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Geometric Properties of Harmonic Function Affiliated With Fractional Operator

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Abstract. This paper's goal is to discover new results for the harmonic univalent functions $\mathfrak{F} = \upsilon + \overline{\eta}$ defined in the open unit disc $\rho = \{\mathfrak{z} : |\mathfrak{z}| < 1\}$. Examining $\mathfrak{R}\mathfrak{S}$ indicates the set of all analytic harmonic functions of form $\mathfrak{F}$ in the open unit disc ρ. The convolution featuring the Mittag-Leffler function and fractional operator is applied to generate the family of harmonic univalent $V_{\mathfrak{R}\mathfrak{S}}$. Motivated by Kamali [9], we present a novel of kamali class with $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ brand-new class of harmonic univalent functions $\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}$ inspiring inequality. Analysing Mittag-Leffler function convolution with modified tremblay operator inequality as a necessary and sufficient condition for univalent harmonic functions related to specific generalised Mittag-Leffler functions to be in the function class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ is the aim of this research. Moreover, we discover extreme points, a distortion theorem, convolution properties, and convex combinations for the functions in $V_{\mathfrak{R}\mathfrak{S}}(\delta)$.

1. Introduction

The intricate relationships between geometric functions, hypergeometric functions, and harmonic functions have been extensively studied in mathematical analysis, as evidenced by various seminal papers ([1], [2], [3], [4], [5]). This research aims to delve deeper into these connections by focusing on the Mittag-Leffler function and its convolution with the modified Tremblay operator. Specifically, we aim to explore the necessary and sufficient conditions for univalent harmonic functions, associated with particular generalized Mittag-Leffler functions, to belong to the function class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$.

Geometric functions, known for their role in mapping geometric shapes within the complex plane, and hypergeometric functions, which generalize a broad spectrum of classical functions, both play crucial roles in various applications of complex analysis and mathematical physics. Harmonic

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functions, which satisfy Laplace's equation, are pivotal in modeling equilibrium states in physical phenomena.

The Mittag-Leffler function, a generalization of the exponential function, is particularly significant in the theory of fractional differential equations and processes with memory effects. The modified Tremblay operator, a transformation tool within function spaces, is used to analyze inequalities and function transformations.

This research will analyze the convolution of the Mittag-Leffler function with the modified Tremblay operator and examine its implications for univalent harmonic functions. By identifying the necessary and sufficient conditions for these functions to be part of the class $V_{\Re\mathfrak{S}}(\delta)$, we aim to contribute to the understanding of how special functions interplay with harmonic functions under specific conditions.

Harmonic functions are frequently recognized for being utilized in the study of minimum surfaces and are essential in numerous challenges in appropriate mathematics. The harmonic functions have been investigated by various researchers of differential geometrics, especially Choquest [6], Kneser [10], Lewy [13], and Rado [15]. The fundamental theory of complex harmonic univalent functions $\mathfrak{G}$ defined in the open unit disc $\rho = \{\mathfrak{z} : |\mathfrak{z}| < 1\}$ was created by Clunie and Sheil-Small in 1984 [7]. These are the purposes for which $\mathfrak{G}(0) = \mathfrak{G}_{\mathfrak{z}}(0) - 1 = 0$

Consider that $\Re\mathfrak{S}$ is the family of all harmonic functions of the form $\mathfrak{G} = \upsilon + \overline{\eta}$, where

$$\upsilon(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} c_{\xi}\mathfrak{z}^{\xi}, \overline{\eta}(\mathfrak{z}) = \sum_{\xi=1}^{\infty} d_{\xi}\overline{\mathfrak{z}}^{\xi}, |d_1| = 1 \tag{1}$$

are analytic in the open unit disk ρ. In the meanwhile, let $V_{\Re\mathfrak{S}}$ occur for the family of sense-preserving and harmonic univalent functions $\mathfrak{G} = \upsilon + \overline{\eta}$. Remember that if η is zero, the family $V_{\Re\mathfrak{S}} = V$.

Further, we classify $V^0_{\Re\mathfrak{S}}$ of $V_{\Re\mathfrak{S}}$ as

$$V^0_{\Re\mathfrak{S}} = \{\mathfrak{G} = \upsilon + \overline{\eta} \in V_{\Re\mathfrak{S}}, \eta'(0) = d_1 = 0\}$$

The classes $V^0_{\Re\mathfrak{S}}$ and $V_{\Re\mathfrak{S}}$ were first studied in [5].

