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A Strongly Convergent Hybrid Method to Unify Split Generalized Mixed Equilibrium and Fixed Point Problems

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Abstract. The aim of this paper is twofold, we propose an extension to the split generalized mixed equilibrium problem firstly and introduce an iterative method based on a hybrid extragradient method. The goal is to efficiently find a common solution for both the split generalized mixed equilibrium problem and the fixed point problem concerning a nonexpansive mapping within the context of real Hilbert spaces. We conduct a thorough analysis of the proposed iterative method and establish a strong convergence theorem under certain mild conditions. Moreover, we present various implications derived from our main result and conduct numerical experiments to validate our findings. Our outcomes represent a substantial expansion and generalization of existing iterative methods and results within this field.

1. Introduction

The Ky Fan inequality, known as the equilibrium problem (EP), has been extensively studied in [1,2]. However, it was in 1994 when Blum and Oettli [3] used the term equilibrium problem. Their work focused on discussing existence theorems and variational principles for EP, which have a significant impact on various branches of pure and applied sciences. In [3], it has been demonstrated that the theory of EP offers a natural, novel, and unified framework for solving a wide range of problems in nonlinear analysis, optimization, economics, finance, game theory, image reconstruction, ecology, transportation, network analysis, physics, and engineering. Notably, the EP encompasses various mathematical problems, including mathematical programming problems, complementarity problems, variational inequality problems, saddle point problems, Nash equilibrium problems in noncooperative games, minmax inequality problems, minimization problems, and fixed point problems, which are discussed as special cases in [3–5] and the references therein.

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In recent years, EP have garnered significant attention in the development of efficient and implementable numerical techniques. These techniques encompass various approaches such as projection methods and their variants, the auxiliary principle technique, the proximal point algorithm, and the descent framework. These methods are designed to solve EP and their related problems effectively. The proximal point method (PPM), originally introduced and investigated by Martinet [6] for monotone variational inequality problems, was subsequently extended by Rockafellar [7] to monotone operators. Moudafi [8] further extended the PPM to EP involving monotone bifunctions. For further insights into related work, please refer to [9, 10] and the references cited therein.

In the field of functional analysis, there is considerable interest in determining the fixed points of nonexpansive mappings. To address this, several iterative methods have been devised to find a common element that simultaneously satisfies a set of EP (specifically, split equilibrium problems) and a set of fixed points belonging to a finite number of nonexpansive mappings. This topic has been extensively explored in the literature, including works such as [11–18] and the references cited therein.

Firstly, consider the real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, which possess a scalar product denoted as $\langle\cdot,\cdot\rangle$ and a norm denoted as $|\cdot|$. Furthermore, let $\mathscr{K}_1$ and $\mathscr{K}_2$ represent nonempty, closed, and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. If we have a sequence κ_n within $\mathscr{H}_1$, the notation $\kappa_n \to \kappa$ indicates strong convergence, while $\kappa_n \rightharpoonup \kappa$ denotes weak convergence of the sequence κ_n.

A mapping $S : \mathscr{K}_1 \to \mathscr{K}_1$ is said to be `nonexpansive` if $\|S\kappa - S\upsilon\| \leq \|\kappa - \upsilon\|$, $\forall \kappa, \upsilon \in \mathscr{K}_1$. The `fixed point problem` (in short, FPP) for a nonexpansive mapping S is:

$$\text{Find } \kappa \in \mathscr{K}_1 \text{ such that } \kappa \in \mathrm{Fix}(S). \tag{1.1}$$

We denote $\mathrm{Fix}(S)$, the set of solutions of FPP(1.1).

Next, we consider the `split Generalized mixed equilibrium problem` (in short, $\mathrm{S_pGMEP}$): Find $\bar{\kappa} \in \mathscr{K}_1$ such that

$$F(\bar{\kappa}, \kappa) + \langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle + \varphi(\kappa) - \varphi(\bar{\kappa}) \geq 0, \quad \forall \kappa \in \mathscr{K}_1, \tag{1.2}$$

and such that

$$\bar{\upsilon} = B\bar{\kappa} \in \mathscr{K}_2 \text{ satisfies } G(\bar{\upsilon}, \upsilon) + \langle g\bar{\upsilon}, \upsilon - \bar{\upsilon}\rangle + \psi(\upsilon) - \psi(\bar{\upsilon}) \geq 0, \quad \forall \upsilon \in \mathscr{K}_2, \tag{1.3}$$

where $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ and $G : \mathscr{K}_2 \times \mathscr{K}_2 \to \mathbb{R}$ are bifunctions, $f : \mathscr{K}_1 \to \mathscr{H}_1$, $g : \mathscr{K}_2 \to \mathscr{H}_2$ are nonlinear mappings, $\varphi : \mathscr{K}_1 \to \mathbb{R} \cup \{+\infty\}$, $\psi : \mathscr{K}_2 \to \mathbb{R} \cup \{+\infty\}$ are nonlinear functionals and $B : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. $\mathrm{S_P}$GMEP(1.2)-(1.3) represents a pair of Generalized mixed equilibrium problems in two different spaces, Their solution sets are denoted by Sol(GMEP(1.2)) and Sol(GMEP(1.3)) respectively. The solution set of $\mathrm{S_P}$GMEP(1.2)-(1.3) is denoted by Sol($\mathrm{S_P}$GMEP(1.2)-(1.3))= $\{\bar{\kappa} \in \mathscr{K}_1 : \bar{\kappa} \in \mathrm{Sol(GMEP(1.2))} : \text{and } B\bar{\kappa} \in \mathrm{Sol(GMEP(1.3))}\}$.

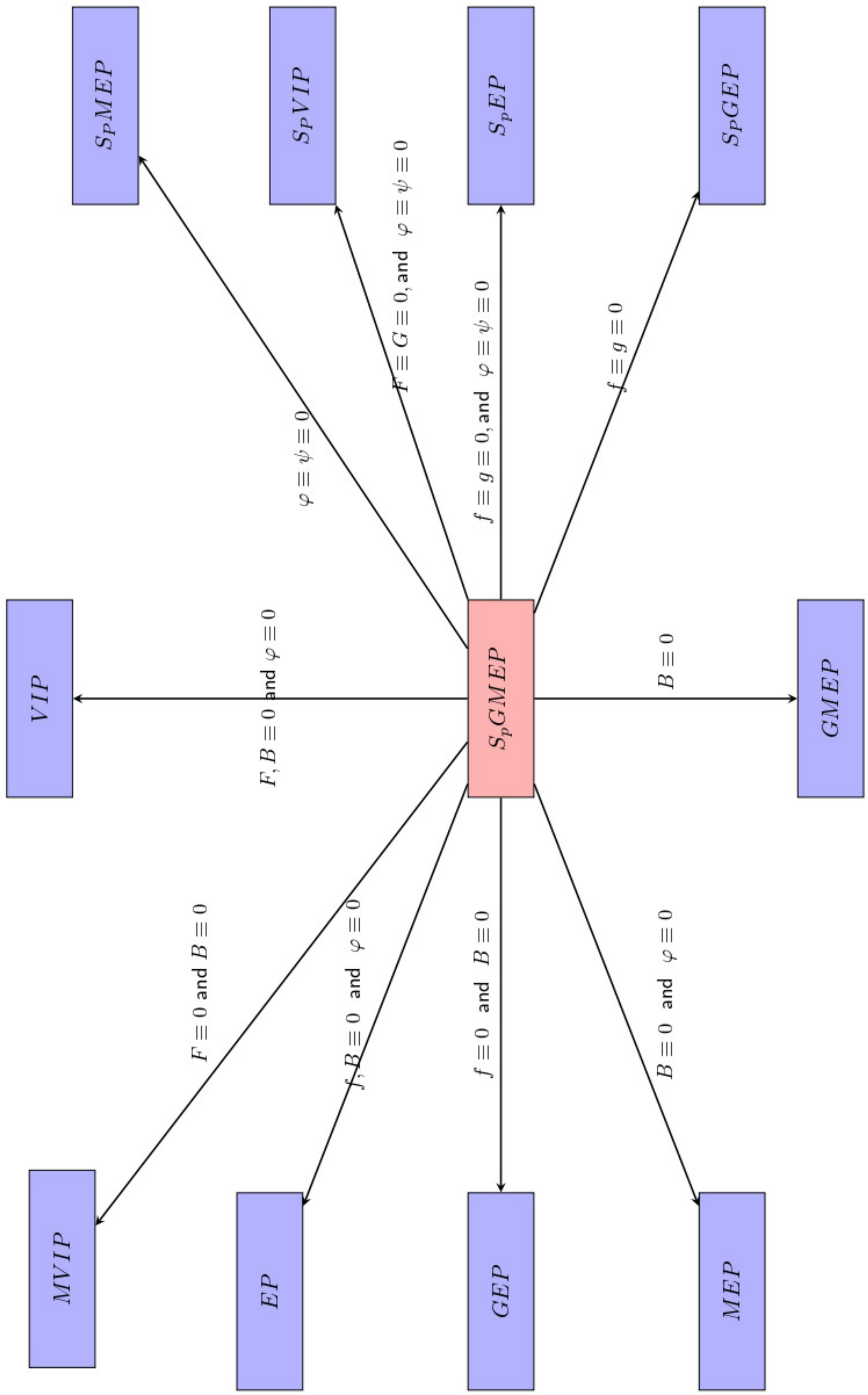

Figure 1: Some Special Cases of $\mathrm{S_P GMEP}$(1.2)-(1.3)

S.No.	**Problem**	**Statement: Find $\bar{\kappa} \in \mathcal{K}_1$ such that**	**Solution Notation**
1	S_pMEP	$F(\bar{\kappa},\kappa) + \langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.4) and such that $\bar{v} = B\bar{\kappa} \in \mathcal{K}_2$ solves $G(\bar{v}, v) + \langle g\bar{v}, v - \bar{v}\rangle \geq 0, \quad \forall v \in \mathcal{K}_2,$ (1.5)	Sol(S$_P$MEP(1.4)-(1.5))
2	S_PVIP	$\langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.6) and such that $\bar{v} = B\bar{\kappa} \in \mathcal{K}_2$ solves $\langle g\bar{v}, v - \bar{v}\rangle \geq 0, \quad \forall v \in \mathcal{K}_2,$ (1.7)	Sol(S$_P$VIP(1.6)-(1.7)).
3	S_PEP	$F(\bar{\kappa},\kappa) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.8) and such that $\bar{v} = B\bar{\kappa} \in \mathcal{K}_2$ solves $G(\bar{v}, v) \geq 0, \quad \forall v \in \mathcal{K}_2,$ (1.9)	Sol(S_PEP(1.8)-(1.9)).
4	S_pGEP	$F(\bar{\kappa},\kappa) + \varphi(\kappa) - \varphi(\bar{\kappa}) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.10) and such that $\bar{v} = B\bar{\kappa} \in \mathcal{K}_2$ solves $G(\bar{v}, v) + \psi(v) - \psi(\bar{v}) \geq 0, \forall v \in \mathcal{K}_2,$ (1.11)	Sol(S$_P$GEP(1.10)-(1.11).
5	$GMEP$	$F(\bar{\kappa},\kappa) + \langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle + \varphi(\kappa) - \varphi(\bar{\kappa}) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.12)	Sol(GMEP(1.12)).
6	MEP	$F(\bar{\kappa},\kappa) + \langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.13)	Sol(MEP(1.13)).
7	GEP	$F(\bar{\kappa},\kappa) + \varphi(\kappa) - \varphi(\bar{\kappa}) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.14)	Sol(GEP(1.14)).
8	EP	$F(\bar{\kappa},\kappa) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.15)	Sol(EP(1.15)).
9	$MVIP$	$\langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle + \varphi(\kappa) - \varphi(\bar{\kappa}) \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.16)	Sol(MVIP(1.16)).
10	VIP	$\langle f\bar{\kappa}, \kappa - \bar{\kappa}\rangle \geq 0, \quad \forall \kappa \in \mathcal{K}_1,$ (1.17)	Sol(VIP(1.17)).

