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Lifts of Almost Product Structures From Manifolds to Tangent Bundles

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Abstract. In the present paper I have explored the properties of complete and horizontal lifts of an almost product structure. Its partial integrability and integrability conditions on the tangent bundle are part of this study. Along with different theorems on the lifts the prolongation of an almost product structure on the third tangent bundle T_3M has been explored.

1. Introduction

Let TM be the tangent bundle of a manifold M. To investigate the properties of geometrical structures such as integrability, curvature, lie derivative etc. tangent bundle is used. Yano and Ishihara [22] introduced and studied tangent bundles on almost complex structures. Khan et. al. [3] have researched lifts of an almost product structure over an almost r-contact structure along with TM. Many researchers like Dida et. al. [4–6], Khan and De [11, 17], Khan [12–15], Omran et. al. [18], Peyghan et. al. [19], Tekkoyun [20] and Yano and Ishihara [22] have done tremendous work on geometric structures and connections, which laid the foundation for numerous research study. Kankarej and Singh [9] studied lifts on its cotangent bundles.
In this paper I have studied the lifts of an almost product structure on manifolds on the tangent bundle and established its partial integrability and integrability conditions. In the later part of the paper the prolongation of an almost product structure on the third tangent bundle T_3M is also studied.

Goldenberg et. al. [7, 8] introduced the polynomial structure of degree n

$$Q(F) = F^n + a_n F^{n-1} + ... + a_2 F + a_1 I,$$

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where F is the tensor field of type (1,1) and I is the identity tensor field on a differentiable manifold. Let M be an n-dimensional differentiable manifold. A non-null tensor field F of type (1, 1) on M is called an almost product structure if it satisfies the equation

$$F^2 = I, \tag{1.1}$$

The pair (M, F) is called an almost product-manifold where F is of constant rank n everywhere in M.

Let l and m be operators defined as

$$\begin{aligned} (a)\; l &= F^2, \\ (b)\; m &= I - F^2. \end{aligned} \tag{1.2}$$

The operators l and m defined in the equation (1.2) satisfy the following identities:

$$\begin{aligned} l + m &= 0, \\ l^2 &= l,\; m^2 = m,\; lm = ml = 0, \\ Fl &= lF = F,\; Fm = mF = 0. \end{aligned} \tag{1.3}$$

Thus there exist two complementary distributions D_l and D_m corresponding to the projection tensors l and m respectively in M.

2. The complete lift of an almost product-structure in the tangent bundle

Let M be an m-dimensional differentiable manifold of class C^∞ and $T_p(M)$ the tangent space at a point p of M then $TM = \cup_{p\in M} T_p(M)$ is a tangent bundle over the manifold M. The tangent bundle TM of M is a differentiable manifold of dimension $2n$. Let $\Im^r_s$ denote the set of tensor field of class C^∞ and type (r, s) in M and $\Im^r_s(TM)$ denote the corresponding set of tensor fields in TM [10]. Let F, G be elements of $\Im^1_1(M)$. Then we have [21]

$$(FG)^C = F^C G^C. \tag{2.1}$$

Putting $F = G$ in the equation (2.1), we obtain

$$(F^2)^C = (F^C)^2. \tag{2.2}$$

Also,

$$(F + G)^C = F^C + G^C. \tag{2.3}$$

Now operating the complete lifts on both sides of the equation (1.1),

$$(F^2)^C = I^C, (F^2)^C = I. \tag{2.4}$$

In the view of equations (1.1), (2.4) and [21], we can conclude that the rank of F^C is $2n$ if and only if the rank of F is n. Therefore , we can state following theorems:

Theorem 2.1. *If $F \in \Im^1_1(M)$ be a almost product-structure in M, then its complete lift F^C is also an almost product-structure in TM.*

Theorem 2.2. *The almost product-structure F is of rank n in M if and only if its complete lift F^C is of rank $2n$ in TM.*

Let F be an almost product-structure of rank n in M. Then the complete lift l^C of l and m^C of m are complementary projection tensors in TM. Therefore there exist two complementary distributions D_{l^C} and D_{m^C} determined by l^C and m^C respectively in TM. The distributions D_{l^C} and D_{m^C} are respectively the complete lifts of D_l^C and D_m^C of D_l and D_m [3].