First,the extrapolation of $E_{\alpha}(\mathfrak{z})$ stated by Wiman [21] is the two-parametric M-L function of $\mathfrak{z} \in C$ defined by the series,

$$E_{\alpha}(\mathfrak{z}) = \sum_{\xi=0}^{\infty} \frac{\mathfrak{z}^{\xi}}{\Gamma(1+\alpha\xi)}, \alpha \in C, \alpha \geq 0, \mathfrak{z} \in C \tag{2}$$

$$E_{\alpha,\beta}(\mathfrak{z}) = \sum_{\xi=0}^{\infty} \frac{\mathfrak{z}^{\xi}}{\Gamma(\beta+\alpha\xi)(\delta)_{\xi}}, \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, \mathfrak{z} \in C \tag{3}$$

Numerical calculations of the Mittag-Leffler function (2) and some of its numerous generalisations across the entire complex plane have only recently been made (see, for example, [9, 16, 18]). Prabhakar [14] developed an extension of the Mittag-Leffler function $E_{\alpha,\beta}(\mathfrak{z})$ of (3) using the series representation in the way mentioned below:

$$E^{\gamma}_{\alpha,\beta}(\mathfrak{z}) = \sum_{\xi=0}^{\infty} \frac{(\gamma)_\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi} \frac{\mathfrak{z}^\xi}{\xi!}, \alpha,\beta,\gamma \in C, R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, \mathfrak{z} \in C \tag{4}$$

Where $(\gamma)_\xi$ denotes the families pochhammer symbol, since

$$(1)_\xi = \xi!, (\xi \in N_0),$$

specified in the terms of the well-known Gamma function and for $(k, m \in C)$ by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} == \begin{cases} 1, k=0; \lambda \in C/\{0\} \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+m-1), k=m \in \mathbb{N}; \lambda \in C \end{cases} \tag{5}$$

We obvious have the following unique situations:

$$E^1_{\alpha,\beta}(\mathfrak{z}) = E_{\alpha,\beta}(\mathfrak{z}); E^1_{\alpha,1}(\mathfrak{z}) = E_\alpha(\mathfrak{z}).$$

In fact, the generalised Mittag-leffler function $E^{\gamma}_{\alpha,\beta}(\mathfrak{z})$ itself is actually a very specific example of a quite well studied function ${}_p\Psi_q$, as shown below (see also Eq. (1.9.1) [12], p. 45) , as previously noted by Srivastava and Saxena [18], (p. 201, Eq. (1.6)). In this instance and subsequent discussions, ${}_p\Psi_q$ signifies the Wright (or, more fittingly, the Fox-Wright) expansion of the hypergeometric ${}_pF_q$ function, as described by (see, for instance, to [17], p. 21).

$${}_p\Psi_q \begin{pmatrix} [(\mathfrak{a}_1, \mathfrak{A}_1) & ... & (\mathfrak{a}_p, \mathfrak{A}_p) \\ \\ (\mathfrak{b}_1, \mathfrak{B}_1) & ... & (\mathfrak{b}_q, \mathfrak{B}_q)] \end{pmatrix} \mathfrak{z} := \sum_{m=0}^{\infty} \frac{\Gamma(\mathfrak{a}_1 + \mathfrak{A}_1 m)...\Gamma(\mathfrak{a}_p + \mathfrak{A}_p m)}{\Gamma(\mathfrak{b}_1 + \mathfrak{B}_1 m)...\Gamma(\mathfrak{b}_q + \mathfrak{B}_q m)} \frac{\mathfrak{z}^m}{m!} \tag{6}$$

$R(\mathfrak{A}_i) > 0, (i = 1,2...p); R(\mathfrak{B}_i) > 0, (i = 1,2...q)$, in which we have assumed,in general that $\mathfrak{a}_i, \mathfrak{A}_i \in C(i = 1,2,...,p)$ and $\mathfrak{b}_i, \mathfrak{B}_i \in C(i = 1,2,...,q)$ and that the equality in the convergence condition hold true only for suitably bounded values of $|\mathfrak{z}|$. Salim [20] revealed the function in the form $E^{\gamma,\delta}_{\alpha,\beta}$in the following form, improving the M-L function to four additional parameters.

$$E^{\gamma,\delta}_{\alpha,\beta}(\mathfrak{z}) = \sum_{\xi=0}^{\infty} \frac{(\gamma)_\xi \mathfrak{z}^\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi},$$

Where $\mathfrak{z}, \alpha, \beta, \gamma, \delta \in C, R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, R(\delta) > 0$.
Recently, Salim and Faraj [5] introduced a new generalization of Mittag-Leffler type function as

$$E^{\gamma,\delta}_{\alpha,\beta}(\mathfrak{z}) = \sum_{\xi=0}^{\infty} \frac{(\gamma)_\xi \mathfrak{z}^\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi}, \tag{7}$$

Where $\mathfrak{z}, \alpha, \beta, \gamma, \delta \in C, min\Big(R(\alpha), R(\beta), R(\gamma), R(\delta)\Big) > 0$.
We introduced a new generalization of the Mittag Leffler type of harmonic function.

$$E^{\gamma,\delta}_{\alpha,\beta}(\mathfrak{z}) = \sum_{\xi=2}^{\infty} \frac{(\gamma)_\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi} \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} \frac{(\gamma)_\xi \mathfrak{z}^\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi} \bar{\mathfrak{z}}^\xi,$$

