\varrho}{4}\right| \ , \ \beta \left|\frac{\varpi}{4} - \frac{\varrho}{4}\right|\Big) \\
&= 3\Big(\left|\mathcal{S}\varpi - \mathcal{S}\varrho\right| \ , \ \beta \left|\mathcal{S}\varpi - \mathcal{S}\varrho\right|\Big) \\
&= 3 \ d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho),
\end{aligned}$$

i.e.,

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \preceq \frac{1}{3}\Big(d_{\mathcal{B}}(\mathcal{S}\varpi, \varpi) + d_{\mathcal{B}}(\mathcal{S}\varrho, \varrho)\Big).$$

It follows that

$$d_{\mathcal{B}}(\mathcal{S}\varpi,\mathcal{S}\varrho) \;\preceq\; \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi,\varpi)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varrho)\Big).$$

Therefore all conditions of Theorem 3.2 are verified and 0 is a unique **FP** of $\mathcal{S}$.

Theorem 3.3. Let $(\Delta,\mathcal{B},d_{\mathcal{B}})$ be a complete **CMS** over **BA** $\mathcal{B}$. Let $\mathcal{S}:\Delta\longrightarrow\Delta$ and $d_{\mathcal{B}}:\Delta\times\Delta\longrightarrow\mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi,\mathcal{S}\varrho) \preceq \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi,\varrho)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varpi)\Big), \qquad \text{for all } \varpi,\varrho\in\Delta,$$

where $\psi:\mathcal{B}_+\longrightarrow\mathcal{B}_+$ and $\rho\left(\psi(\varpi)\right)<\dfrac{1}{3}$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Proof. Let ϖ_0 be an arbitrary element in Δ. Define a sequence $\{\varpi_n\}$ by $\mathcal{S}^n\varpi_0=\mathcal{S}\varpi_{n-1}=\varpi_n,\ n\geq 1$. From our contraction condition, we get

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) &= d_{\mathcal{B}}(\mathcal{S}\varpi_n,\mathcal{S}\varpi_{n-1})\\
&\preceq \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi_n,\varpi_{n-1})+d_{\mathcal{B}}(\mathcal{S}\varpi_{n-1},\varpi_n)\Big)\\
&= \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi_{n-1})+d_{\mathcal{B}}(\varpi_n,\varpi_n)\Big)\\
&\preceq \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi_n)+d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big)\\
&\preceq \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi_n)\Big)+\psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big).
\end{aligned}$$

It follows that

$$(\mathcal{I}_{\mathcal{B}}-\psi)\,d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) \;\preceq\; \psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big).$$

By multiplying $(I-\psi)^{-1}$

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1},\varpi_n) &\preceq (\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\Big)\\
&\ \vdots\\
&\preceq \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^n\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big).
\end{aligned}$$

Now, we show that

$$\rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right)<1, \qquad \text{if} \quad \rho\left(\psi(\varpi)\right)<\frac{1}{3}.$$

Therefore,

$$\begin{aligned}
\rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right) &\preceq \rho\left((\mathcal{I}_{\mathcal{B}}-\psi)^{-1}(\varpi)\right).\,\rho\left(\psi(\varpi)\right)\\
&= \rho\left(\sum_{n=0}^{\infty}\psi^n(\varpi)\right).\,\rho\left(\psi(\varpi)\right)
\end{aligned}$$

$$\begin{aligned}
&\preceq \left(\sum_{n=0}^{\infty}\rho\left(\psi^{n}(\varpi)\right)\right).\,\rho\left(\psi(\varpi)\right)\\
&= \left(\sum_{n=0}^{\infty}\Big[\rho\left(\psi(\varpi)\right)\Big]^{n}\right).\,\rho\left(\psi(\varpi)\right)\\
&\preceq \frac{\rho\left(\mathcal{I}_{\mathcal{B}}\right)}{\rho\left(\mathcal{I}_{\mathcal{B}}\right)-\rho\left(\psi(\varpi)\right)}\cdot\rho\left(\psi(\varpi)\right)\\
&= \frac{1}{1-\frac{1}{3}}\cdot\frac{1}{3} \prec 1.
\end{aligned}$$

Thus, for $n > m$, we get

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_n,\varpi_m) &\preceq d_{\mathcal{B}}(\varpi_n,\varpi_{n-1}) + d_{\mathcal{B}}(\varpi_{n-1},\varpi_{n-2}) + \dots + d_{\mathcal{B}}(\varpi_{m+1},\varpi_m)\\
&\preceq \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{n-1}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big) + \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{n-2}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)\\
&\quad + \dots + \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{m}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)\\
&\preceq \sum_{l=m}^{n-1}\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{l}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big) \longrightarrow 0_{\mathcal{B}}, \quad \text{for all } d_{\mathcal{B}}(\varpi_1,\varpi_0) \succ 0_{\mathcal{B}}, \quad n,m \longrightarrow \infty.
\end{aligned}$$

Hence $\{\varpi_n\}$ is a **CS** in Δ. Based on the completeness of Δ, there exists $\varpi^* \in \Delta$ such that $\lim_{n\to\infty}\varpi_n = \varpi^*$. Therefore, one has

$$\begin{aligned}
0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi_n) + d_{\mathcal{B}}(\mathcal{S}\varpi_n,\mathcal{S}\varpi^*)\\
&\preceq d_{\mathcal{B}}(\varpi^*,\varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi_n,\varpi^*) + d_{\mathcal{B}}(\mathcal{S}\varpi^*,\varpi_n)\Big)\\
&\preceq d_{\mathcal{B}}(\varpi^*,\varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi^*)\Big) + \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi^*,\varpi_n)\Big).
\end{aligned}$$

Taking $\lim_{n\to\infty}$, we get

$$\begin{aligned}
d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*,\varpi^*) + \psi\Big(d_{\mathcal{B}}(\varpi^*,\varpi^*)\Big) + \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi^*,\varpi^*)\Big)\\
&\preceq 0_{\mathcal{B}} + 0_{\mathcal{B}} + \psi\Big(d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)\Big),
\end{aligned}$$

that is,

$$(\mathcal{I}_{\mathcal{B}}-\psi)\Big(d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)\Big) \preceq 0_{\mathcal{B}}.$$

Since $(\mathcal{I}_{\mathcal{B}}-\psi)$ is invertible, then

$$(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,(\mathcal{I}_{\mathcal{B}}-\psi)\Big(d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)\Big) \preceq (\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,(0_{\mathcal{B}}).$$

Thus, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) \preceq 0_{\mathcal{B}}$, which is a contradiction. Then, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) = 0_{\mathcal{B}}$, i.e., $\varpi^* = \mathcal{S}\varpi^*$ is a **FP** of $\mathcal{S}$.

Now, if $(\varrho^* \neq 0) \neq \varpi^*$ is another **FP** of the mapping $\mathcal{S}$, then

$$\begin{aligned} 0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \varrho^*) &= d_{\mathcal{B}}(\mathcal{S}\varpi^*, \mathcal{S}\varrho^*) \\ &\preceq \psi\Big(d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varrho^*) + d_{\mathcal{B}}(\mathcal{S}\varrho^*, \varpi^*)\Big) \\ &\preceq \psi\Big(d_{\mathcal{B}}(\varpi^*, \varrho^*)\Big) + \psi\Big(d_{\mathcal{B}}(\varrho^*, \varpi^*)\Big), \end{aligned}$$

that is,

$$(\mathcal{I}_{\mathcal{B}} - 2\,\psi)\Big(d_{\mathcal{B}}(\varpi^*, \varrho^*)\Big) \preceq 0_{\mathcal{B}}.$$

Since $\psi(\varpi) = 0$ ifff $a = 0$ and $\mathcal{I}_{\mathcal{B}}(\varpi) \neq 2\,\psi(\varpi)$. Then,

$$d_{\mathcal{B}}(\varpi^*, \varrho^*) = 0_{\mathcal{B}} \quad \Longrightarrow \quad \varpi^* = \varrho^*.$$

This implies that the **FP** is unique.