Table 1. Different Problems and Solution Notation

S.No.	Problems Title	Abbreviation	Studied by
1	Split Mixed Equilibrium Problem	S_pMEP	Rizvi [19]
2	Split Variational Inequality problem	S_pVIP	Censor *et al.* [20].
3	Split Equilibrium Problem	S_pEP	Moudafi [21] and Kazmi and Rizvi [16].
4	Split Generalized Equilibrium Problem	S_pGEP	New
5	Generalized Mixed Equilibrium Problem	$GMEP$	Peng and Yao [22].
6	Mixed Equilibrium Problem	MEP	Moudafi and Théra [23] and Moudafi [24]
7	Generalized Equilibrium Problem	GEP	Ceng and Yao [11].
8	Equilibrium Problem	EP	Blum and Oettli [3].
9	Mixed Variational Inequality Problem	$MVIP$	Noor [25].
10	Variational Inequality Problem	VIP	Hartman and Stampacchia [26].

Table 2. Abbreviation and Researchers

SpGMEP encompasses the split variational Inequality problem, which is an extension of split zero problems and split feasibility problems, as discussed in [20]. Further details can be found in references such as [20, 21, 27–29]. This formalism is widely employed in the modeling of various inverse problems, including phase retrieval and other real-world scenarios such as sensor networks in computerized tomography and data compression. For more information, refer to sources like [30–32].

Extensive research has been conducted in the literature regarding iterative methods for approximating Fixed points of a nonexpansive mapping S. The advancements in this field primarily revolve around two types of iterative methods: the Mann and the Halpern iterative method.

The Mann iterative algorithm, originally introduced by Mann [33] that generates a sequence recursively.

$$\kappa_{n+1} = \alpha_n \kappa_n + (1-\alpha_n) S\kappa_n, \quad n \geq 0, \tag{1.18}$$

where the initial guess $\kappa_0 \in \mathscr{K}_1$ and $\{\alpha_n\}$ is a sequence in $(0,1)$. Subsequently, Halpern [34] proposed an iterative method that generates a sequence using the recursive formula:

$$\kappa_{n+1} = \alpha_n u + (1-\alpha_n) S\kappa_n, \quad n \geq 0, \tag{1.19}$$

where the initial guess $\kappa_0 \in \mathscr{K}_1$ and anchor $u \in \mathscr{K}_1$ are arbitrary (but Fixed) and the sequence $\{\alpha_n\}$ is contained in $(0,1)$, for finding a Fixed point of a nonexpansive mapping S.

Korpelevich proposed an iterative method in 1976 [35] which is commonly referred to as the extragradient iterative method for solving VIP (1.17):

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1, \\ u_n = P_{\mathscr{K}_1}(\kappa_n - \lambda f \kappa_n), \\ \kappa_{n+1} = P_{\mathscr{K}_1}(\kappa_n - \lambda f u_n), \end{cases} \tag{1.20}$$

where $\lambda > 0$, and $P_{\mathscr{K}_1}$ is the metric projection of $\mathscr{H}_1$ onto $\mathscr{K}_1$. He proved that the sequence generated by (1.20) converge strongly to a solution of VIP(1.17), if f is a monotone and Lipschitz continuous mappinng.

An iterative method was presented and analyzed by Takahashi and Toxoda in 2003, which uses the Mann iterative method to find a solution that satisfies both VIP(1.17) and FPP(1.1) for a nonexpansive mapping:

$$\kappa_{n+1} = \alpha_n \kappa_n + (1-\alpha_n) S P_{\mathscr{K}_1}(\kappa_n - \lambda_n f \kappa_n), \quad n \geq 0, \tag{1.21}$$

where $\kappa_0 \in \mathscr{K}_1$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\nu)$. They proved that the sequence generated by (1.21) converge weakly to a common solution of VIP(1.17) and FPP(1.1) for a nonexpansive mapping, if f is inverse strongly monotone mapping.

Nakajo and Takahashi introduced and analyzed a hybrid iterative method in 2003 [37], specifically designed for solving FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1 \subseteq \mathscr{H}_1, \\ \upsilon_n = \alpha_n \kappa_n + (1-\alpha_n) S\kappa_n, \\ \mathscr{C}_n = \{\zeta \in \mathscr{K}_1 : \|\upsilon_n - \zeta\| \leq \|\kappa_n - \zeta\|\}, \\ \mathscr{Q}_n = \{\zeta \in \mathscr{K}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n \rangle \geq 0\}, \\ \kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n} \kappa. \end{cases} \tag{1.22}$$

They proved that the sequence $\{\kappa_n\}$ generated by (1.22) converges strongly to $P_{Fix(S)}\kappa_0$, where $P_{Fix(S)}$ denotes the metric projection from $\mathscr{H}_1$ onto $Fix(S)$.

Iiduka and Takahashi investigated and presented the hybrid iterative method in their work in [38], which is aimed at finding a simultaneous solution to VIP(1.17) and FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1 \subseteq \mathscr{H}_1, \\ \upsilon_n = \alpha_n \kappa_n + (1-\alpha_n) SP_{\mathscr{K}_1}(\kappa_n - \lambda_n f\kappa_n), \\ \mathscr{C}_n = \{\zeta \in \mathscr{K}_1 : \|\upsilon_n - \zeta\| \leq \|\kappa_n - \zeta\|\}, \\ \mathscr{Q}_n = \{\zeta \in \mathscr{K}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n \rangle \geq 0\}, \\ \kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n} \kappa. \end{cases} \tag{1.23}$$

for every $n = 0, 1, 2, ...,$ where $0 \leq \alpha_n \leq c < 1$ and $0 < a \leq \lambda_n \leq b < 2\nu$. They proved that if f is inverse strongly monotone mapping, then the sequence $\{\kappa_n\}$, generated by (1.23) converge strongly to a common solution of VIP(1.17) and FPP(1.1) for a nonexpansive mapping.

Nadezkhina and Takahashi expanded and Generalized the findings of Iiduka and Takahashi in 2006 [39], taking a different approach by combining the hybrid iterative method and extragradient iterative method. Their objective was to obtain a unified method for finding a common solution to VIP(1.17) and FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1 \subseteq \mathscr{H}_1, \\ u_n = P_{\mathscr{K}_1}(\kappa_n - \lambda_n f\kappa_n) \\ \upsilon_n = \alpha_n \kappa_n + (1-\alpha_n) SP_{\mathscr{K}_1}(\kappa_n - \lambda_n f u_n), \\ \mathscr{C}_n = \{\zeta \in \mathscr{K}_1 : \|\upsilon_n - \zeta\| \leq \|\kappa_n - \zeta\|\}, \\ \mathscr{Q}_n = \{\zeta \in \mathscr{K}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n \rangle \geq 0\}, \\ \kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n} \kappa. \end{cases} \tag{1.24}$$

for every $n = 0, 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$ and $\alpha_n \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$, and $\{\zeta_n\}$ generated by (1.24) converge strongly to VIP(1.17) and FPP(1.1), if f is monotone and Lipschitz continuous mapping.

Tada and Takahashi presented and analyzed the relaxed hybrid iterative method in 2007 [40], which aims to find a simultaneous solution to EP(1.15) and FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1 \subseteq \mathscr{H}_1, \\ u_n \in \mathscr{K}_1 \quad \text{such that} \quad F(u_n, v) + \dfrac{1}{r_n}\langle v - u_n, u_n - \kappa_n\rangle \geq 0, \quad \forall v \in \mathscr{K}_1, \\ v_n = \alpha_n \kappa_n + (1-\alpha_n) S u_n, \\ \mathscr{C}_n = \{\zeta \in \mathscr{K}_1 : \|v_n - \zeta\| \leq \|\kappa_n - \zeta\|\}, \\ \mathscr{Q}_n = \{\zeta \in \mathscr{K}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n\rangle \geq 0\}, \\ \kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa. \end{cases} \tag{1.25}$$

for every $n = 0, 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$ and $\alpha_n \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{\kappa_n\}$, $\{v_n\}$, and $\{\zeta_n\}$ generated by (1.25) converge strongly to EP(1.15) and FPP(1.1) for a nonexpansive mapping.

In 2007, Moudafi proposed the following Mann iterative method [41] for finding a shared solution to MEP(1.13) and FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1, \\ v_n = T^F_{r_n}(\kappa_n - r_n f \kappa_n), \\ \kappa_{n+1} = \beta_n \kappa_n + (1-\beta_n) S v_n], \end{cases} \tag{1.26}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ is a sequence in $(0, 1)$. He proved that the sequence generated by (1.26) converge weakly to a common solution of MEP(1.13) and FPP(1.1) for a nonexpansive mapping, if f is inverse strongly monotone mapping. For related work, see [42].

Takahashi and Takahashi further developed and expanded upon the findings of Moudafi in 2007 [43], taking a different approach by combining the Mann iterative method and Halpern iterative method. Their objective was to obtain a unified method for finding a shared solution to MEP(1.13) and FPP(1.1) for a nonexpansive mapping:

$$\begin{cases} \kappa_0 = \kappa \in \mathscr{K}_1, \\ v_n = T^F_{r_n}(\kappa_n - r_n f \kappa_n), \\ \kappa_{n+1} = \beta_n \kappa_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) v_n], \end{cases} \tag{1.27}$$

where $\{r_n\} \subset (0, 2\nu)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $(0, 1)$. (1.27) exhibits strong convergence towards a common solution of MEP(1.13) and FPP(1.1) for a nonexpansive mapping. This convergence result holds when f is an inverse strongly monotone mapping.For related work, see [44].