Definition 1.1:
If $\mathfrak{G} \in V_{\mathfrak{R}\mathfrak{S}}$, then the Tremblay fractional operator $T_{\mathfrak{z}}^{\epsilon,\nu}$ of a function $\mathfrak{G}$ is defined, for all $\mathfrak{z} \in U$,by

$$T_{\mathfrak{z}}^{\epsilon,\nu}\mathfrak{G}(\mathfrak{z}) = \frac{\Gamma(\nu)}{\Gamma(\epsilon)}\mathfrak{z}^{1-\nu}D_{\mathfrak{z}}^{\epsilon-\nu}\mathfrak{z}^{\epsilon-1}\mathfrak{G}(\mathfrak{z}), (0 \leq \nu \leq 1, 0 \leq \epsilon \leq 1, 0 \leq \nu - \epsilon < 1, \epsilon > \nu),$$

It is clear that for $\nu = \epsilon = 1$, we obtain

$$T_{\mathfrak{z}}^{1,1}\mathfrak{G}(\mathfrak{z}) = \mathfrak{G}(\mathfrak{z}).$$

In [8], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 1.2:
If $\mathfrak{G} \in V_{\mathfrak{R}\mathfrak{S}}$. Then the modified Tremblay operator denoted by $T_{\mathfrak{z}}^{\epsilon,\nu} : V_{\mathfrak{R}\mathfrak{S}} \to V_{\mathfrak{R}\mathfrak{S}}$ and defined as:

$$T_{\mathfrak{z}}^{\epsilon,\nu} = \frac{\nu}{\epsilon}T_{\mathfrak{z}}^{\epsilon,\nu}\mathfrak{G}(\mathfrak{z}) = \frac{\Gamma(\nu+1)}{\Gamma(\epsilon+1)}\mathfrak{z}^{1-\nu}D_{\mathfrak{z}}^{\epsilon-\nu}\mathfrak{z}^{\epsilon-1}\mathfrak{G}(\mathfrak{z})$$

$$\mathfrak{z} + \sum_{\xi=2}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}c_{\xi}\mathfrak{z}^{\xi} + \sum_{\xi=1}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}d_{\xi}\overline{\mathfrak{z}}^{\xi}$$

$(0 \leq \nu \leq 1, 0 \leq \epsilon \leq 1, 0 \leq \nu - \epsilon < 1, \epsilon > \nu)$ where $T^{\epsilon,\nu}$ is denoted the Tremblay fractional derivative operator defined by previous definition. For more information about Tremblay operator see [19]. The modified Tremblay operator is defined as follows

$$T_{\mathfrak{z}}^{\epsilon,\nu}\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}c_{\xi}\mathfrak{z}^{\xi} + \sum_{\xi=1}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}d_{\xi}\overline{\mathfrak{z}}^{\xi} \tag{8}$$

Recently, Salim and Faraj [5] introduced a new generalization of Mittag-Leffler function associated with fractional differential operator, we discovered (7) and (8) an M-L function convolution using a modified trembley operator.

$$E_{\alpha,\beta}^{\gamma,\delta}(\mathfrak{z}) * T_{\mathfrak{z}}^{\epsilon,\nu}\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \Big(\sum_{\xi=2}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_{\xi}}{\Gamma(\beta+\alpha\xi)(\delta)_{\xi}}\Big)c_{\xi}\mathfrak{z}^{\xi} + \Big(\sum_{\xi=1}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_{\xi}}{\Gamma(\beta+\alpha\xi)(\delta)_{\xi}}\Big)d_{\xi}\overline{\mathfrak{z}}^{\xi} \tag{9}$$

where,we assume

$$E_{\alpha,\beta}^{\gamma,\delta}(\mathfrak{z}) * T_{\mathfrak{z}}^{\epsilon,\nu}\mathfrak{G}(\mathfrak{z}) = \mathfrak{P}_{\alpha,\beta,\mathfrak{z}}^{\gamma,\delta,\epsilon,\nu}\mathfrak{z}\mathfrak{G}(\mathfrak{z}) \tag{10}$$

$$\Big(\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_{\xi}}{\Gamma(\beta+\alpha\xi)(\delta)_{\xi}}\Big) = \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}$$