The example below support Theorem 3.3:

Example 3.3. Assume the same assumptions of Example 3.1 then, we have

$$\begin{aligned} d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) &= \Big(\big|\mathcal{S}\varpi - \mathcal{S}\varrho\big|\ ,\ \beta\,\big|\mathcal{S}\varpi - \mathcal{S}\varrho\big|\Big) \\ &= \frac{1}{4}\,\Big(\big|\varpi - \varrho\big|\ ,\ \beta\,\big|\varpi - \varrho\big|\Big) \\ &= \frac{1}{4}\,\Big(\big|\varpi - \mathcal{S}\varpi + \mathcal{S}\varpi - \varrho\big|\ ,\ \beta\,\big|\varpi - \mathcal{S}\varpi + \mathcal{S}\varpi - \varrho\big|\Big) \\ &\preceq \frac{1}{4}\,\Big(|\varpi - \mathcal{S}\varpi| + \big|\mathcal{S}\varpi - \varrho\big|\ ,\ \beta\,\big(|\varpi - \mathcal{S}\varpi| + \big|\mathcal{S}\varpi - \varrho\big|\big)\Big) \\ &= \frac{1}{4}\,\Big[\Big(|\varpi - \mathcal{S}\varpi|\ ,\ \beta\ |\varpi - \mathcal{S}\varpi|\Big)\Big] + \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varpi - \varrho\big|\ ,\ \beta\ \big|\mathcal{S}\varpi - \varrho\big|\Big)\Big] \\ &= \frac{1}{4}\,\Big[\Big(\big|\varpi - \mathcal{S}\varrho + \mathcal{S}\varrho - \mathcal{S}\varpi\big|\ ,\ \beta\ \big|\varpi - \mathcal{S}\varrho + \mathcal{S}\varrho - \mathcal{S}\varpi\big|\Big)\Big] \\ &\qquad + \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varpi - \varrho\big|\ ,\ \beta\ \big|\mathcal{S}\varpi - \varrho\big|\Big)\Big] \\ &\preceq \frac{1}{4}\,\Big[\Big(\big|\varpi - \mathcal{S}\varrho\big|\ ,\ \beta\ \big|\varpi - \mathcal{S}\varrho\big|\Big)\Big] + \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varrho - \mathcal{S}\varpi\big|\ ,\ \beta\ \big|\mathcal{S}\varrho - \mathcal{S}\varpi\big|\Big)\Big] \\ &\qquad + \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varpi - \varrho\big|\ ,\ \beta\ \big|\mathcal{S}\varpi - \varrho\big|\Big)\Big] \\ &= \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varrho - \varpi\big|\ ,\ \beta\ \big|\mathcal{S}\varrho - \varpi\big|\Big)\Big] + \frac{1}{4}\,\Big[\Big(\big|\mathcal{S}\varpi - \mathcal{S}\varrho\big|\ ,\ \beta\ \big|\mathcal{S}\varpi - \mathcal{S}\varrho\big|\Big)\Big] \end{aligned}$$

$$+ \ \frac{1}{4}\left[\left(\left|\mathcal{S}\varpi - \varrho\right| \ , \ \beta\ \left|\mathcal{S}\varpi - \varrho\right|\right)\right]$$

$$= \ \frac{1}{4}\left[d_{\mathcal{B}}(\mathcal{S}\varrho, \varpi) + d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) + d_{\mathcal{B}}(\mathcal{S}\varpi, \varrho)\right],$$

i.e.,

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \ \preceq \ \frac{1}{3}\left(d_{\mathcal{B}}(\mathcal{S}\varpi, \varrho) + d_{\mathcal{B}}(\mathcal{S}\varrho, \varpi)\right).$$

Then

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \ \preceq \ \psi\left(d_{\mathcal{B}}(\mathcal{S}\varpi, \varrho) + d_{\mathcal{B}}(\mathcal{S}\varrho, \varpi)\right).$$

Therefore all conditions of Theorem 3.3 are satisfied and 0 is a unique **FP** of $\mathcal{S}$.

Theorem 3.4. Let $(\Delta, \mathcal{B}, d_{\mathcal{B}})$ be a complete **CMS** with **BA** $\mathcal{B}$. Let $\mathcal{S} : \Delta \longrightarrow \Delta$ and $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \ \preceq \ \psi\left(d_{\mathcal{B}}(\varpi, \varrho) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi, \varpi) \ . \ d_{\mathcal{B}}(\mathcal{S}\varrho, \varrho)}{1 + d_{\mathcal{B}}(\varpi, \varrho)}\right), \qquad \text{for all } \varpi, \varrho \in \Delta,$$

where $\psi : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$ and $\rho\left(\psi(\varpi)\right) < \dfrac{1}{5}$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Proof. Let ϖ_0 be an arbitrary element in Δ. Define a sequence $\{\varpi_n\}$ by $\mathcal{S}^n\varpi_0 = \mathcal{S}\varpi_{n-1} = \varpi_n, \ n \geq 1$. From our contraction condition, we have

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) &= d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi_{n-1}) \\
&\preceq \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi_n, \varpi_n) \ . \ d_{\mathcal{B}}(\mathcal{S}\varpi_{n-1}, \varpi_{n-1})}{1 + d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})}\right) \\
&= \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + \frac{d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) \ . \ d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})}{1 + d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})}\right) \\
&\preceq \psi\left(d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) + d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\right) \\
&\preceq \psi\left(d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n)\right) \ + \ \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\right).
\end{aligned}$$

Then

$$\left(\mathcal{I}_{\mathcal{B}} - \psi\right) d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) \ \preceq \ \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\right).$$

By multiplying $(\mathcal{I} - \psi)^{-1}$

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) \ &\preceq \ \left[(\mathcal{I}_{\mathcal{B}} - \psi)^{-1}\,\psi\right]\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\right) \\
&\vdots
\end{aligned}$$

$$\preceq \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{n}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big).$$

Now, we verify that

$$\rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right)<1, \qquad \text{if} \quad \rho\left(\psi(\varpi)\right)<\frac{1}{5}.$$

Therefore,

$$\begin{aligned}
\rho\left(\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big](\varpi)\right) &\preceq \rho\Big((\mathcal{I}_{\mathcal{B}}-\psi)^{-1}(\varpi)\Big).\,\rho\left(\psi(\varpi)\right)\\
&= \rho\Big(\sum_{n=0}^{\infty}\psi^{n}(\varpi)\Big).\,\rho\left(\psi(\varpi)\right)\\
&\preceq \Big(\sum_{n=0}^{\infty}\rho\left(\psi^{n}(\varpi)\right)\Big).\,\rho\left(\psi(\varpi)\right)\\
&= \Big(\sum_{n=0}^{\infty}\Big[\rho\left(\psi(\varpi)\right)\Big]^{n}\Big).\,\rho\left(\psi(\varpi)\right)\\
&\preceq \frac{\rho\left(\mathcal{I}_{\mathcal{B}}\right)}{\rho\left(\mathcal{I}_{\mathcal{B}}\right)-\rho\left(\psi(\varpi)\right)}.\,\rho\left(\psi(\varpi)\right)\\
&= \frac{1}{1-\frac{1}{5}}.\,\frac{1}{5}=\frac{1}{4}\;<\;1.
\end{aligned}$$

Thus, for $n>m$, we get

$$\begin{aligned}
& d_{\mathcal{B}}(\varpi_n,\varpi_m)\\
\preceq\; & d_{\mathcal{B}}(\varpi_n,\varpi_{n-1})\,+\,d_{\mathcal{B}}(\varpi_{n-1},\varpi_{n-2})\,+\,...\,+\,d_{\mathcal{B}}(\varpi_{m+1},\varpi_m)\\
\preceq\; & \Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{n-1}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)+\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{n-2}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)\\
& \qquad +...+\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{m}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)\\
\preceq\; & \sum_{l=m}^{n-1}\Big[(\mathcal{I}_{\mathcal{B}}-\psi)^{-1}\,\psi\Big]^{l}\Big(d_{\mathcal{B}}(\varpi_1,\varpi_0)\Big)\longrightarrow 0_{\mathcal{B}}, \quad \text{for all } d_{\mathcal{B}}(\varpi_1,\varpi_0)\succ 0_{\mathcal{B}}, \quad n,m\longrightarrow\infty.
\end{aligned}$$