In 2014, Kazmi and Rizvi presented and examined an implicit iterative method that utilizes the viscosity technique. This method was designed to determine a shared solution of S$_P$EP(1.6)-(1.7) and FPP for a nonexpansive semigroup. Their study focused on analyzing the properties and convergence behavior of this method.

$$\begin{cases} u_t = T^F_{r_t}(\kappa_t + \delta A^*(T^G_{r_t} - I)A\kappa_t), \\ \kappa_t = t\gamma Q(\kappa_t) + (I - tB)\dfrac{1}{s_t}\displaystyle\int_0^{s_t} T(s) u_t ds, \end{cases} \tag{1.28}$$

where Q is the contraction mapping on $\mathscr{H}_1$, and (r_t), (u_t) and (κ_t) are the nets, and $t \in [0,1]$. They proved that the nets generated by (1.28) converge strongly to a common solution of S$_P$EP(1.6)-(1.7) and FPP for a nonexpansive semigroup. For related work, see [16, 17].

It is important to note that the investigation of extragradient iterative methods for solving split equilibrium problems is an area that remains largely unexplored. Hence, in this research paper, we propose and analyze a hybrid extragradient iterative method. The objective is to find a common element that belongs to the solution sets of split Generalized Mixed equilibrium problem (SpGMEP) and FPP for nonexpansive mappings. We aim to examine the properties and convergence behavior of this method in detail.

Motivated by the notable contributions of Nakajo and Takahashi [37], Tada and Takahashi [40], Moudafi [23], Kazmi and Rizvi [15], and Takahashi and Takahashi [43], as well as the ongoing research in this field, we propose and analyze an iterative method for finding a common solution to S$_P$MEP(1.2)-(1.3) and FPP(1.11) for a nonexpansive mapping in a real Hilbert space. The method is based on a combination of the Mann iterative method and the Halpern iterative method. We establish a strong convergence theorem for the proposed iterative method and derive several consequences from these theorems. The results and techniques presented in this study extend and generalize the corresponding results and techniques reported in previous works [15, 23, 37, 40, 43].

2. Preliminaries

In this section, we provide a brief overview of the relevant concepts and results that will be utilized in the subsequent sections. For every point $\kappa \in \mathscr{H}_1$, there exists a unique nearest point in $\mathscr{K}_1$ denoted by $P_{\mathscr{K}_1}\kappa$ such that

$$\|\kappa - P_{\mathscr{K}_1}\kappa\| \le \|\kappa - \upsilon\|, \ \forall \kappa \in \mathscr{K}_1.$$

$P_{\mathscr{K}_1}$ is called the `metric projection` of $\mathscr{H}_1$ onto $\mathscr{K}_1$. It is well known that $P_{\mathscr{K}_1}$ is nonexpansive mapping and satisfies

$$\langle \kappa - \upsilon, P_{\mathscr{K}_1}\kappa - P_{\mathscr{K}_1}\upsilon \rangle \ge \|P_{\mathscr{K}_1}\kappa - P_{\mathscr{K}_1}\upsilon\|^2, \ \forall \kappa, \upsilon \in \mathscr{H}_1. \tag{2.1}$$

Moreover, $P_{\mathscr{K}_1}\kappa$ satisfies the following properties:

$$\langle \kappa - P_{\mathscr{K}_1}\kappa, \upsilon - P_{\mathscr{K}_1}\kappa \rangle \le 0, \tag{2.2}$$

and

$$\|\kappa - \upsilon\|^2 \ge \|\kappa - P_{\mathscr{K}_1}\kappa\|^2 + \|\upsilon - P_{\mathscr{K}_1}\kappa\|^2, \ \forall \kappa \in \mathscr{H}_1, \ \upsilon \in \mathscr{K}_1. \tag{2.3}$$

Assumption 1. [22] *Let $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(\kappa, \kappa) = 0, \ \forall \kappa \in \mathscr{K}_1$;
(ii) *F is monotone, i.e.,* $F(\kappa, \upsilon) + F(\upsilon, \kappa) \le 0, \ \forall \kappa \in \mathscr{K}_1$;
(iii) *For each* $\kappa, \upsilon, \zeta \in \mathscr{K}_1$, $\limsup_{t \to 0} F(t\zeta + (1-t)\kappa, \upsilon) \le F(\kappa, \upsilon)$;
(iv) *For each* $\kappa \in \mathscr{K}_1$, $\upsilon \to F(\kappa, \upsilon)$ *is convex and lower semicontinuous.*

(v) *For each* $\kappa \in \mathscr{H}_1$ *and* $r > 0$, *there exists a bounded subset* $D_\kappa \subseteq K_1$ *and* $\upsilon_\kappa \in \mathscr{K}_1$ *such that for any* $\zeta \in \mathscr{K}_1 \setminus D_\kappa$

$$F(\zeta, \upsilon_\kappa) + \varphi(\upsilon_\kappa) - \varphi(\zeta) + \frac{1}{r}\langle \upsilon_\kappa - \zeta, \zeta - \kappa \rangle < 0;$$

(vi) $\mathscr{K}_1$ *is a bounded set.*

Lemma 2.1. [22] *Assume that* $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ *and let* $\varphi : \mathscr{K}_1 \to \mathbb{R}$ *be a proper lower semicontinuous and convex function satisfying Assumption* 1. *For* $r > 0$ *and for all* $\kappa \in \mathscr{H}_1$, *define a mapping* $T_r^F : \mathscr{H}_1 \to \mathscr{K}_1$ *as follows:*

$$T_r^F \kappa = \{\zeta \in \mathscr{K}_1 : F(\zeta, \upsilon) + \varphi(\upsilon) - \varphi(\zeta) + \frac{1}{r}\langle \upsilon - \zeta, \zeta - \kappa \rangle \geq 0, \ \forall \upsilon \in \mathscr{K}_1\}.$$

Then, the following results hold:

(i) $T_r^F(\zeta)$ *is nonempty for each* $\zeta \in \mathscr{H}_1$;

(ii) T_r^F *is single-valued;*

(iii) T_r^F *is firmly nonexpansive, i.e.,*

$$\|T_r^F \kappa - T_r^F \upsilon\|^2 \leq \langle T_r^F \kappa - T_r^F \upsilon, \kappa - \upsilon \rangle, \ \forall \kappa, \upsilon \in \mathscr{H}_1;$$

(iv) $\mathrm{Fix}(T_r^F) = \mathrm{Sol}(\mathrm{GMEP}(1.2))$;

(v) Sol(GMEP(1.2)) *is closed and convex.*

Moreover, assume that $G : \mathscr{K}_2 \times \mathscr{K}_2 \to \mathbb{R}$ and let $\phi : \mathscr{K}_2 \to \mathbb{R}$ be a proper lower semicontinuous and convex function satisfying Assumption 1. For $s > 0$ and for all $w \in \mathscr{H}_2$, define a mapping $T_s^G : \mathscr{H}_2 \to \mathscr{K}_2$ as follows:

$$T_s^G w = \{d \in \mathscr{K}_2 : \ \exists \zeta \in T_r^F \kappa \text{ such that } B\zeta = d \ \text{ and } G(d, e) + \phi(e) - \phi(d) + \frac{1}{s}\langle d - e, e - w \rangle \geq 0, \ \forall e \in \mathscr{K}_2\}.$$

Then, we can easily observe that T_s^G is single-valued and firmly nonexpansive, Sol(GMEP(1.4)) is closed and convex and $\mathrm{Fix}(T_s^G) = \mathrm{Sol}(\mathrm{GMEP}(1.4))$.

Lemma 2.2. [45,46] *In real Hilbert space* $\mathscr{H}_1$, *the following hold:*

(i) *The identity*

$$\|\lambda \kappa + (1 - \lambda)\upsilon\|^2 = \lambda\|\kappa\|^2 + (1 - \lambda)\|\upsilon\|^2 - \lambda(1 - \lambda)\|\kappa - \upsilon\|^2, \tag{2.4}$$

for all $\kappa, \upsilon \in H$ *and* $\lambda \in (0, 1)$;

(ii) *(Opial's condition) For any sequence* $\{\kappa_n\}$ *with* $\kappa_n \rightharpoonup \kappa$ *the Inequality*

$$\liminf_{n\to\infty} \|\kappa_n - \kappa\| < \liminf_{n\to\infty} \|\kappa_n - \upsilon\| \tag{2.5}$$

holds for every $\upsilon \in \mathscr{H}_1$ *with* $\upsilon \neq \kappa$;

(iii)

$$\|\kappa + \upsilon\|^2 \leq \|\kappa\|^2 + 2\langle \upsilon, \kappa + \upsilon \rangle, \ \forall \kappa, \upsilon \in \mathscr{H}_1;$$

Remark 2.1. *It follows from Lemma* 2.1 *(i)-(ii) that*

$$rF(T_r\kappa, \upsilon) + r\left[\varphi(\upsilon) - \varphi(T_r\kappa)\right] + \langle T_r\kappa - \kappa, \upsilon - T_r\kappa \rangle \geq 0, \ \forall \upsilon \in \mathscr{K}_1, \ \kappa \in \mathscr{H}_1. \tag{2.6}$$

Further Lemma 2.1 (iii) implies the nonexpansivity of T_r, i.e.,

$$\|T_r\kappa - T_r\upsilon \le \|\kappa - \upsilon\|, \ \forall \kappa, \upsilon \in \mathscr{H}_1. \tag{2.7}$$

Furthermore (2.6) implies the following Inequality

$$\|T_r\kappa - \upsilon\|^2 \le \|\kappa - \upsilon\|^2 - \|T_r\kappa - \kappa\|^2 + 2rF(T_r\kappa, \upsilon), \ \forall \upsilon \in \mathscr{K}_1, \kappa \in \mathscr{H}_1 \tag{2.8}$$

The bifunction $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ is called 2-*monotone*, if

$$F(\kappa, \upsilon) + F(\upsilon, \zeta) + F(\zeta, \kappa) \le 0, \ \forall \kappa, \upsilon, \zeta \in \mathscr{K}_1. \tag{2.9}$$

by taking $\upsilon = \zeta$, it is clear that every 2-monotone bifunction is a monotone bifunction. For example, if $F(\kappa, \upsilon) = \kappa(\upsilon - \kappa)$, then F is a 2-monotone bifunction.

3. Hybrid extragradient approximation method

In this section, we present the proof of the strong convergence theorem for the hybrid extragradient method. This method effectively addresses the problem of finding a common element that belongs to both the solution set of SpGMEP(1.2)-(1.3) and the Fixed point set of FPP(1.1) for a nonexpansive mapping.