Kamali formulated a collection of functions in $V_{\mathfrak{R}\mathfrak{S}}$ existence that satisfies the following inequality,

$$\Re\left(\frac{\delta\mathfrak{z}^2(\mathfrak{G}(\mathfrak{z}))''' + (2\delta+1)\mathfrak{z}(\mathfrak{G}(\mathfrak{z}))'' + (\mathfrak{G}(\mathfrak{z}))'}{\delta\mathfrak{z}^2(\mathfrak{G}(\mathfrak{z}))'' + \mathfrak{z}(\mathfrak{G}(\mathfrak{z}))'}\right) > \beta$$

where $0 \leq \beta < 1, 0 \leq \delta < 1$. Inspired by Kamali [11], we present a novel kamali class with $V_{\mathfrak{K}\mathfrak{S}}(\delta)$ brand-new class of harmonic univalent function, $\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z})$ inspiring the following inequality,

$$\left(\frac{\delta\mathfrak{z}^2(\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}(\mathfrak{G}\mathfrak{z}))''' + (2\delta+1)\mathfrak{z}(\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z}))'' + (\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z}))'}{\delta\mathfrak{z}^2(\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z}))'' + \mathfrak{z}(\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z}))'}\right) > \beta \tag{11}$$

The connectivity of geometric functions and hypergeometric functions with harmonic functions is seen through some of these papers [1–5]. Analysing Mittag-Leffler function convolution with modified Trembley operator inequality as a necessary and sufficient condition for univalent harmonic functions related to specific generalized Mittag-Leffler functions are to be in the function class $V_{\mathfrak{K}\mathfrak{S}}(\delta)$ is the aim of this research. Moreover, we discover extreme points, a distortion theorem, and convolution properties, and convex combinations for the functions in $V_{\mathfrak{K}\mathfrak{S}}(\delta)$.

2. Coefficient properties

We begin by establishing a necessary and sufficient coefficient condition for functions $\mathfrak{G}(\mathfrak{z})$ that belong to the class $V_{\mathfrak{K}\mathfrak{S}}(\delta)$. This class of functions is characterized by specific properties that are influenced by the parameter δ. To determine whether a function $\mathfrak{G}(\mathfrak{z})$ is a member of $V_{\mathfrak{K}\mathfrak{S}}(\delta)$, we analyze its coefficients through the lens of the Mittag-Leffler function and the modified Tremblay operator.

Theorem 2.1. *Let (1) as the value of $\mathfrak{G} = \upsilon + \overline{\eta}$. If $G(z) \in VKS(\delta)$,*

$$\left(\sum_{\xi=2}^{\infty}\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)(\xi-1) + 1 - (\delta(\xi-1)+1)\Big)\right)\sum_{\xi=2}^{\infty}\xi c_\xi +$$
$$\left(\sum_{\xi=1}^{\infty}\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)(\xi-1) + 1 - (\delta(\xi-1)+1)\Big)\right)\sum_{\xi=1}^{\infty}\xi d_\xi$$

$$\leq 1 \tag{12}$$

Proof: *We implement equation(9)*

$$\mathfrak{P}^{\gamma,\delta,\epsilon,\nu}_{\alpha,\beta,\mathfrak{z}}\mathfrak{z}\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \Big(\sum_{\xi=2}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi}\Big)c_\xi\mathfrak{z}^\xi + \Big(\sum_{\xi=1}^{\infty}\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi}\Big)d_\xi\overline{\mathfrak{z}}^\xi$$

where,

$$\Big(\frac{\Gamma(\xi+\nu)\Gamma(\nu+1)}{\Gamma(\xi+\epsilon)\Gamma(\epsilon+1)}\Big)\Big(\frac{(\gamma)_\xi}{\Gamma(\beta+\alpha\xi)(\delta)_\xi}\Big) = \Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}$$

$$\left|\frac{\sum_{\xi=2}^{\infty}\left(\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}(\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1c_\xi\xi(\mathfrak{z})^{\xi-1}\right)+1+\left(\sum_{\xi=1}^{\infty}\left(\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}(\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1d_\xi\xi\overline{(\mathfrak{z})}^{\xi-1}\right)\right.}{\sum_{\xi=2}^{\infty}\left(\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}(\delta((\xi-1)+1)c_\xi\xi(\mathfrak{z})^{\xi}\right)+\mathfrak{z}+\sum_{\xi=1}^{\infty}\left(\Lambda^{\nu,\gamma}_{\alpha,\beta,\xi}(\delta(\xi-1)+1)\xi d_\xi\overline{(\mathfrak{z})}^{\xi}\right)}\right|$$

$$< 1$$

$$\Bigg|\sum_{\xi=2}^{\infty}\Big(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\left(\delta(\xi-2)(\xi-1)\right)+(2\delta+1)(\xi-1)+1\Big)c_\xi\xi(\mathfrak{z})^{\xi-1}+1$$

$$+\left(\sum_{\xi=1}^{\infty}\Big(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\left(\delta(\xi-2)(\xi-1)\right)+(2\delta+1)(\xi-1)+1\Big)d_\xi\xi\overline{(\mathfrak{z})}^{\xi-1}\right)$$

$$-\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta(\xi-1)+1)c_\xi\xi(\mathfrak{z})^{\xi}-\mathfrak{z}-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta(\xi-1)+1)\xi d_\xi\overline{(\mathfrak{z})}^{\xi}\Bigg|<1$$