Hence $\{\varpi_n\}$ is a **CS** in Δ. Based on the completeness of Δ, there exists $\varpi^*\in\Delta$ such that $\lim_{n\to\infty}\varpi_n=\varpi^*$. Therefore, one has

$$\begin{aligned}
0_{\mathcal{B}}\preceq d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi_n)\,+\,d_{\mathcal{B}}(\mathcal{S}\varpi_n,\mathcal{S}\varpi^*)\\
&\preceq d_{\mathcal{B}}(\varpi^*,\varpi_{n+1})\,+\,\psi\left(d_{\mathcal{B}}(\varpi_n,\varpi^*)+\frac{\mathcal{S}\varpi_n,\varpi_n)\,.\,d_{\mathcal{B}}(\mathcal{S}\varpi^*,\varpi^*)}{1+d_{\mathcal{B}}(\varpi_n,\varpi^*)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi^*) + \frac{d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) \,.\, d_{\mathcal{B}}(\varpi^*, \varpi^*)}{1 + d_{\mathcal{B}}(\varpi_n, \varpi^*)}\right) \\
&\preceq d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\Big(d_{\mathcal{B}}(\varpi_n, \varpi^*)\Big).
\end{aligned}
$$

Taking $\lim_{n\to\infty}$, we get

$$
\begin{aligned}
d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \varpi^*) + \psi\Big(d_{\mathcal{B}}(\varpi^*, \varpi^*)\Big) \\
&\preceq 0_{\mathcal{B}} + \psi(0_{\mathcal{B}}) = 0_{\mathcal{B}}.
\end{aligned}
$$

Thus, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) \preceq 0_{\mathcal{B}}$, which is a contradiction. Then, $d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) = 0_{\mathcal{B}}$, i.e., $\varpi^* = \mathcal{S}\varpi^*$ is a **FP** of $\mathcal{S}$.

Now, if $(\varrho^* \neq 0) \neq \varpi^*$ is another **FP** of the mapping $\mathcal{S}$, then

$$
\begin{aligned}
0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \varrho^*) &= d_{\mathcal{B}}(\mathcal{S}\varpi^*, \mathcal{S}\varrho^*) \\
&\preceq \psi\left(d_{\mathcal{B}}(\varpi^*, \varrho^*) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varpi^*) \,.\, d_{\mathcal{B}}(\mathcal{S}\varrho^*, \varrho^*)}{1 + d_{\mathcal{B}}(\varpi^*, \varrho^*)}\right) \\
&= \psi\Big(d_{\mathcal{B}}(\varpi^*, \varrho^*)\Big),
\end{aligned}
$$

that is,

$$
(\mathcal{I}_{\mathcal{B}} - \psi)\Big(d_{\mathcal{B}}(\varpi^*, \varrho^*)\Big) \preceq 0_{\mathcal{B}}.
$$

Since $(\mathcal{I}_{\mathcal{B}} - \psi)$ is invertible, then

$$
(\mathcal{I}_{\mathcal{B}} - \psi)^{-1} \, (\mathcal{I}_{\mathcal{B}} - \psi)\Big(d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*)\Big) \preceq (\mathcal{I}_{\mathcal{B}} - \psi)^{-1} \, (0_{\mathcal{B}}).
$$

Then,

$$
d_{\mathcal{B}}(\varpi^*, \varrho^*) = 0_{\mathcal{B}} \quad \Longrightarrow \quad \varpi^* = \varrho^*.
$$

This implies that the **FP** is unique.

In Theorem 3.4, if $\psi = \lambda\, \mathcal{I}$ then we obtain the result below:

Corollary 3.1. Let $(\Delta, \mathcal{B}, d_{\mathcal{B}})$ be a complete **CMS** with **BA** $\mathcal{B}$. Let $\mathcal{S} : \Delta \longrightarrow \Delta$ and $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}_+$ be mappings satisfy

$$
d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \preceq \lambda\left(d_{\mathcal{B}}(\varpi, \varrho) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi, \varpi) \,.\, d_{\mathcal{B}}(\mathcal{S}\varrho, \varrho)}{1 + d_{\mathcal{B}}(\varpi, \varrho)}\right), \qquad \text{for all } \varpi, \varrho \in \Delta,
$$

where $0 \leq \lambda < 1$ is a constant and $\rho\,(\varpi) < \dfrac{1}{5}$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Theorem 3.5. Let $(\Delta, \mathcal{B}, d_{\mathcal{B}})$ be a complete **CMS** over **BA** $\mathcal{B}$. Let $\mathcal{S} : \Delta \longrightarrow \Delta$ and $d_{\mathcal{B}} : \Delta \times \Delta \longrightarrow \mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \preceq \psi\left(d_{\mathcal{B}}(\varpi, \varrho) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi, \varrho) \,.\, d_{\mathcal{B}}(\mathcal{S}\varrho, \varpi)}{1 + d_{\mathcal{B}}(\varpi, \varrho)}\right), \qquad \text{for all } \varpi, \varrho \in \Delta,$$

where $\psi : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$. Then, $\mathcal{S}$ has a unique **FP** in Δ.

Proof. Let ϖ_0 be an arbitrary element in Δ. Define a sequence $\{\varpi_n\}$ by $\mathcal{S}^n\varpi_0 = \mathcal{S}\varpi_{n-1} = \varpi_n,\ n \geq 1$. From our contraction condition, we get

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) &= d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi_{n-1}) \\
&\preceq \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi_n, \varpi_{n-1}) \,.\, d_{\mathcal{B}}(\mathcal{S}\varpi_{n-1}, \varpi_n)}{1 + d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})}\right) \\
&= \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + \frac{d_{\mathcal{B}}(\varpi_{n+1}, \varpi_{n-1}) \,.\, d_{\mathcal{B}}(\varpi_n, \varpi_n)}{1 + d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})}\right) \\
&\preceq \psi\Big(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\Big).
\end{aligned}$$

Then

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_{n+1}, \varpi_n) &\preceq \psi\Big(d_{\mathcal{B}}(\varpi_n, \varpi_{n-1})\Big) \\
&\vdots \\
&\preceq \psi^n\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big).
\end{aligned}$$

Then, for $n > m$, we find

$$\begin{aligned}
d_{\mathcal{B}}(\varpi_n, \varpi_m) &\preceq d_{\mathcal{B}}(\varpi_n, \varpi_{n-1}) + d_{\mathcal{B}}(\varpi_{n-1}, \varpi_{n-2}) + \ldots + d_{\mathcal{B}}(\varpi_{m+1}, \varpi_m) \\
&\preceq \psi^{n-1}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) + \psi^{n-2}\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) + \ldots + \psi^m\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \\
&\preceq \sum_{k=m}^{n-1} \psi^k\Big(d_{\mathcal{B}}(\varpi_1, \varpi_0)\Big) \longrightarrow 0_{\mathcal{B}}, \qquad \text{for all } d_{\mathcal{B}}(\varpi_1, \varpi_0) \succ 0_{\mathcal{B}}, \quad n, m \longrightarrow \infty.
\end{aligned}$$