3. Integrability conditions of an almost product-structure in the tangent bundle

Let F be the almost product-structure with $F^2 = I$. Then the Nijenhuis tensor N of F is a tensor of type (1,2) given by [16]

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]. \tag{3.1}$$

Let N^C be the Nijenhuis tensor of F^C in TM, then we have

$$\begin{aligned} N^C(X^C,Y^C) &= [F^CX^C,F^CY^C] - F^C[F^CX^C,Y^C] \\ &- F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C]. \end{aligned} \tag{3.2}$$

Let $X,Y \in \mathfrak{I}_0^1(M)$ and $F \in \mathfrak{I}_1^1(M)$, then by property of complete lifts we have

$$\begin{aligned} [X^C,Y^C] &= [X,Y]^C, \\ (X+Y)^C &= X^C + Y^C, \\ F^CX^C &= (FX)^C. \end{aligned} \tag{3.3}$$

From equations (1.3) and (3.3), we get

$$\begin{aligned} F^Cl^C &= (Fl)^C = F^C, \\ F^Cm^C &= (Fm)^C = 0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let $X,Y \in \mathfrak{I}_0^1(M)$, then the complete lifts l^C and m^C holds following identities :*

$$N^C(m^CX^C,m^CY^C) = F^C[m^CX^C,m^CY^C], \tag{3.5}$$

$$m^CN^C(X^C,Y^C) = m^C[F^CX^C,F^CY^C], \tag{3.6}$$

$$m^C(l^CX^C,l^CY^C) = m^C[F^CX^C,F^CY^C], \tag{3.7}$$

$$m^CN^C((F^2)^CX^C,(F^2)^CY^C) = m^CN^C(l^CX^C,l^CY^C). \tag{3.8}$$

Proof: The proof of equations (3.5) to (3.8) follow from equations (1.3), (3.4) and (3.1).

Theorem 3.2. *Let $X,Y \in \mathfrak{I}_0^1(M)$, the following identities are equivalent*

$$\begin{aligned} (a)\ m^CN^C(X^C,Y^C) &= 0, \\ (b)\ m^CN^C(l^CX^C,l^CY^C) &= 0, \\ (c)\ m^CN^C((F^2)^CX^C,(F^2)^CY^C) &= 0. \end{aligned}$$

Proof: From equation (3.8), we have

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2)^C X^C, (F^2)^C Y^C) = 0.$$

Now the right side of the equations (3.6), (3.7) are equal. Thus we conclude from last equation that identities (a), (b), and (c) are equivalent.

Theorem 3.3. *The complete lift D_m^C in TM of a distribution D_m in M is integrable if D_m is integrable in M.*

Proof: By [21] the distribution D_m is integral if and only if

$$l[mX, mY] = 0, \tag{3.9}$$

for all $X, Y \in \Im(M)$, where $l = I - m$. Operating complete lift of both sides and using (3.5), we get

$$l^C[m^C X^C, m^C Y^C] = 0, \tag{3.10}$$

for all $X, Y \in \Im(M)$, where $l^C = (I-m)^C = I - m^C$ is the projection tensor complementary to m^C. Thus the condition (3.9) implies (3.10).

Theorem 3.4. *The complete lift D_m^C in TM of a distribution D_m in M is integrable if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$, for all $X, Y \in \Im_0^1(M)$.*

Proof: By [21] the distribution D_m is integral in M if and only if

$$N(mX, mY) = 0,$$

for all $X, Y \in \Im(M)$. By virtue of condition (3.5), we have

$$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C).$$

In view of (3.10), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0. \tag{3.11}$$

Also, we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0. \tag{3.12}$$

Adding equations (3.11) and (3.12), we get

$$(l^C + m^C) N^C(m^C X^C, m^C Y^C) = 0.$$

Since $l^C + m^C = I^C = I$, we have

$$N^C(m^C X^C, m^C Y^C) = 0.$$

Theorem 3.5. *Let the distribution D_l be integrable in M, that is $mN(X,Y) = 0$ for all $X, Y \in \Im_0^1(M)$. Then the distribution D_l^C is integrable in TM if and only if the one of the conditions of Theorem (3.2) is satisfied.*

Proof: The distribution D_l is integral in M if and only if

$$mN(lX, lY) = 0.$$

Thus distribution D_l^C is integrable in TM if and only if

$$m^C N^C(l^C X^C, l^C Y^C) = 0.$$

Thus the theorem can be proved from equation (3.8).