We take $|\mathfrak{z}|=r, 0\le r<1$

$$\sum_{\xi=2}^{\infty}\Big(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1c_\xi\xi\Big)+1+\Bigg(\sum_{\xi=1}^{\infty}\Big(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta(\xi-2)(\xi-1))$$

$$+(2\delta+1)(\xi-1)+1d_\xi\xi\Big)-\sum_{\xi=2}^{\infty}\Big(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta((\xi-1)+1)c_\xi\xi\Big)-1-\sum_{\xi=1}^{\infty}(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}(\delta(\xi-1)+1)\xi d_\xi)\Bigg)<1$$

Which implies the result,

$$\left(\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1\Big)-\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\sum_{\xi=2}^{\infty}\xi c_\xi$$

$$+\left(\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1\Big)-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\sum_{\xi=1}^{\infty}\xi d_\xi\le 1$$

In order to preserve brevity in this piece of work, we will assume,

$$k_\xi(\delta)=\left(\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)(\xi-1)+1\Big)-\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)$$

Unless particularly specified. The coefficient estimates for functions in $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ *are provided in the following result.*

Theorem 2.2. *Let* $\mathfrak{G}(\mathfrak{z})\in V_{\mathfrak{R}\mathfrak{S}}(\delta)$

$$c_\xi=\frac{1}{\xi k_\xi(\delta)},\xi=2,3,... \tag{13}$$

$$d_\xi=\frac{1}{\xi k_\xi(\delta)},\xi=1,2,... \tag{14}$$

For the functions $\mathfrak{G}_\xi(\mathfrak{z})$ *given the outcome is accurate.*

$$\mathfrak{G}_\xi(\mathfrak{z})=\mathfrak{z}+\sum_{\xi=2}^{\infty}\frac{1}{\xi k_\xi(\delta)}\mathfrak{z}^\xi+\sum_{\xi=1}^{\infty}\frac{1}{\xi k_\xi(\delta)}\overline{\mathfrak{z}}^\xi \tag{15}$$

Proof: *If* $\mathfrak{G}_\xi(\mathfrak{z})\in V_{\mathfrak{R}\mathfrak{S}}(\delta)$*, then we have, for each* ξ

$$\xi k_\xi(\delta)c_\xi\le\xi k_\xi(\delta)d_\xi\le\sum_{\xi=2}^{\infty}\xi k_\xi(\delta)c_\xi\le\sum_{\xi=1}^{\infty}\xi k_\xi(\delta)d_\xi\le 1.$$

Furthermore, we have

$$c_\xi \le \sum_{\xi=2}^{\infty} \frac{1}{\xi k_\xi(\delta)}$$

$$d_\xi \le \sum_{\xi=1}^{\infty} \frac{1}{\xi k_\xi(\delta)}$$

Since

$$\mathfrak{G}_\xi(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} \frac{1}{\xi k_\xi(\delta)} \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} \frac{1}{\xi k_\xi(\delta)} \bar{\mathfrak{z}}^\xi$$

accomplishes the specifications of Theorem 2.1, $\mathfrak{G}_\xi(\mathfrak{z}) \in V_{\mathfrak{K}\mathfrak{S}}(\delta)$ *and for this function, equality is achieved.*

3. Distortion bounds

Theorem 3.1. *Let* $\mathfrak{G}(\mathfrak{z}) \in V_{\mathfrak{K}\mathfrak{S}}(\delta)$,*therefore*

$$r - \frac{1}{k_\xi(\delta)} r - \frac{1}{k_\xi(\delta)} \bar{r} \le |\mathfrak{G}(\mathfrak{z})| \le r + \frac{1}{k_\xi(\delta)} r + \frac{1}{k_\xi(\delta)} \bar{r} \tag{16}$$

The outcome is accurate for

$$\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \frac{1}{k_\xi(\delta)} \mathfrak{z} + \frac{1}{k_\xi(\delta)} \bar{\mathfrak{z}} \tag{17}$$

Proof: *If* $\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} \frac{1}{\xi k_\xi(\delta)} \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} \frac{1}{\xi k_\xi(\delta)} \bar{\mathfrak{z}}^\xi$, *we take*

$$|\mathfrak{G}(\mathfrak{z})| \le \mathfrak{z} + \sum_{\xi=2}^{\infty} \frac{1}{\xi k_\xi(\delta)} \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} \frac{1}{\xi k_\xi(\delta)} \bar{\mathfrak{z}}^\xi \le r + r \sum_{\xi=2}^{\infty} c_\xi + \bar{r} \sum_{\xi=1}^{\infty} d_\xi$$

Then,

$$\sum_{\xi=2}^{\infty} c_\xi \le \frac{1}{k_\xi(\delta)}, \sum_{\xi=1}^{\infty} d_\xi \le \frac{1}{k_\xi(\delta)}$$

This contributed to us to,

$$|\mathfrak{G}(\mathfrak{z})| \le r + \frac{1}{k_\xi(\delta)} r + \frac{1}{k_\xi(\delta)} \bar{r}$$

Likewise,

$$|\mathfrak{G}(\mathfrak{z})| \ge r - \frac{1}{k_\xi(\delta)} r - \frac{1}{k_\xi(\delta)} \bar{r}$$

The outcome is accurate for $\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \frac{1}{k_\xi(\delta)} \mathfrak{z} + \frac{1}{k_\xi(\delta)} \bar{\mathfrak{z}}$.