Theorem 3.1. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces and $\mathscr{K}_1 \subseteq \mathscr{H}_1$ and $\mathscr{K}_2 \subseteq \mathscr{H}_2$ are nonempty, closed and convex subsets. Let $B : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. Assume that $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ and $G : \mathscr{K}_2 \times \mathscr{K}_2 \to \mathbb{R}$ are 2-monotone bifunctions satisfying Assumption 1 and G is upper semicontinuous in first argument. Let $f : \mathscr{K}_1 \to \mathscr{H}_1$, and $g : \mathscr{K}_2 \to \mathscr{H}_2$ are ν_1, ν_2-inverse strongly monotone mappings and let $\varphi : \mathscr{K}_1 \to \mathbb{R} \cup \{+\infty\}$, $\psi : \mathscr{K}_2 \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functionals and $S : \mathscr{K}_1 \to \mathscr{K}_1$ be a nonexpansive mapping such that $\Omega :=$ SpMEP(1.2)-(1.3)$\cap\mathrm{Fix}(S) \neq \emptyset$. For a given $\kappa_0 \in \mathscr{K}_1$ arbitrarily, let the iterative sequences $\{\kappa_n\}$, $\{\upsilon_n\}$, and $\{\zeta_n\}$ are generated by*

$$\kappa_0 = \kappa \in \mathscr{K}_1,$$

$$\upsilon_n = U(\kappa_n + \gamma_n B^*(V - I)B\kappa_n), \tag{3.1}$$

$$\zeta_n = \alpha_n\kappa_n + (1 - \alpha_n)SU(\kappa_n + \gamma_n B^*(V - I)B\upsilon_n), \tag{3.2}$$

$$\mathscr{C}_n = \{\zeta \in \mathscr{H}_1 : \|\zeta_n - \zeta\|^2 \le \|\kappa_n - \zeta\|^2\}, \tag{3.3}$$

$$\mathscr{Q}_n = \{\zeta \in \mathscr{H}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n \rangle \ge 0\}, \tag{3.4}$$

$$\kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa, \tag{3.5}$$

where $U := T^F_{r_n}(I - r_n f)$, $V := T^G_{r_n}(I - r_n g)$, $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \nu)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\gamma_n \in \left(0, \dfrac{1}{\|B\|^2}\right)$, where $\nu = \min\{\nu_1, \nu_2\}$. Then the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$ and $\{\zeta_n\}$ converge strongly to $d = P_\Omega\kappa$.

Proof. Since $f : \mathscr{K}_1 \to \mathscr{H}_1$ be an ν_1-inverse strongly monotone mapping then for any $\kappa, \upsilon \in \mathscr{K}_1$ and $r_n \in [a, b]$, we have

$$\begin{aligned} \|(I - r_n f)\kappa - (I - r_n f)\upsilon\|^2 &= \|(\kappa - \upsilon) - r_n(f\kappa - f\upsilon)\|^2 && (3.6) \\ &\le \|\kappa - \upsilon\|^2 - r_n(2\nu_1 - r_n)\|f\kappa - f\upsilon\|^2 && (3.7) \\ &\le \|\kappa - \upsilon\|^2, && (3.8) \end{aligned}$$

which shows that $(I - r_n f)$ is nonexpansive. Similarly, we can show that $(I - r_n g)$ is nonexpansive, and since by Lemma 2.1 T_n is also nonexpansive. Hence $U := T^F_{r_n}(I - r_n f)$, and $V := T^G_{r_n}(I - r_n g)$ are nonexpansive. Since $\Omega \neq \emptyset$, it follows from Lemma 2.1 that $\mathrm{Sol}(\mathrm{GMEP}(1.2)) = \mathrm{Fix}(T^F_{r_n}(I - r_n f))$ and $\mathrm{Sol}(\mathrm{GMEP}(1.3)) = \mathrm{Fix}(T^G_{r_n}(I - r_n g))$ are closed and convex sets. Therefore Ω is nonempty, closed and convex and hence $P_\Omega \kappa$ is well defined.

First, we show that the sequence $\{\kappa_n\}$ generated by (3.1)-(3.5) is well defined. Indeed, it is obvious that $\mathscr{Q}_n$ is closed and convex for every n. Since

$$\mathscr{C}_n := \{\zeta \in \mathscr{H}_1 : \|\zeta_n - \kappa_n\|^2 + 2\langle \zeta_n - \kappa_n, \kappa_n - \zeta \rangle \le 0\}, \tag{3.9}$$

we observe that $\mathscr{C}_n$ is closed and convex for every n. Hence $\mathscr{C}_n \cap \mathscr{Q}_n$ are closed and convex for all n. We claim that $\mathscr{C}_n \cap \mathscr{Q}_n$ is nonempty for all n. For this, it is enough to show that $\Omega \subset \mathscr{C}_n \cap \mathscr{Q}_n$ for every n. Let $\bar{\kappa} \in \Omega$ then $\bar{\kappa}$ is a solution of SPGMEP(1.2)-(1.3), which means

$$\|\upsilon_n - \bar{\kappa}\| = \|U(\kappa_n + \gamma_n B^*(V - I)B\kappa_n) - \bar{\kappa}\|.$$

We estimate

$$\begin{aligned} \|\upsilon_n - \bar{\kappa}\|^2 &= \|U(\kappa_n + \gamma_n B^*(V - I)B\kappa_n) - \bar{\kappa}\|^2 \\ &= \|U(\kappa_n + \gamma_n B^*(V - I)B\kappa_n) - U\bar{\kappa}\|^2 \\ &\le \|\kappa_n + \gamma_n B^*(V - I)B\kappa_n - \bar{\kappa}\|^2 \\ &\le \|\kappa_n - \bar{\kappa}\|^2 + \gamma_n^2\|B^*(V - I)B\kappa_n\|^2 \\ &\quad + 2\gamma_n\langle \kappa_n - \bar{\kappa}, B^*(V - I)B\kappa_n \rangle, && (3.10) \end{aligned}$$

which implies that

$$\begin{aligned} \|\upsilon_n - \bar{\kappa}\|^2 &\le \|\kappa_n - \bar{\kappa}\|^2 + \gamma_n^2\langle (V - I)B\kappa_n, BB^*(V - I)B\kappa_n \rangle \\ &\quad + 2\gamma_n\langle \kappa_n - \bar{\kappa}, B^*(V - I)B\kappa_n \rangle. && (3.11) \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \gamma_n^2\langle (V - I)B\kappa_n, BB^*(V - I)B\kappa_n \rangle &\le L\gamma_n^2\langle (V - I)B\kappa_n, (V - I)B\kappa_n \rangle \\ &= L\gamma_n^2\|(V - I)B\kappa_n\|^2, && (3.12) \end{aligned}$$

where L is a spectral radius of BB^* and B^* is a adjoint operator of B. Moreover, we obtain

$$\begin{aligned} 2\gamma_n\langle \kappa_n-\bar{\kappa}, B^*(V-I)B\kappa_n\rangle &= 2\gamma_n\langle B(\kappa_n-\bar{\kappa}), (V-I)B\kappa_n\rangle \\ &= 2\gamma_n\langle B(\kappa_n-\bar{\kappa})+(V-I)B\kappa_n-(V-I)B\kappa_n, (V-I)B\kappa_n\rangle \\ &= 2\gamma_n\Big\{\langle VB\kappa_n-B\bar{\kappa}, (V-I)B\kappa_n\rangle-\|(V-I)B\kappa_n\|^2\Big\} \\ &\le 2\gamma_n\Big\{\frac{1}{2}\|(V-I)B\kappa_n\|^2-\|(V-I)B\kappa_n\|^2\Big\} \\ &\le -\gamma_n\|(V-I)B\kappa_n\|^2. \end{aligned} \tag{3.13}$$

On combining (3.11), (3.12) and (3.13), we obtain

$$\|\upsilon_n-\bar{\kappa}\|^2 \le \|\kappa_n-\bar{\kappa}\|^2+\gamma_n(L\gamma_n-1)\|(V-I)B\kappa_n\|^2. \tag{3.14}$$

Since $\gamma_n \in \left(0, \frac{1}{\|B\|^2}\right)$, we obtain

$$\|\upsilon_n-\bar{\kappa}\|^2 \le \|\kappa_n-\bar{\kappa}\|^2. \tag{3.15}$$

Further, it follows from Proposition 2.1 (v) in [21] that the mapping $I-V$ is ν-inverse strongly monotone with $\nu > \frac{1}{2}$. Therefore, we have

$$\begin{aligned} \langle B^*(I-V)B\kappa-B^*(I-V)B\upsilon, \kappa-\upsilon\rangle &= \langle (I-V)B\kappa-(I-V)B\upsilon, B\kappa-B\upsilon\rangle \\ &\ge \nu\|(I-V)B\kappa-(I-V)B\upsilon\|^2 \\ &\ge \frac{\nu}{L}\|B^*(I-V)B\kappa-B^*(I-V)B\upsilon\|^2. \end{aligned}$$

Hence $\gamma_n B^*(I-V)B$ is $\frac{\nu}{\gamma_n L}$-inverse strongly monotone. Since $\gamma_n \in \left(0, \frac{1}{\|B\|^2}\right)$, therefore its complement $I-\gamma_n B^*(I-V)B$ is $\frac{\nu}{\gamma_n L}$ is averaged and hence nonexpansive. For more details, see [21].

Setting $\mathscr{F} := B^*(V-I)B$, then

$$t_n := U(\kappa_n-\gamma_n\mathscr{F}\upsilon_n). \tag{3.16}$$

Applying (2.8), with $\kappa_n-\gamma_n\mathscr{F}\upsilon_n$ and $\bar{\kappa}$, we have

$$\begin{aligned} \|t_n-\bar{\kappa}\|^2 &\le \|\kappa_n-\gamma_n\mathscr{F}\upsilon_n-\bar{\kappa}\|^2-\|t_n-(\kappa_n-\gamma_n\mathscr{F}\upsilon_n)\|^2+2\gamma_n F(t_n,\bar{\kappa}) \\ &= \|\kappa_n-\bar{\kappa}\|^2-\|t_n-\kappa_n\|^2+2\gamma_n\langle \mathscr{F}\upsilon_n, \bar{\kappa}-t_n\rangle+2\gamma_n F(t_n,\bar{\kappa}) \\ &= \|\kappa_n-\bar{\kappa}\|^2-\|t_n-\kappa_n\|^2+2\gamma_n\Big[\langle \mathscr{F}\upsilon_n-\mathscr{F}\bar{\kappa}, \bar{\kappa}-\upsilon_n\rangle \\ &\quad +\langle \mathscr{F}\bar{\kappa}, \bar{\kappa}-\upsilon_n\rangle-\langle \mathscr{F}\upsilon_n, t_n-\upsilon_n\rangle\Big]+2\gamma_n F(t_n,\bar{\kappa}). \end{aligned}$$