Hence $\{\varpi_n\}$ is a **CS** in Δ. Based on the completeness of Δ, there exists $\varpi^* \in \Delta$ such that $\lim_{n \to \infty} \varpi_n = \varpi^*$. Therefore, one has

$$\begin{aligned}
0_{\mathcal{B}} \preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi^*) &\preceq d_{\mathcal{B}}(\varpi^*, \mathcal{S}\varpi_n) + d_{\mathcal{B}}(\mathcal{S}\varpi_n, \mathcal{S}\varpi^*) \\
&\preceq d_{\mathcal{B}}(\varpi^*, \varpi_{n+1}) + \psi\left(d_{\mathcal{B}}(\varpi_n, \varpi^*) + \frac{d_{\mathcal{B}}(\mathcal{S}\varpi_n, \varpi^*) \,.\, d_{\mathcal{B}}(\mathcal{S}\varpi^*, \varpi_n)}{1 + d_{\mathcal{B}}(\varpi_n, \varpi^*)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\preceq\ d_{\mathcal{B}}(\varpi^*,\varpi_{n+1})\ +\ \psi\left(d_{\mathcal{B}}(\varpi_n,\varpi^*)+\frac{d_{\mathcal{B}}(\varpi_{n+1},\varpi^*)\,.\,d_{\mathcal{B}}(\varpi_n,\varpi^*)}{1+d_{\mathcal{B}}(\varpi_n,\varpi^*)}\right)\\
&\preceq\ d_{\mathcal{B}}(\varpi^*,\varpi_{n+1})\ +\ \psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi^*)+d_{\mathcal{B}}(\varpi_{n+1},\varpi^*)\Big)\\
&\preceq\ d_{\mathcal{B}}(\varpi^*,\varpi_{n+1})\ +\ \psi\Big(d_{\mathcal{B}}(\varpi_n,\varpi^*)\Big)+\psi\Big(d_{\mathcal{B}}(\varpi_{n+1},\varpi^*)\Big).
\end{aligned}
$$

Taking $\lim_{n\to\infty}$, we get

$$
\begin{aligned}
d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)\ &\preceq\ d_{\mathcal{B}}(\varpi^*,\varpi^*)\ +\ \psi\Big(d_{\mathcal{B}}(\varpi^*,\varpi^*)\Big)\ +\ \psi\Big(d_{\mathcal{B}}(\varpi^*,\varpi^*)\Big)\\
&=\ 0_{\mathcal{B}}\ +\ \psi(0_{\mathcal{B}})\ +\ \psi(0_{\mathcal{B}})\ =\ 0_{\mathcal{B}}.
\end{aligned}
$$

Thus, $d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)\preceq 0_{\mathcal{B}}$, that is a contradiction. Then, $d_{\mathcal{B}}(\varpi^*,\mathcal{S}\varpi^*)=0_{\mathcal{B}}$, i.e., $\varpi^*=\mathcal{S}\varpi^*$ is a **FP** of $\mathcal{S}$.
Now, if $(\varrho^*\neq 0)\neq\varpi^*$ is another **FP** of the mapping $\mathcal{S}$, then

$$
\begin{aligned}
0_{\mathcal{B}}\ \preceq\ d_{\mathcal{B}}(\varpi^*,\varrho^*)\ &=\ d_{\mathcal{B}}(\mathcal{S}\varpi^*,\mathcal{S}\varrho^*)\\
&\preceq\ \psi\left(d_{\mathcal{B}}(\varpi^*,\varrho^*)+\frac{d_{\mathcal{B}}(\mathcal{S}\varpi^*,\varrho^*)\,.\,d_{\mathcal{B}}(\mathcal{S}\varrho^*,\varpi^*)}{1+d_{\mathcal{B}}(\varpi^*,\varrho^*)}\right)\\
&\preceq\ \psi\left(d_{\mathcal{B}}(\varpi^*,\varrho^*)+\frac{d_{\mathcal{B}}(\varpi^*,\varrho^*)\,.\,d_{\mathcal{B}}(\varrho^*,\varpi^*)}{1+d_{\mathcal{B}}(\varpi^*,\varrho^*)}\right)\\
&=\ 2\,\psi\Big(d_{\mathcal{B}}(\varpi^*,\varrho^*)\Big),
\end{aligned}
$$

that is,

$$(\mathcal{I}_{\mathcal{B}}-2\,\psi)\Big(d_{\mathcal{B}}(\varpi^*,\varrho^*)\Big)\preceq 0_{\mathcal{B}}.$$

Since $\psi(\varpi)=0_{\mathcal{B}}$ if $\varpi=0$ and $\mathcal{I}_{\mathcal{B}}(\varpi)\neq 2\,\psi(\varpi)$. Then,

$$d_{\mathcal{B}}(\varpi^*,\varrho^*)=0_{\mathcal{B}}\quad\Longrightarrow\quad\varpi^*=\varrho^*.$$

This implies that the **FP** is unique.
In Theorem 3.5, if $\psi=\lambda\,\mathcal{I}$ then we have the result below:

Corollary 3.2. Let $(\Delta,\mathcal{B},d_{\mathcal{B}})$ be a complete **CMS** over **BA** $\mathcal{B}$. Let $\mathcal{S}:\Delta\longrightarrow\Delta$ and $d_{\mathcal{B}}:\Delta\times\Delta\longrightarrow\mathcal{B}_+$ be mappings satisfy

$$d_{\mathcal{B}}(\mathcal{S}\varpi,\mathcal{S}\varrho)\ \preceq\ \lambda\left(d_{\mathcal{B}}(\varpi,\varrho)+\frac{d_{\mathcal{B}}(\mathcal{S}\varpi,\varrho)\,.\,d_{\mathcal{B}}(\mathcal{S}\varrho,\varpi)}{1+d_{\mathcal{B}}(\varpi,\varrho)}\right),\qquad\text{for all }\varpi,\varrho\in\Delta,$$

where $0\leq\lambda<1$ is a constant. Then, $\mathcal{S}$ has a unique **FP** in Δ.

4. Solve Urysohn integral equations

Here, we apply Theorem 3.2 to discuss the existence and uniqueness of solution to the following Urysohn **IEs**:

$$\varpi(t) = h(t) + \int_0^1 K_1(t,s,\varpi(s))\,ds, \tag{4.1}$$

$$\varrho(t) = h(t) + \int_0^1 K_2(t,s,\varrho(s))\,ds, \tag{4.2}$$

where $K_1, K_2 : [0,1]\times[0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ and $h : [0,1] \longrightarrow \mathbb{R}$.

Let $\mathcal{B}$ and $\mathcal{P}$ are defined as Example 3.1. Let $\Delta = C([0,1],\mathbb{R})$ where $C[0,1]$ denotes the set of all real continuous funtions (**RCFs**), $\mathcal{P}$ is a cone and $d_{\mathcal{B}} : \Delta\times\Delta \longrightarrow \mathcal{B}$ is defined as follows:

$$\begin{aligned} d_{\mathcal{B}}(\varpi,\varrho) &= \left(\left\|\varpi-\varrho\right\|_\infty, \beta\left\|\varpi-\varrho\right\|_\infty\right) \\ &= \left(\sup_{t\in[0,1]}\left|\varpi(t)-\varrho(t)\right|, \beta\sup_{t\in[0,1]}\left|\varpi(t)-\varrho(t)\right|\right) \in \mathcal{P}, \end{aligned}$$

where $\varpi,\varrho\in\Delta$. Then, $(\Delta,\mathcal{B},d)$ is a complete **CMS** over **BA**.
Now, we define $\psi : \mathcal{B}_+ \longrightarrow \mathcal{B}_+$ as

$$\psi(r) = \mu\, r,$$

where $\mu\in[0,1)$ and $\mathcal{B}_+ = \mathbb{R}^2_+$.
Theorem 4.1. Define the mapping $\mathcal{S} : \Delta \longrightarrow \Delta$ by

$$\begin{aligned} \mathcal{S}\,\varpi(t) &= h(t) + \int_0^1 K_1(t,s,\varpi(s))\,ds, \\ \mathcal{S}\,\varrho(t) &= h(t) + \int_0^1 K_2(t,s,\varrho(s))\,ds, \end{aligned}$$

for all $t\in[0,1]$. If the following inequality

$$\left|K_1(t,s,\varpi(s)) - K_2(t,s,\varrho(s))\right| \le \mu\,\left|\varpi(t)-\varrho(t)\right|,$$

holds, the **IEs** (4.1) and (4.2) have a unique solution.
Proof. It is clear that finding the solution of the equations (4.1) and (4.2) is equivalent to finding the **FP** of the mapping $\mathcal{S}$.
Now, consider