Exectly like the following also is applicable:

Theorem 3.2. *Let* $\mathfrak{G}(\mathfrak{z}) \in V_{\mathfrak{K}\mathfrak{S}}(\delta)$, *therefore*

$$1 - \frac{1}{k_\xi(\delta)} - \frac{1}{k_\xi(\delta)} \le |\mathfrak{G}'(\mathfrak{z})| \le 1 + \frac{1}{k_\xi(\delta)} + \frac{1}{k_\xi(\delta)} \tag{18}$$

The outcome is accurate for the function given by (16).

4. Convolution and convex combination

In this section, we establish the invariance properties of the function class $\mathfrak{G}(\mathfrak{z})$ within the class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$. Specifically, we prove that the class $\mathfrak{G}(\mathfrak{z})$ retains its membership in $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ under the operations of convolution and convex combination.

Convolution and convex combination are fundamental operations in the analysis of functions, and proving that $\mathfrak{G}(\mathfrak{z})$ is closed under these operations is crucial for understanding the stability and structural properties of this function class.

Invariance under convolution indicates that the combined effect of two functions within the class also belongs to the same class, preserving key properties. Similarly, invariance under convex combination ensures that any weighted average of functions in the class remains within the class, reflecting the robustness of $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ under linear combinations.

By proving these invariance properties, we demonstrate the robustness and stability of the class $\mathfrak{G}(\mathfrak{z})$ under these common function operations, providing deeper insight into the structure and applications of functions within $V_{\mathfrak{R}\mathfrak{S}}(\delta)$.

If harmonic functions $\mathfrak{G}$ of the form

$$\mathfrak{G}(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} |c_\xi| \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} |d_\xi| \overline{\mathfrak{z}}^\xi \tag{19}$$

$$G(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} |C_\xi| \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} |D_\xi| \overline{\mathfrak{z}}^\xi \tag{20}$$

The definition of the convolution of $G(\mathfrak{z})$ and $\mathfrak{G}(\mathfrak{z})$ is

$$(\mathfrak{G} * G)(\mathfrak{z}) = \mathfrak{G}(\mathfrak{z}) * G(\mathfrak{z}) = \sum_{\xi=2}^{\infty} |c_\xi||C_\xi| \mathfrak{z}^\xi + \sum_{\xi=1}^{\infty} |d_\xi||D_\xi| \overline{\mathfrak{z}}^\xi \tag{21}$$

Theorem 4.1. *If $0 \leq \delta \leq \zeta < 1$, and $G \in V_{\mathfrak{R}\mathfrak{S}}(\zeta)$, $(\mathfrak{G} * G) \in V_{\mathfrak{R}\mathfrak{S}}(\delta) \subset V_{\mathfrak{R}\mathfrak{S}}(\zeta)$*
When the convolution

$$(\mathfrak{G} * G) \in V_{\mathfrak{R}\mathfrak{S}}(\delta) \subset V_{\mathfrak{R}\mathfrak{S}}(\zeta) \tag{22}$$

Proof: *Then the convolution $(\mathfrak{G} * G)$ is given by (21). We wish to show that the coefficients $(\mathfrak{G} * G)$ of satisfy the required condition given in Theorem 5 For $V_{\mathfrak{R}\mathfrak{S}}(\zeta)$ we note that $|C_\xi| \leq 1$ and $|D_\xi| \leq 1$.Now, for the convolution function $(\mathfrak{G} * G)$,we obtain*

$$\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty} |c_\xi||C_\xi| +$$

$$\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty} |d_\xi||D_\xi|$$

$$\leq \left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty}|c_\xi| +$$
$$\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty}|d_\xi|$$

Therefore,

$$\left(\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty}|c_\xi| +\right.$$
$$\left.\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty}|d_\xi|\right) \leq 1$$

Then $0 \leq \delta \leq \zeta < 1$, *and* $G \in V_{\mathfrak{R}\mathfrak{S}}(\zeta)$, $(\mathfrak{G} * G) \in V_{\mathfrak{R}\mathfrak{S}}(\delta) \subset V_{\mathfrak{R}\mathfrak{S}}(\zeta)$.