Since $\mathscr{F}$ is $\dfrac{\nu}{\gamma_n L}$-inverse strongly monotone, then $\mathscr{F}$ is monotone and $\dfrac{\gamma_n L}{\nu}$-Lipschitz continuous. Using (2.6), (3.1) and monotonicity of $\mathscr{F}$ in the above Inequality, we get

$$\begin{aligned}
\|t_n-\bar{\kappa}\|^2 &\leq \|\kappa_n-\bar{\kappa}\|^2-\|t_n-\kappa_n\|^2+2\gamma_n\langle \mathscr{F}\upsilon_n,\upsilon_n-t_n\rangle \\
&\quad +2\gamma_n\Big[F(\bar{\kappa},\upsilon_n)+F(t_n,\bar{\kappa})\Big] \\
&\leq \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2-2\langle \kappa_n-\upsilon_n,\upsilon_n-t_n\rangle \\
&\quad +2\gamma_n\langle \mathscr{F}\upsilon_n,\upsilon_n-t_n\rangle+2\gamma_n\Big[F(\bar{\kappa},\upsilon_n)+F(t_n,\bar{\kappa})\Big] \\
&= \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2-2\langle \upsilon_n-(\kappa_n-\gamma_n\mathscr{F}\kappa_n),t_n-\upsilon_n\rangle \\
&\quad +2\gamma_n\langle \mathscr{F}\kappa_n-\mathscr{F}\upsilon_n,t_n-\upsilon_n\rangle+2\gamma_n\Big[F(\bar{\kappa},\upsilon_n)+F(t_n,\bar{\kappa})\Big] \\
&= \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2+2\gamma_n\langle \mathscr{F}\kappa_n-\mathscr{F}\upsilon_n,t_n-\upsilon_n\rangle \\
&\quad +2\gamma_n\Big[F(\bar{\kappa},\upsilon_n)+F(\upsilon_n,t_n)+F(t_n,\bar{\kappa})\Big].
\end{aligned}$$

Since F is a 2-monotone bifunction, we have

$$\begin{aligned}
\|t_n-\bar{\kappa}\|^2 &\leq \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2 \\
&\quad +2\gamma_n\frac{L}{\nu}\|\kappa_n-\upsilon_n\|\|t_n-\upsilon_n\| \qquad (3.17)\\
&\leq \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2 \\
&\quad +\|\upsilon_n-t_n\|^2+\left(\frac{\gamma_n L}{\nu}\right)^2\|\kappa_n-\upsilon_n\|^2 \\
&\leq \|\kappa_n-\bar{\kappa}\|^2-\left(1-\left(\frac{\gamma_n L}{\nu}\right)^2\right)\|\kappa_n-\upsilon_n\|^2. \qquad (3.18)
\end{aligned}$$

Since $\gamma_n\in\left(0,\dfrac{1}{\|B\|^2}\right)$, and $\nu>\frac{1}{2}$. Hence, we obtain

$$\|t_n-\bar{\kappa}\|^2\leq\|\kappa_n-\bar{\kappa}\|^2. \qquad (3.19)$$

Since $\bar{\kappa}\in\Omega$ then $\bar{\kappa}=S\bar{\kappa}$. Next using (3.2) and (3.19), we get the following estimate

$$\begin{aligned}
\|\zeta_n-\bar{\kappa}\|^2 &= \|\alpha_n\kappa_n+(1-\alpha_n)St_n-\bar{\kappa}\|^2 \\
&= \|\alpha_n(\kappa_n-\bar{\kappa})+(1-\alpha_n)(St_n-\bar{\kappa})\|^2 \\
&= \alpha_n\|\kappa_n-\bar{\kappa}\|^2+(1-\alpha_n)\|St_n-\bar{\kappa}\|^2-\alpha_n(1-\alpha_n)\|St_n-\bar{\kappa}\|^2 \\
&\leq \alpha_n\|\kappa_n-\bar{\kappa}\|^2+(1-\alpha_n)\|St_n-\bar{\kappa}\|^2 \\
&\leq \alpha_n\|\kappa_n-\bar{\kappa}\|^2+(1-\alpha_n)\|t_n-\bar{\kappa}\|^2 \qquad (3.20)\\
&\leq \alpha_n\|\kappa_n-\bar{\kappa}\|^2+(1-\alpha_n)\|\kappa_n-\bar{\kappa}\|^2 \\
&= \|\kappa_n-\bar{\kappa}\|^2. \qquad (3.21)
\end{aligned}$$

Therefore $\bar{\kappa}\in\mathscr{C}_n$ and consequently $\Omega\subseteq\mathscr{C}_n$. Further, since $\Omega\subseteq C_0$ and $\Omega\subseteq Q_0=H$, it follows that $\Omega\subset C_0\cap Q_0$ and hence $C_0\cap Q_0$ is nonempty, closed and convex set. Therefore $\kappa_1=P_{C_0\cap Q_0}\kappa$ is well defined. Now, suppose that $\Omega\subseteq C_{n-1}\cap Q_{n-1}$ for some $n>1$. Let $\kappa_n=P_{C_{n-1}\cap Q_{n-1}}\kappa$. Again, since $\Omega\subseteq$

$\mathscr{C}_n$ and for any $\bar{\kappa} \in \Omega$, it follows from ( ) that $\langle \kappa - \kappa_n, \kappa_n - \bar{\kappa} \rangle = \langle \kappa - P_{C_{n-1} \cap Q_{n-1}} \kappa, P_{C_{n-1} \cap Q_{n-1}} \kappa - \bar{\kappa} \rangle \geq 0$, and hence $\bar{\kappa} \in \mathscr{Q}_n$. Therefore $\Omega \subseteq \mathscr{C}_n \cap \mathscr{Q}_n$ for every $n = 0, 1, 2, \ldots$ and hence $\kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n} \kappa$ is well defined for every $n = 0, 1, 2, \ldots$. Thus the sequence $\{\kappa_n\}$ is well defined.

Let $d = P_\Omega \kappa$. From $\kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n} \kappa$ and $d \in \Omega \subset \mathscr{C}_n \cap \mathscr{Q}_n$, we have

$$\|\kappa_{n+1} - \kappa\| \leq \|d - \kappa\|, \tag{3.22}$$

for every $n = 0, 1, 2, \ldots$. Therefore $\{\kappa_n\}$ is bounded. Further, it follows from (3.19) and (3.21) that the sequences $\{\zeta_n\}$ and $\{t_n\}$ are bounded. From (3.4) and (3.5), we have that $\kappa_{n+1} \in \mathscr{C}_n \cap \mathscr{Q}_n$ and $\kappa_n = P_{\mathscr{Q}_n} \kappa$. Therefore

$$\|\kappa_n - \kappa\| \leq \|\kappa_{n+1} - \kappa\|, \tag{3.23}$$

for every $n = 0, 1, 2, \ldots$. It follows from (3.22) and (3.23) that the sequence $\{\|\kappa_n - \kappa\|\}$ is nondecreasing and bounded and hence convergent. Therefore $\lim_{n\to\infty} \|\kappa_n - \kappa\|$ exists. Since $\kappa_n = P_{\mathscr{Q}_n} \kappa$ and $\kappa_{n+1} \in \mathscr{Q}_n$, using (2.3), we have

$$\|\kappa_{n+1} - \kappa_n\|^2 \leq \|\kappa_{n+1} - \kappa\|^2 - \|\kappa_n - \kappa\|^2,$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|\kappa_{n+1} - \kappa_n\| = 0. \tag{3.24}$$

Since $\kappa_{n+1} \in \mathscr{C}_n$, it follows from (3.9) that

$$\begin{aligned} \|\zeta_n - \kappa_n\|^2 &\leq 2\langle \zeta_n - \kappa_n, \kappa_{n+1} - \kappa_n \rangle \\ &\leq 2\|\zeta_n - \kappa_n\| \|\kappa_{n+1} - \kappa_n\|. \end{aligned}$$

Therefore

$$\|\zeta_n - \kappa_n\| \leq 2\|\kappa_{n+1} - \kappa_n\|,$$

and hence, using (3.24), we have

$$\lim_{n\to\infty} \|\zeta_n - \kappa_n\| = 0. \tag{3.25}$$

It follows from (3.18) and (3.20) that

$$\begin{aligned} \|\kappa_n - v_n\|^2 &\leq \left[(1-\alpha_n)\left(1 - \left(\frac{\gamma_n L}{\nu}\right)^2\right)\right]^{-1} \left(\|\kappa_n - \bar{\kappa}\|^2 - \|\zeta_n - \bar{\kappa}\|^2\right) \\ &= \left[(1-\alpha_n)\left(1 - \left(\frac{\gamma_n L}{\nu}\right)^2\right)\right]^{-1} \left(\|\kappa_n - \bar{\kappa}\| - \|\zeta_n - \bar{\kappa}\|\right)\left(\|\kappa_n - \bar{\kappa}\| + \|\zeta_n - \bar{\kappa}\|\right) \\ &\leq \left[(1-\alpha_n)\left(1 - \left(\frac{\gamma_n L}{\nu}\right)^2\right)\right]^{-1} \|\kappa_n - \zeta_n\| \left(\|\kappa_n - \bar{\kappa}\| - \|\zeta_n - \bar{\kappa}\|\right). \end{aligned}$$

Since $\{\kappa_n\}$ and $\{\zeta_n\}$ are bounded and $\lim_{n\to\infty} \|\zeta_n - \kappa_n\| = 0$, we have

$$\lim_{n\to\infty} \|\kappa_n - v_n\| = 0. \tag{3.26}$$

Further, it follows from (3.14) that

$$\begin{aligned}\gamma_n(1-L\gamma_n)\|(V-I)B\kappa_n\|^2 &\leq \|\kappa_n-\bar{\kappa}\|^2-\|\upsilon_n-\bar{\kappa}\|^2\\&\leq (\|\kappa_n-\bar{\kappa}\|-\|\zeta_n-\bar{\kappa}\|)(\|\kappa_n-\bar{\kappa}\|+\|\zeta_n-\bar{\kappa}\|)\\&\leq \|\kappa_n-\zeta_n\|(\|\kappa_n-\bar{\kappa}\|-\|\zeta_n-\bar{\kappa}\|).\end{aligned}$$

Since $\gamma_n \in \left(0, \frac{1}{\|B\|^2}\right)$ and using (3.26), we get

$$\lim_{n\to\infty}\|(V-I)B\kappa_n\|=0. \tag{3.27}$$

by the same process as in (3.17), we have

$$\begin{aligned}\|t_n-\bar{\kappa}\|^2 &\leq \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2+\frac{2\gamma_nL}{\nu}\|\kappa_n-\upsilon_n\|\|t_n-\upsilon_n\|\\&\leq \|\kappa_n-\bar{\kappa}\|^2-\|\kappa_n-\upsilon_n\|^2-\|\upsilon_n-t_n\|^2+\|\kappa_n-\upsilon_n\|^2\\&\quad+\left(\frac{\gamma_nL}{\nu}\right)^2\|t_n-\upsilon_n\|^2\\&= \|\kappa_n-\bar{\kappa}\|^2-\left(1-\left(\frac{\gamma_nL}{\nu}\right)^2\right)\|\upsilon_n-t_n\|^2.\end{aligned} \tag{3.28}$$