$$\begin{aligned} d_{\mathcal{B}}(\mathcal{S}x,\mathcal{S}\varrho) &= \left(\left\|\mathcal{S}\varpi-\mathcal{S}\varrho\right\|_\infty, \beta\left\|\mathcal{S}\varpi-\mathcal{S}\varrho\right\|_\infty\right) \\ &= \left(\sup_{t\in[0,1]}\left|\mathcal{S}\varpi(t)-\mathcal{S}\varrho(t)\right|, \beta\sup_{t\in[0,1]}\left|\mathcal{S}\varpi(t)-\mathcal{S}\varrho(t)\right|\right) \end{aligned}$$

$$
\begin{aligned}
&= \Bigg(\sup_{t\in[0,1]} \left| \int_0^1 K_1(t,s,\varpi(s))\,ds - \int_0^1 K_2(t,s,\varrho(s))\,ds \right|, \\
&\qquad \beta \sup_{t\in[0,1]} \left| \int_0^1 K_1(t,s,\varpi(s))\,ds - \int_0^1 K_2(t,s,\varrho(s))\,ds \right| \Bigg) \\
&\preceq \left(\sup_{t\in[0,1]} \int_0^1 |K_1 - K_2|\,ds,\ \beta \sup_{t\in[0,1]} \int_0^1 |K_1 - K_2|\,ds \right) \\
&\preceq \mu \left(\sup_{t\in[0,1]} \int_0^1 \big|\varpi(t) - \varrho(t)\big|\,ds,\ \beta \sup_{t\in[0,1]} \int_0^1 \big|\varpi(t) - \varrho(t)\big|\,ds \right) \\
&\preceq \mu \left(\sup_{t\in[0,1]} \big|\varpi(t) - \varrho(t)\big| \int_0^1 ds,\ \beta \sup_{t\in[0,1]} \big|\varpi(t) - \varrho(t)\big| \int_0^1 ds \right) \\
&= \mu \left(\big\|\varpi(t) - \varrho(t)\big\|_\infty,\ \beta \big\|\varpi(t) - \varrho(t)\big\|_\infty \right) \\
&= \mu\, d_{\mathcal{B}}(\varpi,\varrho) \;=\; \psi(d_{\mathcal{B}}(\varpi,\varrho)),
\end{aligned}
$$

where $\mu \in [0,1)$. Therefore all requirements of Theorem 3.1 hold, then the problems (4.1) and (4.2) have a unique solution.

Now, we can apply the results shown in Theorem 3.3 to obtain the same results of Theorem 4.1.

Theorem 4.2. Assume that all conditions of Theorem 4.1 are true. Define $\psi : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ as $\psi(r) = \lambda\, r$. where $\lambda = \dfrac{\mu}{2} \in [0,1)$. Then the proposed equations have a unique solution.

Proof.

$$
\begin{aligned}
d_{\mathcal{B}}(\mathcal{S}x,\mathcal{S}\varrho) &= \left(\big\|\mathcal{S}\varpi - \mathcal{S}\varrho\big\|_\infty,\ \beta \big\|\mathcal{S}\varpi - \mathcal{S}\varrho\big\|_\infty \right) \\
&= \left(\sup_{t\in[0,1]} \big|\mathcal{S}\varpi(t) - \mathcal{S}\varrho(t)\big|,\ \beta \sup_{t\in[0,1]} \big|\mathcal{S}\varpi(t) - \mathcal{S}\varrho(t)\big| \right) \\
&= \Bigg(\sup_{t\in[0,1]} \left| \int_0^1 K_1(t,s,\varpi(s))\,ds - \int_0^1 K_2(t,s,\varrho(s))\,ds \right|, \\
&\qquad \beta \sup_{t\in[0,1]} \left| \int_0^1 K_1(t,s,\varpi(s))\,ds - \int_0^1 K_2(t,s,\varrho(s))\,ds \right| \Bigg) \\
&\preceq \left(\sup_{t\in[0,1]} \int_0^1 |K_1 - K_2|\,ds,\ \beta \sup_{t\in[0,1]} \int_0^1 |K_1 - K_2|\,ds \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mu\left(\sup_{t\in[0,1]}\int_0^1 \left|\varpi(t)-\varrho(t)\right| ds,\ \beta \sup_{t\in[0,1]}\int_0^1 \left|\varpi(t)-\varrho(t)\right| ds\right)\\
&= \mu\left(\sup_{t\in[0,1]} \left|\varpi(t)-\varrho(t)\right| \int_0^1 ds,\ \beta \sup_{t\in[0,1]} \left|\varpi(t)-\varrho(t)\right| \int_0^1 ds\right)\\
&= \mu\left(\left\|\varpi-\varrho\right\|_\infty,\ \beta\left\|\varpi-\varrho\right\|_\infty\right)\\
&= \mu\left(\frac{\left\|\varpi-\varrho\right\|_\infty}{2}+\frac{\left\|\varpi-\varrho\right\|_\infty}{2},\right.\\
&\qquad\qquad \left.\beta\left(\frac{\left\|\varpi-\varrho\right\|_\infty}{2}+\frac{\left\|\varpi-\varrho\right\|_\infty}{2}\right)\right)\\
&= \frac{\mu}{2}\left[\left(\left\|\mathcal{S}\varpi-\varrho\right\|_\infty,\ \beta\left\|\mathcal{S}\varpi-\varrho\right\|_\infty\right)\right.\\
&\qquad\qquad \left.+\left(\left\|\varpi-\mathcal{S}\varrho\right\|_\infty,\ \beta\left\|\varpi-\mathcal{S}\varrho\right\|_\infty\right)\right]\\
&= \frac{\mu}{2}\left[d_{\mathcal{B}}(\mathcal{S}\varpi,\varrho)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varpi)\right]\\
&= \lambda\left[d_{\mathcal{B}}(\mathcal{S}\varpi,\varrho)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varpi)\right]\\
&= \psi\left(d_{\mathcal{B}}(\mathcal{S}\varpi,\varrho)+d_{\mathcal{B}}(\mathcal{S}\varrho,\varpi)\right).
\end{aligned}
$$

Hence, all stipulations of Theorem 3.3 are verified. Then the considered problems have a unique solution.

5. Solving a Caputo fractional derivative

In this part, we will implicate some theoretical results to study the existence and uniqueness of the solution to a **CFD** of order α, it takes the form

$$
{}^{C}D^{\alpha}(\varpi(t)) = f(t,\varpi(t)) \qquad (0 < t < 1,\ 1 < \alpha \leq 2), \tag{5.1}
$$

via the integral boundry conditions

$$
\varpi(0) = 0, \quad \varpi(1) = \int_0^{\eta} \varpi(s)\, ds \quad (0 < \eta < 1), \qquad \text{with} \quad \varpi \in C([0,1],\mathbb{R}),
$$

where $C([0,1],\mathbb{R})$ is the set of all **RCFs** from $[0,1]$ into $\mathbb{R}$, ${}^{C}D^{\alpha}$ represents the **CFD** of order α and $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a **CF** (see [57]).

Now, we present our main theorem in this section.