Theorem 4.2. *Consider a convex linear combination, class* $V_{\mathfrak{R}\mathfrak{S}}(\zeta)$ *is closed.*

Proof: *If* $k = 1, 2, \ldots$, *suppose that* $\mathfrak{G}_k \in V_{\mathfrak{R}\mathfrak{S}}(\zeta)$

$$\mathfrak{G}_k(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty}|c_{\xi,k}|\mathfrak{z}^{\xi} + \sum_{\xi=1}^{\infty}|d_{\xi,k}|\overline{\mathfrak{z}}^{\xi} \tag{23}$$

$$\left(\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty}|c_{\xi,k}| +\right.$$
$$\left.\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty}|d_{\xi,k}|\right) \leq 1$$

If $\sum_{\xi=1}^{\infty} t_k = 1, 0 \leq t_k < 1$,*the convex combination of* $\mathfrak{G}_k$,*it might be written as*

$$\sum_{\xi=1}^{\infty} t_k\mathfrak{G}_k(\mathfrak{z}) = \mathfrak{z} + \sum_{\xi=2}^{\infty} t_k|c_{\xi,k}|\mathfrak{z}^{\xi} + \sum_{\xi=1}^{\infty} t_k|d_{\xi,k}|\overline{\mathfrak{z}}^{\xi} \tag{24}$$

By the theorem(5) we take,

$$\left(\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty} t_k|c_{\xi,k}| +\right.$$
$$\left.\left(\frac{\sum_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty} t_k|d_{\xi,k}|\right) \leq 1$$

$$\left(\sum_{k=1}^{\infty} t_k \left(\frac{\sum\limits_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum\limits_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty} c_{\xi,k} + \right.$$
$$\left.\left(\frac{\sum\limits_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum\limits_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty} d_{\xi,k}\right) \leq 1$$

We substitute $\sum\limits_{k=1}^{\infty} t_k = 1$

$$\left(\left(\frac{\sum\limits_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum\limits_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=2}^{\infty} |c_{\xi,k}| + \right.$$
$$\left.\left(\frac{\sum\limits_{\xi=2}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1)) + (2\delta+1)((\xi-1)+1)\Big) - \sum\limits_{\xi=1}^{\infty} \Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)\sum_{\xi=1}^{\infty} |d_{\xi,k}|\right) \leq 1$$

Therefore $\sum\limits_{k=1}^{\infty} t_k, \mathfrak{G}_k(\mathfrak{z}) \in V_{\mathfrak{R}\mathfrak{S}}(\delta) \subset V_{\mathfrak{R}\mathfrak{S}}(\zeta)$.

Corollary 4.1. *Convex linear combinations result in the class* $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ *is closed.*
Proof: *Let* $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ *be the class that contains the functions* $\mathfrak{G}_k(\mathfrak{z})(k = 1, 2, ...)$ *described by (23). After that is finished, the function* $\mathfrak{h}(\mathfrak{z})$ *defined by*

$$\mathfrak{h}(\mathfrak{z}) = \omega\mathfrak{G}_k(\mathfrak{z}) + (1-\omega)\mathfrak{G}_k\mathfrak{z}, 0 \leq \omega \leq 1 \tag{25}$$

be in the class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$

The following theorem explained the neighbourhood result for the class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$.

5. Neighbourhood for the class $V_{\mathfrak{R}\mathfrak{S}}(\delta)$

Following our discussion on the invariance properties of the function class $\mathfrak{G}(\mathfrak{z})$ within $V_{\mathfrak{R}\mathfrak{S}}(\delta)$, we next focus on the inclusion of all neighborhoods related to the inclusion relation. This step employs Ruscheweyh's approach to analyze and explain the (n, β)-neighborhood of a function $\mathfrak{G}(\mathfrak{z}) \in V_{\mathfrak{R}\mathfrak{S}}(\delta)$.
Ruscheweyh's approach provides a systematic method to define and understand the neighborhoods of functions within certain classes, facilitating the study of their stability and inclusion properties. By examining the (n, β)-neighborhood, we can determine how small perturbations in the coefficients affect the membership of functions within $V_{\mathfrak{R}\mathfrak{S}}(\delta)$. This analysis is crucial for understanding the robustness of functions in this class under various transformations and perturbations.

Theorem 5.1. *Let*

$$\beta=\left(\left(\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\right)+\right. \tag{26}$$
$$\left.\left(\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\right)\right)\leq 1$$

that is $V_{\Re\Im}(\delta)\in N_{n,\beta}(e)$

Proof:*If* $\mathfrak{G}\in V_{\Re\Im}(\delta)$*,then*

$$\sum_{\xi=2}^{\infty}\left(\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\xi c_\xi+$$
$$\sum_{\xi=1}^{\infty}\left(\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\xi d_\xi\leq 1$$

Hence,

$$\left(\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\sum_{\xi=2}^{\infty}\xi c_\xi+$$
$$\left(\sum_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)\right)\sum_{\xi=1}^{\infty}\xi d_\xi\leq 1$$

and which implies

$$\sum_{\xi=2}^{\infty}\xi c_\xi+\sum_{\xi=1}^{\infty}\xi d_\xi$$
$$\leq\left(\left(\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\right)\right.$$
$$\left.+\left(\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\right)\right)=\beta$$

6. Extreme points

After that, applying the expression clco $V_{\Re\Im}(\delta)$, we derive the extreme points of the closed convex hulls of $V_{\Re\Im}(\delta)$.