Theorem 3.6. *Complete lift F^C of a almost product-structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: The almost product-structure F in M is partially integrable if and only if

$$N(lX, lY) = 0, \forall X, Y \in \mathfrak{I}_0^1(M). \tag{3.13}$$

From equations (1.3) and (3.1), we get

$$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0.$$

Also from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2)^C X^C, (F^2)^C Y^C) = 0.$$

Theorem 3.7. *Complete lift F^C of a almost product-structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: A necessary and sufficient condition for a almost product-structure in M to be integrable is

$$(N(X, Y)) = 0 \tag{3.14}$$

for all $X, Y \in \mathfrak{I}_0^1(M)$. From equation (3.1), we get

$$N^C(X^C, Y^C) = (N(X, Y))^C.$$

Therefore, from equation (3.14) we obtain the result.

4. The horizontal lift of an almost product-structure in the tangent bundle

Theorems on horizontal lift of the almost product-structure are as given below:

Suppose we have tensor fields S and $\nabla_\gamma S$ in M and TM respectively with affine connection ∇ in the terms of partial differential equations given by [1,2,21]

$$S = S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial x^h}\otimes dx^k\otimes\ldots\otimes dx^j,$$

$$\nabla_\gamma S = y^l\nabla_\gamma S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial y^h}\otimes dx^k\otimes\ldots\otimes dx^j$$

corresponding to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$ by [21].
We define the horizontal lift S^H of a tensor field S in M to TM by

$$S^H = S^C - \nabla_\gamma S.$$

Theorem 4.1. *Let $F \in \Im^1_1(M)$ be an almost product-structure in M, then its horizontal lift $F^H$is also almost product-structure in TM.*

Proof: If $P(t)$ is a polynomial in one variable t, then by [21] we have

$$(P(F))^H = P(F^H), \tag{4.1}$$

for all $F \in \Im^1_1(M)$. Operating the horizontal lifts of both sides of the equation (1.1), we get

$$(F^2)^H = I^H, (F^2)^H = I. \tag{4.2}$$

which shows that F^H is an almost product-structure in TM. From (1.1) and (4.2), we can easily say that the rank of F^H is $2n$ if and only if the rank of F is n. Therefore, we have the following theorem:

Theorem 4.2. *The almost product-structure F is of rank n in M if and only if its complete lift F^H is of rank $2n$ in TM.*

Let m be a projection tensor field of type (1,1) in M defined by (1.3), then there exists in M a distribution D determined by m. Also,

$$m^2 = m.$$

In view of (4.1), we get

$$(m^H)^2 = m^H.$$

Thus, m^H is also a projection in TM. Hence there exists in TM a distribution D^H corresponding to m^H, which is called the horizontal lift of the distribution D.

5. Prolongation of an almost product-structure on third tangent bundle T_3M

Let T_3M be the third order tangent bundle over M and let F^{III} be the third lift on F in T_3M. Then for any $F, G \in \Im_1^1(M)$, we have

$$\begin{aligned}(G^{III}F^{III})X^{III} &= G^{III}(F^{III}X^{III}) \\ &= (G^{III}(FX)^{III}) \\ &= (G(FX))^{III} \\ &= (GF)^{III}X^{III},\end{aligned} \quad (5.1)$$

for all $X \in \Im_0^1(M)$. Thus we have

$$G^{III}F^{III} = (GF)^{III}$$

If $P(t)$ is a polynomial in one variable t, then we have by [21]

$$(P(F))^{III} = P(F^{III}). \quad (5.2)$$

for all $F \in \Im_1^1(M)$.

Theorem 5.1. *Let $F \in \Im_1^1(M)$ be a almost product-structure in M, then the third lift F^{III} is also almost product-structure in T_3M.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have by [21]

$$(P(F))^{III} = P(F^{III}), \quad (5.3)$$

for all $F \in \Im_1^1(M)$. Operating the third lifts of both sides of the equation (1.1), we get

$$\begin{aligned}(F^2)^{III} &= I^{III}, \\ (F^2)^{III} &= I.\end{aligned}$$

which shows that F^{III} is a almost product-structure in T_3M.

Theorem 5.2. *The third lift F^{III} is integrable in T_3M if and only if F is integrable in M.*

Proof: Let N^{III} and N be Nijenhuis tensors of F^{III} and F respectively. Then we have

$$N^{III}(X, Y) = (N(X, Y))^{III}. \quad (5.4)$$

since almost product-structure is integrable in M if and only if $N(X, Y) = 0$. Then from (5.4), we get

$$N^{III}(X, Y) = 0. \quad (5.5)$$

Thus F^{III} is integrable if and only if F is integrable in M.

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