Theorem 5.1. Problem (5.1) has a solution provided that the following hypotheses are true:

(A) there exists $\psi\in\Psi$ such that

$$\left|f(t,\varpi)-f(t,\varrho)\right| \;\leq\; \frac{\Gamma(\alpha+1)}{3}\;\psi\Big(\left|\varpi-\varrho\right|,\lambda\left|\varpi-\varrho\right|\Big),$$

for all $\varpi,\varrho\in\mathbb{R}$, $\lambda\geq 0$ and $t\in[0,1]$;

(B) for all $t\in[0,1]$, there exists a mapping $\mathcal{S}:C([0,1],\mathbb{R})\longrightarrow C([0,1],\mathbb{R})$ such that

$$\begin{aligned}\mathcal{S}\varpi(t) \;&=\; \int_0^t \frac{f(s,\varpi(s))}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\,ds\\ &\quad+\;2t\int_0^1\left(\int_0^s \frac{f(r,\varpi(r))}{\Gamma(\alpha)\,(s-r)^{1-\alpha}}\,dr\right)ds;\end{aligned}$$

(C) for all $t\in[0,1]$ and $\varpi,\varrho\in C([0,1],\mathbb{R})$, let the metric $d_{\mathcal{B}}:C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})\rightarrow\mathbb{R}^2_+$ is defined as

$$\begin{aligned}d_{\mathcal{B}}(\varpi,\varrho) \;&=\; \Big(\left\|\varpi-\varrho\right\|_\infty,\,\lambda\left\|\varpi-\varrho\right\|_\infty\Big)\\ &=\;\Big(\sup_{t\in[0,1]}\left|\varpi(t)-\varrho(t)\right|,\,\lambda\sup_{t\in[0,1]}\left|\varpi(t)-\varrho(t)\right|\Big)\in\mathcal{P};\end{aligned}$$

(D) for all $t\in[0,1]$, if $\{\varpi_n\}$ is a sequence in $C([0,1],\mathbb{R})$ and $\varpi\in C([0,1],\mathbb{R})$, then $\varpi_n\longrightarrow\varpi$ in $C([0,1],\mathbb{R})$.

Then, the problem (5.1) has at least one solution.

Proof. Let $\Delta=C([0,1],\mathbb{R})$ where Δ is the Banach space equipped with the supremum norm $\|\varpi\|_\infty=\sup_{t\in[0,1]}\left|\varpi(t)\right|\quad\forall\;\varpi\in\Delta$. Then, $\varpi\in\Delta$ is a solution of (5.1) if and only if $\varpi\in\Delta$ is a solution of the following **IE**:

$$\begin{aligned}\varpi(t) \;&=\; \int_0^t \frac{f(s,\varpi(s))}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\,ds\\ &\quad+\;2t\int_0^1\left(\int_0^s \frac{f(r,\varpi(r))}{\Gamma(\alpha)\,(s-r)^{1-\alpha}}\,dr\right)ds,\quad t\in[0,1].\end{aligned}$$

Define the function $\psi:\mathcal{B}_+\longrightarrow\mathcal{B}_+$ as

$$\psi(p)\;\;\leq\;\;p,$$

where $\mathcal{B}_+ = \mathbb{R}^2_+$. Now, let $\varpi, \varrho \in \Delta$ and $t \in [0,1]$. By assumption **(A)**, we have

$$
\begin{aligned}
&\Big|\mathcal{S}\varpi(t) - \mathcal{S}\varrho(t)\Big| \\
= \; & \Bigg| \int_0^t \frac{f(s,\varpi(s))}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\, ds \; - \; \int_0^t \frac{f(s,\varrho(s))}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\, ds \\
& + \; 2t \int_0^1 \left(\int_0^s \frac{f(r,\varpi(r))}{\Gamma(\alpha)\,(s-r)^{1-\alpha}}\, dr \right) ds \; - \; 2t \int_0^1 \left(\int_0^s \frac{f(r,\varrho(r))}{\Gamma(\alpha)\,(s-r)^{1-\alpha}}\, dr \right) ds \Bigg| \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Big|f(s,\varpi(s)) - f(s,\varrho(s))\Big|}{|t-s|^{1-\alpha}}\, ds \\
& + \; \frac{2t}{\Gamma(\alpha)} \int_0^1 \left| \int_0^s \frac{\Big(f(r,\varpi(r)) - f(r,\varrho(r))\Big)}{(s-r)^{1-\alpha}}\, dr \right| ds \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Big|f(s,\varpi(s)) - f(s,\varrho(s))\Big|}{|t-s|^{1-\alpha}}\, ds \\
& + \; \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s \frac{\Big|f(r,\varpi(r)) - f(r,\varrho(r))\Big|}{|s-r|^{1-\alpha}}\, dr\, ds \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1}\, \frac{\Gamma(\alpha+1)}{3}\, \psi\Big(\big|\varpi(s) - \varrho(s)\big|, \lambda \big|\varpi(s) - \varrho(s)\big|\Big)\, ds \\
& + \; \frac{2t}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s |s-r|^{\alpha-1}\, \frac{\Gamma(\alpha+1)}{3}\, \psi\Big(\big|\varpi(r) - \varrho(r)\big|, \lambda \big|\varpi(r) - \varrho(r)\big|\Big)\, dr \right) ds \\
\leq \; & \frac{\Gamma(\alpha+1)}{3\,\Gamma(\alpha)}\, \psi\Big(\big|\varpi(s) - \varrho(s)\big|, \lambda \big|\varpi(s) - \varrho(s)\big|\Big) \int_0^t |t-s|^{\alpha-1}\, ds \\
& + \; \frac{2t\,\Gamma(\alpha+1)}{3\,\Gamma(\alpha)}\, \psi\Big(\big|\varpi(r) - \varrho(r)\big|, \lambda \big|\varpi(r) - \varrho(r)\big|\Big) \int_0^1 \left(\int_0^s |s-r|^{\alpha-1}\, dr \right) ds \\
\leq \; & \frac{\Gamma(\alpha+1)}{3\,\alpha\,\Gamma(\alpha)}\, \psi\Big(\big|\varpi(s) - \varrho(s)\big|, \lambda \big|\varpi(s) - \varrho(s)\big|\Big) \int_0^t |t-s|^{\alpha-1}\, ds \\
& + \; \frac{2t\,\Gamma(\alpha+1)}{3\,\alpha\,\Gamma(\alpha)}\, \psi\Big(\big|\varpi(r) - \varrho(r)\big|, \lambda \big|\varpi(r) - \varrho(r)\big|\Big) \int_0^1 s^{(\alpha+1)-1}\, (1-s)^{(1)-1}\, ds \\
\leq \; & \psi\Big(\big|\varpi(s) - \varrho(s)\big|, \lambda \big|\varpi(s) - \varrho(s)\big|\Big)\, \frac{t^\alpha}{3} \\
& + \frac{2t}{3}\, B(\alpha+1,\, 1)\; \psi\Big(\big|\varpi(r) - \varrho(r)\big|, \lambda \big|\varpi(r) - \varrho(r)\big|\Big)
\end{aligned}
$$

$$\begin{aligned}
&\leq \psi\left(\left|\sup_{t\in[0,1]} \varpi(t) - \sup_{t\in[0,1]} \varrho(t)\right|, \lambda \left|\sup_{t\in[0,1]} \varpi(t) - \sup_{t\in[0,1]} \varrho(t)\right|\right) \frac{t^{\alpha}}{3} \\
&\quad + \frac{2t\,\Gamma(\alpha+1)\,\Gamma(1)}{3\,\Gamma(\alpha+2)}\ \psi\left(\left|\sup_{t\in[0,1]} \varpi(t) - \sup_{t\in[0,1]} \varrho(t)\right|, \lambda \left|\sup_{t\in[0,1]} \varpi(t) - \sup_{t\in[0,1]} \varrho(t)\right|\right) \\
&\leq \psi\left(\sup_{t\in[0,1]} \left|\varpi(t) - \varrho(t)\right|, \lambda \sup_{t\in[0,1]} \left|\varpi(t) - \varrho(t)\right|\right) \frac{t^{\alpha}}{3} \\
&\quad + \frac{2t}{3\,(\alpha+1)}\ \psi\left(\sup_{t\in[0,1]} \left|\varpi(t) - \varrho(t)\right|, \lambda \sup_{t\in[0,1]} \left|\varpi(t) - \varrho(t)\right|\right) \\
&\leq \psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty}\Big)\left(\frac{t^{\alpha}}{3} + \frac{2t}{3\,(\alpha+1)}\right),
\end{aligned}$$

where B denotes the beta function.