Theorem 6.1. *Consider* $\mathfrak{G}$ *to be established using (21). Then* $\mathfrak{G}\in V_{\Re\Im}(\delta)$ *if and only if*

$$\mathfrak{G}(\mathfrak{z})=\sum_{\xi=1}^{\infty}[H_\xi v_\xi(\mathfrak{z})+I_\xi\overline{\eta}_\xi(\mathfrak{z})], \tag{27}$$

Where

$$v_1(\mathfrak{z})=\mathfrak{z}, v_\xi(\mathfrak{z})=\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\mathfrak{z}^\xi,(\xi=2,3,...),$$

$$\overline{\eta}_\xi(\mathfrak{z})=\mathfrak{z}+\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\overline{\mathfrak{z}}^\xi,(\xi=1,2,...),$$

$H_1=1, \sum\limits_{\xi=2}^{\infty}H_\xi+\sum\limits_{\xi=1}^{\infty}I_\xi\geq 0, H_\xi\geq 0, I_\xi\geq 0$. *The extreme points of* $V_{\mathfrak{R}\mathfrak{S}}(\delta)$ *are especially* v_ξ *and* $\overline{\eta}_\xi$.

Proof: *We have the following form* $\mathfrak{G}$ *functions with form (27):*

$$\mathfrak{G}(\mathfrak{z})=\sum_{\xi=1}^{\infty}[H_\xi v_\xi(\mathfrak{z})+I_\xi\overline{\eta}_\xi(\mathfrak{z})],$$

$$\sum_{\xi=1}^{\infty}(H_\xi+I_\xi)\mathfrak{z}+\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\mathfrak{z}^\xi$$
$$+\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}\overline{\mathfrak{z}}^\xi$$

We get

$$\left(\frac{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)$$
$$\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}H_\xi+$$
$$\left(\frac{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}\right)$$
$$\frac{1}{\sum\limits_{\xi=2}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\sum\limits_{\xi=1}^{\infty}\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}I_\xi$$

$$=\sum_{\xi=2}^{\infty}H_\xi+\sum_{\xi=1}^{\infty}I_\xi\leq 1$$

then $\mathfrak{G}\in coloV_{\mathfrak{R}\mathfrak{S}}(\delta)$. *Consider, however that* $\mathfrak{G}\in coloV_{\mathfrak{R}\mathfrak{S}}(\delta)$ *Setting,*

$$H_\xi=\frac{\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}|c_\xi|, 0\leq H_\xi\leq 1,\xi=1,2,3,...$$

$$I_\xi=\frac{\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big((\delta(\xi-2)(\xi-1))+(2\delta+1)((\xi-1)+1)\Big)-\Lambda_{\alpha,\beta,\xi}^{\nu,\gamma}\Big(\delta(\xi-1)+1\Big)}{1}|d_\xi|, 0\leq I_\xi\leq 1,\xi=1,2,3,...$$

and $H_1 = 1$, $\sum_{\xi=2}^{\infty} H_\xi + \sum_{\xi=1}^{\infty} I_\xi \geq 0$ *observe that, according to* $H_1 \geq 0$. *As a result, we acquire*
$\mathfrak{G}(\mathfrak{z}) = \sum_{\xi=1}^{\infty} [H_\xi \upsilon_\xi(\mathfrak{z}) + I_\xi \overline{\eta}_\xi(\mathfrak{z})$ *as needed.*

7. Applications:

They are used in conformal mapping, which is essential in complex analysis, fluid dynamics, and other areas of applied mathematics. They appear in numerous problems in mathematical physics, such as in the study of differential equations, and are also crucial in probability theory and statistics. They are used to model physical phenomena where equilibrium states are studied, such as temperature distribution in a steady-state. It is used in modeling processes with memory effects and in fractional calculus. It is employed in various inequality analyses and in the study of function spaces and their properties.

8. Conclusion:

In this paper we explained about the harmonic univalent functions $\mathfrak{G} = \upsilon + \overline{\eta}$ defined in the open unit disc ρ. The convolution featuring the Mittag-Leffler function and fractional operator is applied to generate the family of harmonic univalent $V_{\Re\mathfrak{S}}$. Analysing Mittag-Leffler function convolution with modified tremblay operator inequality as a necessary and sufficient condition for univalent harmonic functions related to specific generalized Mittag-Leffler functions to be in the function class $V_{\Re\mathfrak{S}}(\delta)$ is the aim of this research.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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