Further, using (3.28), we have

$$\|\zeta_n-\bar{\kappa}\|^2\leq\|\kappa_n-\bar{\kappa}\|^2-(1-\alpha_n)\left(1-\left(\frac{\gamma_nL}{\nu}\right)^2\right)\|\upsilon_n-t_n\|^2,$$

which implies that

$$\begin{aligned}\|t_n-\upsilon_n\|^2 &\leq \left[(1-\alpha_n)\left(1-\left(\frac{\gamma_nL}{\nu}\right)^2\right)\right]^{-1}\left(\|\kappa_n-\bar{\kappa}\|^2-\|\zeta_n-\bar{\kappa}\|^2\right)\\&= \left[(1-\alpha_n)\left(1-\left(\frac{\gamma_nL}{\nu}\right)^2\right)\right]^{-1}(\|\kappa_n-\bar{\kappa}\|-\|\zeta_n-\bar{\kappa}\|)(\|\kappa_n-\bar{\kappa}\|+\|\zeta_n-\bar{\kappa}\|)\\&\leq \left[(1-\alpha_n)\left(1-\left(\frac{\gamma_nL}{\nu}\right)^2\right)\right]^{-1}(\|\kappa_n-\bar{\kappa}\|+\|\zeta_n-\bar{\kappa}\|)\|\kappa_n-\zeta_n\|.\end{aligned} \tag{3.29}$$

Again, since the sequences $\{\kappa_n\}$ and $\{\zeta_n\}$ are bounded and $\lim_{n\to\infty}\|\zeta_n-\kappa_n\|=0$, it follows from (3.29) that

$$\lim_{n\to\infty}\|t_n-\upsilon_n\|=0. \tag{3.30}$$

Further, it follows from (3.26), (3.30) and the triangle Inequality that

$$\|\kappa_n-t_n\|\leq\|\kappa_n-\upsilon_n\|+\|\upsilon_n-t_n\|^2$$

which yields

$$\lim_{n\to\infty}\|\kappa_n-t_n\|=0. \tag{3.31}$$

Next, we have to show that $\lim_{n\to\infty}\|St_n-t_n\|=0$. Since

$$\zeta_n=\alpha_n\kappa_n+(1-\alpha_n)St_n$$

$$\zeta_n - \kappa_n = \alpha_n \kappa_n + (1-\alpha_n) S t_n - \kappa_n$$
$$= (1-\alpha_n) S t_n - \kappa_n,$$

which implies that

$$(1-\alpha_n)\|St_n - \kappa_n\| \leq \|\zeta_n - \kappa_n\|.$$

Then, we have

$$(1-c)\|St_n - \kappa_n\| \ \leq (1-\alpha_n)\|St_n - \kappa_n\| \ \leq \|\zeta_n - \kappa_n\|.$$

Since $\lim_{n\to\infty} \|\zeta_n - \kappa_n\| = 0$, we have

$$\lim_{n\to\infty} \|St_n - \kappa_n\| = 0.$$

Therefore by the triangle Inequality

$$\|St_n - t_n\| \leq \|St_n - \kappa_n\| + \|\kappa_n - t_n\|.$$

Since $\lim_{n\to\infty} \|St_n - \kappa_n\| = 0$ and $\lim_{n\to\infty} \|\kappa_n - t_n\| = 0$, we have

$$\lim_{n\to\infty} \|St_n - t_n\| = 0. \tag{3.32}$$

Since $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \rightharpoonup \hat{\kappa}$ say. Therefore, it follows from (3.30) that there also exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \rightharpoonup \hat{\kappa}$.

We first show that $\hat{\kappa} \in \mathrm{Fix}(S)$. On contrary, we assume that $\hat{\kappa} \notin \mathrm{Fix}(S)$. From (3.31) and $\kappa_{n_k} \rightharpoonup \hat{\kappa}$, we have $t_{n_k} \rightharpoonup \hat{\kappa}$. Since $S\hat{\kappa} \neq \hat{\kappa}$. It follows from Opial's condition (2.5) that

$$\begin{aligned}
\liminf_{k\to\infty} \|t_{n_k} - \hat{\kappa}\| &< \liminf_{k\to\infty} \|t_{n_k} - S\hat{\kappa}\| \\
&\leq \liminf_{k\to\infty} \left\{ \|t_{n_k} - St_{n_k}\| + \|St_{n_k} - S\hat{\kappa}\| \right\} \\
&\leq \liminf_{k\to\infty} \|t_{n_k} - \hat{\kappa}\|,
\end{aligned}$$

which is a contradiction. Thus, $\hat{\kappa} \in \mathrm{Fix}(S)$.

Next, we show that $\hat{\kappa} \in \mathrm{Sol(MEP(1.2))}$. Since $v_n = T_{r_n}^F(\kappa_n - r_n f \kappa_n)$, for any $v \in \mathscr{K}_1$, we have

$$F(v_n, v) + \langle f\kappa_n, v - v_n\rangle + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - \kappa_n\rangle \geq 0, \ \forall v \in \mathscr{K}_1.$$

It follows from monotonicity of F that

$$\langle f\kappa_n, v - v_n\rangle + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - \kappa_n\rangle \geq F(v, v_n).$$

Replacing n by n_k, we get

$$\langle f\kappa_{n_k}, v - v_{n_k}\rangle + \varphi(v) - \varphi(v_{n_k}) + \left\langle v - v_{n_k}, \frac{v_{n_k} - \kappa_{n_k}}{r_{n_k}} \right\rangle \geq F(v, v_{n_k}). \tag{3.33}$$

Since $\|v_n - \kappa_n\| \to 0$ and $v_n \rightharpoonup \hat{\kappa}$, it is easy to observe that $v_{n_k} \rightharpoonup \hat{\kappa}$. Further, for any t with $0 < t \leq 1$ and $v \in \mathscr{K}_1$, let $v_t = tv + (1-t)\hat{\kappa}$. Since $\hat{\kappa} \in \mathscr{K}_1$, $v \in \mathscr{K}_1$, we have $v_t \in \mathscr{K}_1$. So from (3.33), we have

$$\begin{aligned}\langle v_t - v_{n_k}, f v_t\rangle &\geq \langle v_t - v_{n_k}, f v_t\rangle - \varphi(v_t) + \varphi(v_{n_k}) - \langle f\kappa_{n_k}, v_t - v_{n_k}\rangle \\ &\quad - \left\langle v_t - v_{n_k}, \frac{v_{n_k} - \kappa_{n_k}}{r_{n_k}}\right\rangle + F(v_t, v_{n_k}) \\ &= \langle v_t - v_{n_k}, f v_t - f\kappa_{n_k}\rangle + \langle v_t - v_{n_k}, f v_{n_k} - f\kappa_{n_k}\rangle - \varphi(v_t) + \varphi(v_{n_k}) \\ &\quad - \left\langle v_t - v_{n_k}, \frac{v_{n_k} - \kappa_{n_k}}{r_{n_k}}\right\rangle + F(v_t, v_{n_k}). \end{aligned} \tag{3.34}$$

From Lipschitz continuity of f and $\lim_{n\to\infty} \|v_n - \kappa_n\| = 0$, we obtain $\|f v_{n_k} - f\kappa_{n_k}\| = 0$ as $k \to \infty$. Further, since f is monotone and the weakly lower semicontinuity of φ and $v_{n_k} \rightharpoonup \hat{\kappa}$, it follows that from (3.34) that

$$\langle v_t - \hat{\kappa}, f v_t\rangle \geq -\varphi(v_t) + \varphi(\hat{\kappa}) + F(v_t, \hat{\kappa}). \tag{3.35}$$

Hence, from Assumption 1 and (3.35), we have

$$\begin{aligned} 0 &= F(v_t, v_t) + \varphi(v_t) - \varphi(v_t) \\ &\leq tF(v_t, v) + (1-t)F(v_t, \hat{\kappa}) + t\varphi(v) + (1-t)\varphi(\hat{\kappa}) - \varphi(v_t) \\ &\leq t[F(v_t, v) + \varphi(v) - \varphi(v_t)] + (1-t)[F(v_t, v) + \varphi(\hat{\kappa}) - \varphi(v_t)] \\ &\leq t[F(v_t, v) + \varphi(v) - \varphi(v_t)] + (1-t)t\langle v - \hat{\kappa}, f v_t\rangle, \end{aligned} \tag{3.36}$$

which implies that $F(v_t, v) + \varphi(v) - \varphi(v_t) + (1-t)\langle v - \hat{\kappa}, f v_t\rangle \geq 0$. Letting $t \to 0_+$, we have

$$F(\hat{\kappa}, v) + \varphi(v) - \varphi(\hat{\kappa}) + \langle v - \hat{\kappa}, f\hat{\kappa}\rangle \geq 0, \ \forall v \in \mathscr{K}_1,$$

which implies that $\hat{\kappa} \in \mathrm{Sol}(\mathrm{GMEP}(1.2))$.

Next, we show that $B\hat{\kappa} \in \mathrm{Sol}(\mathrm{GMEP}(1.3))$. Since $\|v_n - \kappa_n\| \to 0$, $v_n \rightharpoonup \hat{\kappa}$ as $n \to \infty$ and $\{\kappa_n\}$ is bounded, there exists a subsequence $\{\kappa_{n_k}\}$ of $\{\kappa_n\}$ such that $\kappa_{n_k} \rightharpoonup \hat{\kappa}$ and since B is a bounded linear operator so that $B\kappa_{n_k} \rightharpoonup B\hat{\kappa}$.

Setting $e_{n_k} = by_{n_k} - T^G_{r_{n_k}}(by_{n_k} - r_{n_k} gby_{n_k})$. It follows that from (3.27) that $\lim_{k\to\infty} e_{n_k} = 0$ and $by_{n_k} - e_{n_k} = T^G_{r_{n_k}}(by_{n_k} - r_{n_k} gby_{n_k})$.

Therefore from Lemma 2.1, we obtain

$$\begin{aligned} G(by_{n_k} - e_{n_k}, \zeta) \quad &+ \quad \psi(v) - \psi(by_{n_k} - e_{n_k}) + \left\langle gby_{n_k}, \zeta - (by_{n_k} - e_{n_k})\right\rangle \\ &+ \frac{1}{r_{n_k}}\langle \zeta - (by_{n_k} - e_{n_k}), (by_{n_k} - e_{n_k}) - by_{n_k}\rangle \geq 0, \ \forall \zeta \in \mathscr{K}_2. \end{aligned} \tag{3.37}$$

Since G is upper semicontinuous in first argument, taking lim sup to above Inequality as $k \to \infty$ and using condition (ii), we obtain

$$G(B\hat{\kappa}, \zeta) + \langle gB\hat{\kappa}, \zeta - B\hat{\kappa}\rangle + \psi(v) - \psi(B\hat{\kappa}) \geq 0, \ \forall \zeta \in \mathscr{K}_2,$$

which means that $B\hat{\kappa} \in \mathrm{Sol}(\mathrm{GMEP}(1.3))$. Therefore $\hat{\kappa} \in$ SpGMEP(1.2)-(1.3).