Taking supremum on both sides, we get

$$\sup_{t\in[0,1]} \left|\mathcal{S}\varpi(t) - \mathcal{S}\varrho(t)\right| \leq \psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty}, \lambda\|\varpi(t) - \varrho(t)\|_{\infty}\Big) \times \sup_{t\in[0,1]} \left(\frac{t^{\alpha}}{3} + \frac{2t}{3\,(\alpha+1)}\right),$$

which implies that

$$\|\mathcal{S}\varpi(t) - \mathcal{S}\varrho(t)\|_{\infty} \leq \psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty},\ \lambda\,\|\varpi(t) - \varrho(t)\|_{\infty}\Big).$$

Consequently,

$$\begin{aligned}
&d_{\mathcal{B}}(\mathcal{S}\varpi, \mathcal{S}\varrho) \\
&= \Big(\left\|\mathcal{S}\varpi - \mathcal{S}\varrho\right\|_{\infty},\ \lambda \left\|\mathcal{S}\varpi - \mathcal{S}\varrho\right\|_{\infty}\Big) \\
&\leq \Big(\psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty}, \lambda\|\varpi(t) - \varrho(t)\|_{\infty}\Big),\ \lambda\,\psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty}, \lambda\|\varpi(t) - \varrho(t)\|_{\infty}\Big)\Big) \\
&= (1,\lambda)\,\psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty},\ \lambda\,\|\varpi(t) - \varrho(t)\|_{\infty}\Big) \\
&\leq \psi\Big(\|\varpi(t) - \varrho(t)\|_{\infty},\ \lambda\,\|\varpi(t) - \varrho(t)\|_{\infty}\Big) \\
&= \psi\Big(d_{\mathcal{B}}(\varpi, \varrho)\Big).
\end{aligned}$$

Then, all assertions of Theorem 3.1 are satisfied.

Remark 5.1. By a similar way of Theorem 5.1, if $\psi : \mathcal{B}_{+} \longrightarrow \mathcal{B}_{+}$ is defined by $\psi(p) \leq p$ where $\mathcal{B}_{+} = \mathbb{R}^2_{+}$. We can show that all conditions of Theorem 3.3 are verified.

Conclusions

In this article, we introduced our findings under novel $\psi-$contraction mapping within the framework of a complete **CMS** over **BA**. We investigated several definitions, corollaries and theorems under ψ-contraction mapping in such spaces. These definitions, corollaries and theorems are regarded as an extension and generalisation of the findings in the literature. On the other hand, we presented several cases of ψ-contraction mapping in main results and applications to the findings of our fundamental theorems using Ursohn integral equations (**UIEs**) and nonlinear **FDEs**. We concentrated on the nonlinear **FDEs** and **UIEs** as applications that mark a substantial contribution to the fields of **FC** and **IEs**. There are numerous guidelines and methods for the future work as well as investigatations and discoveries in this particular discipline such as studying and analyzing variations of ψ-contraction mappings, such as weak ψ-contractions or mixed ψ-contractions to learn more about their properties and applications. Also, we can establish significant developments on the existence and uniqueness of solutions to certain integral and **FDEs** by utilizing the power of ψ-contraction mappings.

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References

[1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et leur Application aux Équations Intégrales, Fund. Math. 3 (1922), 133–181. https://eudml.org/doc/213289.

[2] A. Amini-Harandi, H. Emami, A Fixed Point Theorem for Contraction Type Maps in Partially Ordered Metric Spaces and Application to Ordinary Differential Equations, Nonlinear Anal.: Theory Meth. Appl. 72 (2010), 2238–2242. https://doi.org/10.1016/j.na.2009.10.023.

[3] E. Karapınar, T. Abdeljawad, F. Jarad, Applying New Fixed Point Theorems on Fractional and Ordinary Differential Equations, Adv. Diff. Equ. 2019 (2019), 421. https://doi.org/10.1186/s13662-019-2354-3.

[4] V. Gupta, W. Shatanawi and N. Mani, Fixed Point Theorems for (ψ, β)–Geraghty Contraction Type Maps in Ordered Metric Spaces and Some Applications to Integral and Ordinary Differential Equations, J. Fixed Point Theory Appl. 19 (2016), 1251–1267. https://doi.org/10.1007/s11784-016-0303-2.
[5] H.V. Long, N.T.K. Son, R. Rodríguez-López, Some Generalizations of Fixed Point Theorems in Partially Ordered Metric Spaces and Applications to Partial Differential Equations with Uncertainty, Vietnam J. Math. 46 (2017), 531–555. https://doi.org/10.1007/s10013-017-0254-y.
[6] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[7] H. Beyer, S. Kempfle, Definition of Physically Consistent Damping Laws with Fractional Derivatives, Z. Angew. Math. Mech. 75 (1995), 623–635. https://doi.org/10.1002/zamm.19950750820.
[8] M. Caputo, Linear Models of Dissipation whose Q is almost Frequency Independent–II, Geophys. J. Int. 13 (1967), 529–539. https://doi.org/10.1111/j.1365-246x.1967.tb02303.x.
[9] J.H. He, Some Applications of Nonlinear Fractional Differential Equations and Their Approximations, Bull. Sci. Technol. 15 (1999), 86–90.
[10] A.M. Lopes, J.A. Tenreiro Machado, C.M.A. Pinto, A.M.S.F. Galhano, Fractional Dynamics and MDS Visualization of Earthquake Phenomena, Comp. Math. Appl. 66 (2013), 647–658. https://doi.org/10.1016/j.camwa.2013.02.003.
[11] M.R. Ubriaco, Entropies Based on Fractional Calculus, Phys. Lett. A 373 (2009), 2516–2519. https://doi.org/10.1016/j.physleta.2009.05.026.
[12] J. Prehl, F. Boldt, C. Essex, K. Hoffmann, Time Evolution of Relative Entropies for Anomalous Diffusion, Entropy 15 (2013), 2989–3006. https://doi.org/10.3390/e15082989.
[13] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, Inc., New York, 1993.
[14] R. Alyusof, S. Alyusof, N. Iqbal, M.A. Arefin, Novel Evaluation of the Fractional Acoustic Wave Model with the Exponential-Decay Kernel, Complexity 2022 (2022), 9712388. https://doi.org/10.1155/2022/9712388.
[15] J.A. Tenreiro Machado, Entropy Analysis of Integer and Fractional Dynamical Systems, Nonlinear Dyn. 62 (2010), 371–378. https://doi.org/10.1007/s11071-010-9724-4.
[16] J.A.T. Machado, Optimal Tuning of Fractional Controllers Using Genetic Algorithms, Nonlinear Dyn. 62 (2010), 447–452. https://doi.org/10.1007/s11071-010-9731-5.
[17] J. Prehl, C. Essex, K.H. Hoffmann, Tsallis Relative Entropy and Anomalous Diffusion, Entropy 14 (2012), 701–716. https://doi.org/10.3390/e14040701.
[18] L. Sommacal, P. Melchior, A. Dossat, J. Petit, J.M. Cabelguen, A. Oustaloup, A.J. Ijspeert, Improvement of the Muscle Fractional Multimodel for Low-Rate Stimulation, Biomed. Signal Process. Control 2 (2007), 226–233. https://doi.org/10.1016/j.bspc.2007.07.013.
[19] N. Iqbal, T. Botmart, W.W. Mohammed, A. Ali, Numerical Investigation of Fractional-Order Kersten-Krasil'shchik Coupled KdV–mKdV System with Atangana-Baleanu Derivative, Adv. Cont. Discr. Mod. 2022 (2022), 37. https://doi.org/10.1186/s13662-022-03709-5.
[20] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[21] H. Yasmin, N. Iqbal, A Comparative Study of the Fractional-Order Nonlinear System of Physical Models via Analytical Methods, Math. Probl. Eng. 2022 (2022), 7488996. https://doi.org/10.1155/2022/7488996.
[22] H.A. Hammad, M. De la Sen, Tripled Fixed Point Techniques for Solving System of Tripled-Fractional Differential Equations, AIMS Math. 6 (2020), 2330–2343. https://doi.org/10.3934/math.2021141.
[23] R.A. Rashwan, H.A. Hammad, M.G. Mahmoud, Common Fixed Point Results for Weakly Compatible Mappings Under Implicit Relations in Complex Valued G-Metric Spaces, Inf. Sci. Lett. 8 (2019), 111–119. https://doi.org/10.18576/isl/080305.