This implies $\hat{\kappa} \in \Omega$. From $d = P_\Omega \kappa$, $\hat{\kappa} \in \Omega$ and (3.22), we have

$$\|d-\kappa\| \leq \|\hat{\kappa}-\kappa\| \leq \liminf_{k\to\infty} \|\kappa_{n_k}-\kappa\| \leq \limsup_{k\to\infty} \|\kappa_{n_k}-\kappa\| \leq \|d-\kappa\|.$$

Thus, we have

$$\lim_{k\to\infty} \|\kappa_{n_k}-\kappa\| = \|\hat{\kappa}-\kappa\|.$$

Since $\kappa_{n_k}-\kappa \rightharpoonup \hat{\kappa}-\kappa$ and from Kadec-Klee propertyof Hilbert space, we have $\kappa_{n_k}-\kappa \to \hat{\kappa}-\kappa$ and hence $\kappa_{n_k} \to \hat{\kappa}$. Since by definition of $\mathscr{Q}_n$, we have $\kappa_n = P_{\mathscr{Q}_n}\kappa$ and $d \in \Omega \subset \mathscr{C}_n \cap \mathscr{Q}_n \subset \mathscr{Q}_n$, we conclude that

$$-\|d-\kappa_{n_k}\|^2 = \langle d-\kappa_{n_k}, \kappa_{n_k}-\kappa\rangle + \langle d-\kappa_{n_k}, \kappa-d\rangle \geq \langle d-\kappa_{n_k}, \kappa-d\rangle.$$

Letting $k \to \infty$, we obtain $-\|d-\hat{\kappa}\|^2 \geq \langle d-\hat{\kappa}, \kappa-d\rangle \geq 0$, since $d = P_\Omega\kappa$ and $\hat{\kappa} \in \Omega$. Hence we have $\hat{\kappa} = d$. This implies that $\kappa_n \to d$. Since by (3.23), $\lim_{n\to\infty} \|\kappa_n-\kappa\|$ exists and then by the above computation, we must have $\lim_{n\to\infty} \|\kappa_n-\kappa\| = \|d-\kappa\|$. Now another application of the Kadec-Klee propertyimplies that $\kappa_n \rightharpoonup d$. Then (3.25), (3.26) and (3.31) imply that $\upsilon_n \to d$, $\zeta_n \to d$ and $t_n \to d$. This completes the proof. □

4. Consequences

Now we discuss some consequences of Theorem 3.1.

Corollary 4.1. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces and $\mathscr{K}_1 \subseteq \mathscr{H}_1$ and $\mathscr{K}_2 \subseteq \mathscr{H}_2$ are nonempty, closed and convex subsets. Let $B : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. Assume that $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ and $G : \mathscr{K}_2 \times \mathscr{K}_2 \to \mathbb{R}$ are 2-monotone bifunctions satisfying Assumption 1 and G is upper semicontinuous in first argument. Let $S : \mathscr{K}_1 \to \mathscr{K}_1$ be a nonexpansive mapping such that $\Omega :=$ SpEP(1.8)-(1.9)$\cap \mathrm{Fix}(S) \neq \emptyset$. For a given $\kappa_0 \in \mathscr{K}_1$ arbitrarily, let the iterative sequences $\{\kappa_n\}$, $\{\upsilon_n\}$, and $\{\zeta_n\}$ are generated by*

$$\begin{aligned}
\kappa_0 &= \kappa \in \mathscr{K}_1,\\
\upsilon_n &= T^F_{r_n}(\kappa_n + \gamma_n B^*(T^G_{r_n} - I)B\kappa_n),\\
\zeta_n &= \alpha_n\kappa_n + (1-\alpha_n)ST^F_{r_n}(\kappa_n + \gamma_n B^*(T^G_{r_n} - I)B\upsilon_n),\\
\mathscr{C}_n &= \{\zeta \in \mathscr{H}_1 : \|\zeta_n - \zeta\|^2 \leq \|\kappa_n - \zeta\|^2\},\\
\mathscr{Q}_n &= \{\zeta \in \mathscr{H}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n\rangle \geq 0\},\\
\kappa_{n+1} &= P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa,
\end{aligned}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \nu)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\gamma_n \in \left(0, \frac{1}{\|B\|^2}\right)$, where $\nu = \min\{\nu_1, \nu_2\}$. Then the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$ and $\{\zeta_n\}$ converge strongly to $d = P_\Omega\kappa$.

Proof. The proof follows by taking $f = g = \varphi = \psi = 0$ in Theorem 3.1. □

Corollary 4.2. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces and $\mathscr{K}_1 \subseteq \mathscr{H}_1$ and $\mathscr{K}_2 \subseteq \mathscr{H}_2$ are nonempty, closed and convex subsets. Let $f : \mathscr{K}_1 \to \mathscr{H}_1$, and $g : \mathscr{K}_2 \to \mathscr{H}_2$ are ν_1, ν_2-inverse strongly monotone mappings and let $S : \mathscr{K}_1 \to \mathscr{K}_1$ be a nonexpansive mapping such that $\Omega :=$ SpVIP(1.6)-(1.7)$\cap$Fix$(S) \neq \emptyset$. For a given $\kappa_0 \in \mathscr{K}_1$ arbitrarily, let the iterative sequences $\{\kappa_n\}$, $\{\upsilon_n\}$, and $\{\zeta_n\}$ are generated by*

$$\begin{aligned}
\kappa_0 &= \kappa \in \mathscr{K}_1,\\
\upsilon_n &= U(\kappa_n + \gamma_n B^*(V - I)B\kappa_n),\\
\zeta_n &= \alpha_n\kappa_n + (1-\alpha_n)SU(\kappa_n + \gamma_n B^*(V-I)B\upsilon_n),\\
\mathscr{C}_n &= \{\zeta \in \mathscr{H}_1 : \|\zeta_n - \zeta\|^2 \leq \|\kappa_n - \zeta\|^2\},\\
\mathscr{Q}_n &= \{\zeta \in \mathscr{H}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n\rangle \geq 0\},\\
\kappa_{n+1} &= P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa,
\end{aligned}$$

where where $U := P_{\mathscr{K}_1}(I - r_n f)$, $V := P_{\mathscr{K}_2}(I - r_n g)$, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, \nu)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\gamma_n \in \left(0, \frac{1}{\|B\|^2}\right)$, where $\nu = \min\{\nu_1, \nu_2\}$. Then the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$ and $\{\zeta_n\}$ converge strongly to $d = P_\Omega\kappa$.

Proof. The proof follows by taking $F = G = 0$, and $\varphi = \psi = 0$ in Theorem 3.1, since in this case, we have $T^F_{r_n}(\kappa) = P_{\mathscr{K}_1}\kappa$, $\forall\kappa \in \mathscr{H}_1$ and $T^G_{r_n}(\upsilon) = P_{\mathscr{K}_2}\upsilon$ $\forall\kappa \in \mathscr{H}_2$. □

The following corollary is due to Nadezhkina and Takahashi [39].

Corollary 4.3. *[39] Let $\mathscr{K}_1$ be a nonempty, closed and convex subset of a real Hilbert space $\mathscr{H}_1$; let $f : \mathscr{H}_1 \to \mathscr{H}_1$ be ν-inverse strongly monotone mapping and let $S : \mathscr{K}_1 \to \mathscr{K}_1$ be a nonexpansive mapping such that $\Omega_3 = \mathrm{Sol}(\mathrm{VIP}(1.17)) \cap \mathrm{Fix}(S) \neq \emptyset$. Let the iterative sequences $\{\kappa_n\}$ and $\{\upsilon_n\}$ and $\{\zeta_n\}$ be generated by the following iterative scheme:*

$$\begin{aligned}
\kappa_0 &= \kappa \in \mathscr{K}_1,\\
\upsilon_n &= P_{\mathscr{K}_1}(\kappa_n - r_n f\kappa_n)\\
\zeta_n &= \alpha_n\kappa_n + (1-\alpha_n)SP_{\mathscr{K}_1}(\kappa_n - r_n f\upsilon_n),\\
\mathscr{C}_n &= \{\zeta \in \mathscr{H}_1 : \|\zeta_n - \zeta\|^2 \leq \|\kappa_n - \zeta\|^2\},\\
\mathscr{Q}_n &= \{\zeta \in \mathscr{H}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n\rangle \geq 0\},\\
\kappa_{n+1} &= P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa,
\end{aligned}$$

where $\{r_n\} \subset [a,b]$ for some $a, b \in (0, \nu)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$ and $\{\zeta_n\}$ converge strongly to $d = P_{\Omega_3}\kappa$.

Proof. The proof follows by taking $F = G = 0$, $\varphi = \psi = 0$ and $B = 0$, a nonexpansive mapping in Theorem 3.1, since in this case, we have $T_{r_n}(\kappa) = P_{\mathscr{K}_1}\kappa$, $\forall\kappa \in \mathscr{H}_1$. □

5. Numerical example

In this section, we provide a numerical example which justify the main result.