[24] H.A. Hammad, H. Aydi, H. Işık, M. De la Sen, Existence and Stability Results for a Coupled System of Impulsive Fractional Differential Equations With Hadamard Fractional Derivatives, AIMS Math. 8 (2023), 6913–6941. https://doi.org/10.3934/math.2023350.

[25] H.A. Hammad, M. De la Sen, Analytical Solution of Urysohn Integral Equations by Fixed Point Technique in Complex Valued Metric Spaces, Mathematics 7 (2019), 852. https://doi.org/10.3390/math7090852.

[26] H.A. Hammad, R.A. Rashwan, A. Nafea, M.E. Samei, S. Noeiaghdam, Stability Analysis for a Tripled System of Fractional Pantograph Differential Equations With Nonlocal Conditions, J. Vib. Control 30 (2023), 632–647. https://doi.org/10.1177/10775463221149232.

[27] H.A. Hammad, H. Aydi, M. De la Sen, Generalized Dynamic Process for an Extended Multivalued $F-$Contraction in Metric-Like Spaces With Applications, Alex. Eng. J. 59 (2020), 3817–3825. https://doi.org/10.1016/j.aej.2020.06.037.

[28] H.A. Hammad, M.-F. Bota, L. Guran, Wardowski's Contraction and Fixed Point Technique for Solving Systems of Functional and Integral Equations, J. Funct. Spaces 2021 (2021), 7017046. https://doi.org/10.1155/2021/7017046.

[29] L.G. Huang, X. Zhang, Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings, J. Math. Anal. Appl. 332 (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087.

[30] D. Turkoglu, M. Abuloha, Cone Metric Spaces and Fixed Point Theorems in Diametrically Contractive Mappings, Acta. Math. Sin.-English Ser. 26 (2010), 489–496. https://doi.org/10.1007/s10114-010-8019-5.

[31] H. Huang, S. Xu, Fixed Point Theorems of Contractive Mappings in Cone b-Metric Spaces and Applications, Fixed Point Theory Appl 2013 (2013), 112. https://doi.org/10.1186/1687-1812-2013-112.

[32] M. Abbas, M.A. Khan, S. Radenović, Common Coupled Fixed Point Theorems in Cone Metric Spaces for w-Compatible Mappings, Appl. Math. Comp. 217 (2010), 195–202. https://doi.org/10.1016/j.amc.2010.05.042.

[33] S. Ali Abou Bakr, On Various Types of Cone Metric Spaces and Some Applications in Fixed Point Theory, Int. J. Nonlinear Anal. Appl. 14 (2023), 163–184. https://doi.org/10.22075/ijnaa.2021.24310.2715.

[34] M.A. Alghamdi, S.H. Alnafei, S. Radenović, N. Shahzad, Fixed Point Theorems for Convex Contraction Mappings on Cone Metric Spaces, Math. Comp. Model. 54 (2011), 2020–2026. https://doi.org/10.1016/j.mcm.2011.05.010.

[35] I. Altun, G. Durmaz, Some Fixed Point Theorems on Ordered Cone Metric Spaces, Rend. Circ. Mat. Palermo 58 (2009), 319–325. https://doi.org/10.1007/s12215-009-0026-y.

[36] L. Wangwe, S. Kumar, Common Fixed Point Theorems for F-Kannan–Suzuki Type Mappings in TVS-Valued Cone Metric Space With Some Applications, J. Math. 2022 (2022), 6504663. https://doi.org/10.1155/2022/6504663.

[37] V. Pasquale, Common Fixed Points in Cone Metric Spaces, Rend. Circ. Mat. Palermo 56 (2007), 464–468.

[38] H. Liu, S. Xu, Cone Metric Spaces With Banach Algebras and Fixed Point Theorems of Generalized Lipschitz Mappings, Fixed Point Theory Appl. 2013 (2013), 320. https://doi.org/10.1186/1687-1812-2013-320.

[39] J. Fernandez, N. Malviya, S. Shukla, Cone b-Metric-Like Spaces Over Banach Algebra and Fixed Point Theorems With Application, Asian J. Math. Comp. Res. 18 (2017), 49–66.

[40] J. Fernandez, S. Saelee, K. Saxena, N. Malviya, P. Kumam, The A-Cone Metric Space Over Banach Algebra With Applications, Cogent Math. 4 (2017), 1282690. https://doi.org/10.1080/23311835.2017.1282690.

[41] J. Fernandez, N. Malviya, Zoran D. Mitrović, A. Hussain, V. Parvaneh, Some Fixed Point Results on $\mathcal{N}_b$–Cone Metric Spaces Over Banach Algebra, Adv. Diff. Equ. 2020 (2020), 529. https://doi.org/10.1186/s13662-020-02991-5.

[42] J. Fernandez, N. Malviya, A. Savić, M. Paunović, Z.D. Mitrović, The Extended Cone b-Metric-like Spaces over Banach Algebra and Some Applications, Mathematics 10 (2022), 149. https://doi.org/10.3390/math10010149.

[43] Z.D. Mitrović, N. Hussain, On Results of Hardy-Rogers and Reich in Cone b-Metric Space Over Banach Algebra and Applications, U.P.B. Sci. Bull., Ser. A, 81 (2019), 147–154.

[44] Z.D. Mitrović, H. Aydi, S. Radenović, On Banach and Kannan Type Results in Cone $b_v(s)$-Metric Spaces Over Banach Algebra, Acta Math. Univ. Comen. 89 (2020), 143–152.

[45] Z.D. Mitrović, A. Ahmed, J.N. Salunke, A Cone Generalized b-Metric Like Space Over Banach Algebra and Contraction Principle, Thai J. Math. 19 (2021), 583–592.

[46] S. Radenović, B.E. Rhoades, Fixed Point Theorem for Two Non-Self Mappings in Cone Metric Spaces, Comp. Math. Appl. 57 (2009), 1701–1707. https://doi.org/10.1016/j.camwa.2009.03.058.

[47] Sh. Rezapour, R. Hamlbarani, Some Notes on the Paper "Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings", J. Math. Anal. Appl. 345 (2008), 719–724. https://doi.org/10.1016/j.jmaa.2008.04.049.

[48] K. Roy, S. Panja, M. Saha, A Generalized Fixed Point Theorem in an Extended Cone b-Metric Space Over Banach Algebra With Its Application to Dynamical Programming, Appl. Math. E-Notes 21 (2021), 209–219.

[49] W. Shatanawi, Z. D. Mitrović, N. Hussain, S. Radenović, On Generalized Hardy–Rogers Type α-Admissible Mappings in Cone b-Metric Spaces over Banach Algebras, Symmetry 12 (2020), 81. https://doi.org/10.3390/sym12010081.

[50] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1991.

[51] J. Dugundji, A. Granas, Fixed Point Theory, Vol. 61, Polish Scientific Publishers, Warsaw, Poland, 1982.

[52] L.G. Huang, X. Zhang, Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings, J. Math. Anal. Appl. 332 (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087.

[53] S. Xu, S. Radenović, Fixed Point Theorems of Generalized Lipschitz Mappings on Cone Metric Spaces Over Banach Algebras Without Assumption of Normality, Fixed Point Theory Appl. 2014 (2014), 102. https://doi.org/10.1186/1687-1812-2014-102.

[54] S. Omran, I. Masmali, On the $(\alpha-\psi)-$contractivemappings in C^*-algebra valued $b-$metric spaces and fixed point theorems, J. Math. 2021 (2021), 7865976. https://doi.org/10.1155/2021/7865976.

[55] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Vol. 204, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[56] S.G. Samko, A.A. Kilbas, I. Oleg, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, Yverdon, London, 1993.

[57] D. Baleanu, S. Rezapour, H. Mohammadi, Some Existence Results on Nonlinear Fractional Differential Equations, Phil. Trans. R. Soc. A. 371 (2013), 20120144. https://doi.org/10.1098/rsta.2012.0144.