Example 5.1. *Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle \kappa, \upsilon \rangle = \kappa\upsilon$, $\forall \kappa, \upsilon \in \mathbb{R}$, and induced usual norm $|.|$. Let $\mathscr{K}_1 = [0, +\infty)$ and $\mathscr{K}_2 = (-\infty, 0]$; let $F : \mathscr{K}_1 \times \mathscr{K}_1 \to \mathbb{R}$ and $G : \mathscr{K}_2 \times \mathscr{K}_2 \to \mathbb{R}$ be defined by $F(\kappa, \upsilon) = 3\kappa(\upsilon - \kappa)$, $\forall \kappa, \upsilon \in \mathscr{K}_1$ and $G(u, v) = (u - 3)(v - u)$, $\forall u, v \in \mathscr{K}_2$; let for each $\kappa \in \mathbb{R}$, we define $B(\kappa) = -2\kappa$ and let, for each $\kappa \in \mathscr{K}_1$, $S(\kappa) = \frac{1}{2}\kappa$ and let $f : \mathscr{K}_1 \to \mathscr{H}_1$ and $g : \mathscr{H}_2 \to \mathscr{H}_2$ are defined by $f(\kappa) = 2\kappa$ and $g(\kappa) = 0$, $\forall \kappa \in \mathscr{K}_1$ and $\varphi = \psi = 0$. Let the sequences $\{\kappa_n\}$, $\{\upsilon_n\}$, and $\{\zeta_n\}$ are generated by the iterative schemes*

$$p_n = T^G_{r_n}(I - r_n g)B\kappa_n;\ \upsilon_n = U\left[\kappa_n + \frac{1}{6}B^*(p_n - B\kappa_n)\right]; \tag{5.1}$$

$$\zeta_n = \frac{1}{n+1}\kappa_n + (1 - \frac{1}{n+1})SU\left[\kappa_n + \frac{1}{6}B^*(p_n - B\kappa_n)\right], \tag{5.2}$$

$$\mathscr{C}_n = \{\zeta \in \mathscr{H}_1 : ||\zeta_n - \zeta||^2 \leq ||\kappa_n - \zeta||^2\} \tag{5.3}$$

$$\mathscr{Q}_n = \{\zeta \in \mathscr{H}_1 : \langle \kappa_n - \zeta, \kappa - \kappa_n \rangle \geq 0\} \tag{5.4}$$

$$\kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa \tag{5.5}$$

where $\alpha_n = \dfrac{1}{n+1}$ and $r_n = 1$. Then $\{\kappa_n\}$ converges strongly to $0 \in \Omega$.

Proof. The verification of Assumption 1 for bifunctions F and G, along with the upper semicontinuity of G, can be easily demonstrated. The operator B is bounded and linear on $\mathbb{R}$, and has an adjoint operator B^*, both having a norm of 2. As γ_n belongs to the open interval $\left(0, \frac{1}{4}\right)$, we can choose $\gamma = \frac{1}{6}$. Additionally, it is straightforward to notice that $\Omega :=$ SpGMEP(1.2)-(1.3)$\cap$Fix$(S) \neq \{0\}$. After simplification, schemes (5.1)-(5.5) reduce to

$$\upsilon_n = \frac{1}{4}\left[\frac{-11}{3}\kappa_n + \frac{-3}{3}\right];$$

$$\zeta_n = \frac{1}{n+1}\kappa_n + \left(1 - \frac{1}{n+1}\right)\frac{1}{8}\left[1 - \frac{20}{9}\kappa_n\right],$$

$$\mathscr{C}_n = \left[\frac{\zeta_n + \kappa_n}{2}, \infty\right]$$

$$\mathscr{Q}_n = [\kappa_n, \infty]$$

$$\kappa_{n+1} = P_{\mathscr{C}_n \cap \mathscr{Q}_n}\kappa$$

Following the steps of proof of Theorem 3.1, we obtain that $\{\kappa_n\}$ and $\{\upsilon_n\}$ converge strongly to $0 \in \Omega$. The proof is completed. □

Using Matlab 7.0 software, we conducted a study to analyze the convergence behavior of the sequence κ_n for various initial values. The results of our analysis, depicted in the following figures, demonstrate the strong convergence of κ_n towards the limit 0.

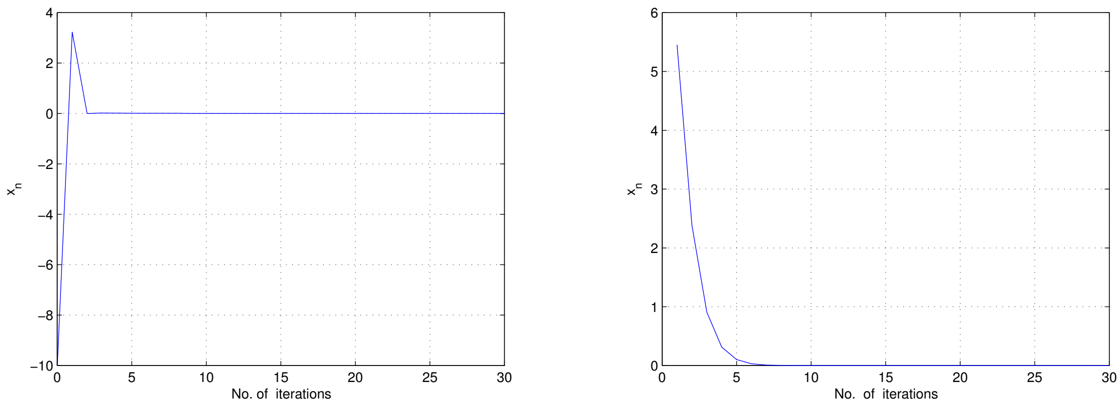

FIGURE 1. Convergence of $\{\kappa_n\}$ for different initial values

Example 5.2. *We consider an example in infinite dimensional Hilbert spaces. Let* $\mathscr{H}_1 = \mathscr{H}_2 = L^2[0,1]$, *with the inner product defined by* $\langle \kappa, \upsilon \rangle = \int_0^1 \kappa(t)\upsilon(t)dt$, *and induced usual norm* $\|\kappa\| = \sqrt{\int_0^1 |\kappa(t)|^2 dt}$, *for all* $\kappa, \upsilon \in L^2[0,1]$. *Let* $F : L^2[0,1] \times L^2[0,1] \to \mathbb{R}$ *and* $G : L^2[0,1] \times L^2[0,1] \to \mathbb{R}$ *be defined by* $F(\kappa, \upsilon) = (\kappa(t) - 2t)(\upsilon(t) - \kappa(t))$, $\forall \kappa, \upsilon \in L^2[0,1]$ *and* $G(u, v) = (u(t) + 4t)(v(t) - u(t))$, $\forall u, v \in L^2[0,1]$. *Then* F *and* G *satisfies Assumption 2.1; let for each* $\kappa \in L^2[0,1]$, *we define* $B(\kappa) = -\frac{9}{4}\kappa(t)$ *and let, for each* $\kappa \in L^2[0,1]$, $S(\kappa) = \frac{1}{2}\kappa(t)$ *and let* $f : L^2[0,1] \to L^2[0,1]$ *and* $g : L^2[0,1] \to L^2[0,1]$ *are defined by* $f(\kappa) = 2\kappa(t)$ *and* $g(\kappa) = 3\kappa(t)$, $\forall \kappa \in L^2[0,1]$. *Then* f *is 2-inverse strongly monotone mapping and* g *is 3-inverse strongly monotone mapping and assume that* $\varphi \equiv \psi = 0$.

Further, let $B : L^2[0,1] \to L^2[0,1]$ be a bounded linear operator defined by $B(\kappa(t)) = -\frac{9}{4}\kappa(t)$ such that $|B\| = \|B^*\| = \frac{9}{4}$. Let $S(\kappa(t) = \kappa(t)$ for all $t \in [0,1]$, then clearly S is nonexpansive mapping with $Fix(S) = (-\infty, \infty)$. Let $X =: \left\{ u \in L^2[0,1]; \int_0^1 u(t)dt = 0 \right\}$, then $P_X(u) = u - \int_0^1 u(s)ds,\ u \in L^2[0,1]$. Furthermore, take $\alpha_n = \frac{1}{n+1}$ and $r_n = 1$ and following the steps of proof of Theorem 3.1, we obtain that $\{\kappa_n\}$ and $\{\upsilon_n\}$ converge strongly to $0 \in \Omega$.

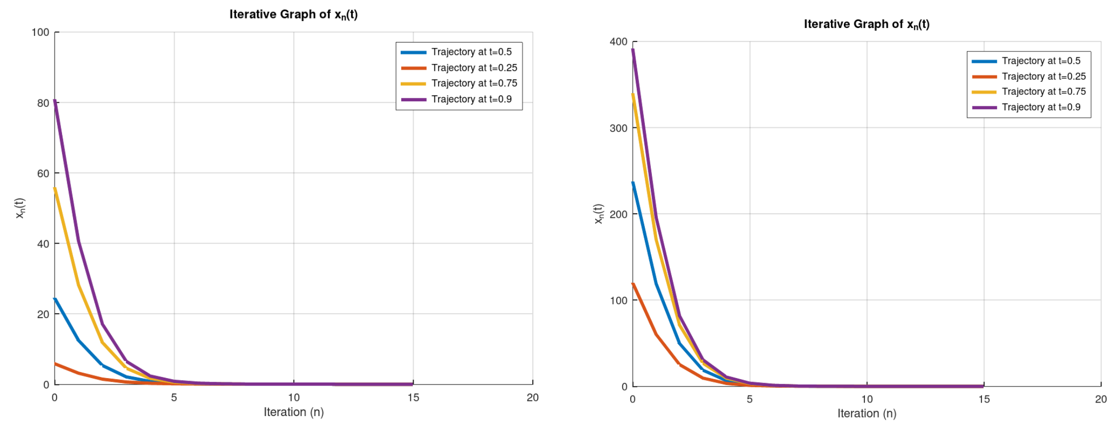

Figure 2. Convergence of $\{\kappa_n\}$ for different initial values

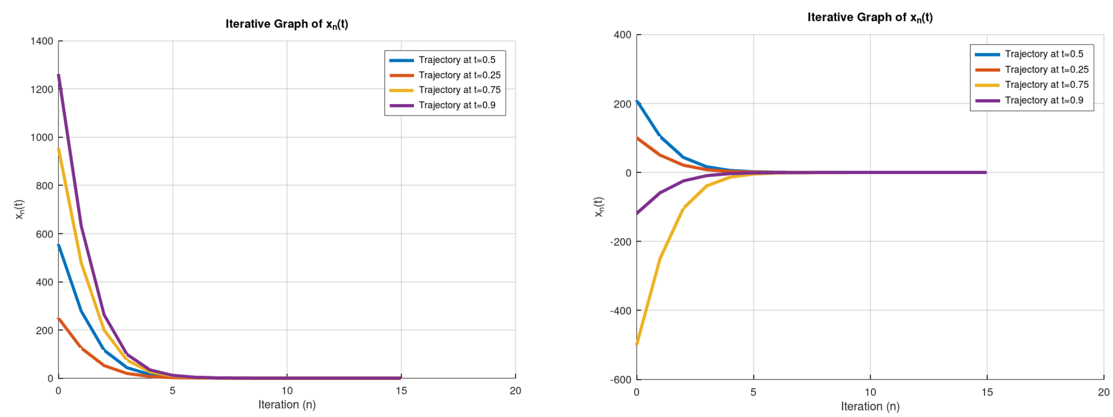

Figure 3. Convergence of $\{\kappa_n\}$ for different initial values

Conclusion: The present study introduces an extension to the split Generalized Mixed Equilibrium Problem, along with a proposed iterative method based on the hybrid extragradient method. This method is utilized to obtain a common solution for the split Generalized Mixed Equilibrium Problem and the Fixed Point Problem for a nonexpansive mapping in the context of real Hilbert spaces. Our proposed method is shown to achieve strong convergence, given certain mild conditions. These results are a generalization and expansion of previously known findings in the field.

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