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A New Fixed Point Approach for Approximating Solutions of Fractional Differential Equations in Banach Spaces

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Abstract. This contribution targets the solution of fractional differential equations (FDEs) via novel iterative approach and in a new class of nonlinear mappings. Our approach is based on the class of (α, β, γ)-nonexpansive mappings and three-step M-iterative scheme. Under various assumptions, we first carry out some weak and strong convergence results in a setting of a Banach spaces. After this, we carry out an application of one our main result to find approximate solution for a broad class of FDEs. Eventually, we we construct a new example of (α, β, γ)-nonexpansive mappings and show that this new mapping is not continuous on its whole domain and hence it is not nonexpansive. Using this example, we perform a numerical simulation of various iterative scheme including our M-iterative scheme. It has been observed the numerical effectiveness of the M-iterative scheme is high as compared to the other iterative schemes. Accordingly, our main outcome is new/extends some known results of the literature.

1. Introduction and Preliminaries

Let $\mathcal{V}$ be a norm linear space and $E \neq \emptyset$ be a subset of $\mathcal{V}$. A mapping $Q : E \to E$ is called a contraction mapping on E if $\forall t, y \in E$, we have

$$\|Qt - Qy\| \leq \alpha \|t - y\|, \text{ for some fixed } \alpha \in [0, 1).$$

Numerical computation of nonlinear operator is attention grabbing field for many researcher. In recent years, the theory of fixed point iterations has found numerous important applications for solving various classes of differential equations, particularly boundary value problems (see,

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e.g., [1–3] and others). Initially, The Banach Contraction Principle (BCP) [8] with the help of Picard [20] iterative scheme make significant contribution in the field of Fixed Point Theory but Picard iteration in general is not effective in the case of nonexpansive mappings. This is shown in example given below.

Example 1.1. *Suppose $E = [0,1]$ and $Qt = -(t-1)$. Then Q is nonexpansive but not contraction. It follows that Q admits a fixed point and the point $s_0 = 0.5$ is the only fixed point of Q. Notice that, for each $t_1 = t \in E - \{0.5\}$, the Picard [20] iteration of Q generates the following sequence,*

$$t, 1-t, t, 1-t, \ldots$$

This sequence does not converge to the fixed point $s_0 = 0.5$ of Q.

With the passage of time, different researchers (cf. Browder [9], Gohde [11] and others) introduced useful assumptions by considering a closed, convex and bounded subset of a uniformly convex Banach (UCB) space so that a self nonexpansive mapping has at least one fixed point. In the same year, Kirk [16] obtained the Browder–Gohde fixed point result in the setting of reflexive Banach spaces. The first notable generalization of nonexpansive mappings is due to Suzuki [24]. Suzuki introduced a condition called condition (C) for mappings and proved that every nonexpansive mappings obviously satisifis the condition (C) but the vice versa is not hold. A mapping with condition (C) is sometimes reffered to as a Suzuki mapping. A self map on a subset E of a Banach space is said to be with condition (C) or Suzuki mapping if for each two element $t, y \in E$, we have

$$\frac{1}{2}||t - Qt|| \leq ||t - y|| \Rightarrow ||Qt - Qy|| \leq ||t - y||.$$

Another generalization of nonexpansive mappings was introduced by Aoyama and Kahsoka [7] as follows. A self map on a subset E of a Banach space is said to be α-nonexpansive if for each two element $t, y \in E$, we have

$$||Qt - Qy||^2 \leq \alpha||t - Qy||^2 + \alpha||y - Qt||^2 + (1 - 2\alpha)||t - y||^2,$$

where $\alpha \in [0,1]$.

Very recently, a new generalization of nonexpansive mappings was presented by Ullah and Ahmad [27] as follows.

Definition 1.1. *[27] A mapping Q on a subset E of a Banach space is called (α, β, γ)–nonexpansive if*

$$||Qt - Qy|| \leq \alpha||t - y|| + \beta||t - Qt|| + \gamma||t - Qy|| \ \forall t, y \in E,$$

and here the scalars $\alpha, \beta, \gamma \in \mathbb{R}^+$ with the conditions $\gamma \in [0,1)$ and $\alpha + \gamma \leq 1$.

As demonstrated in Example 1.1, Picard's iteration generally diverges within the fixed point set of nonexpansive mappings. This example motivate researcher for development of other iterative methods to find fixed point of nonexpansive (also generalized nonexpansive) mappings, like, the iterative methods due to Mann [17] , Ishikawa [12], Noor [18], Agarwal [6], Abbas [4], and Thakur [26]. Ullah and Arshad [28] suggested the iterative scheme M as follows:

$$\left.\begin{array}{l} t_0 \in E, \\ z_n = (1-\psi_n)t_n + \psi_n Q t_n, \\ y_n = Q z_n, \\ t_{n+1} = Q y_n. \end{array}\right\} \tag{1.1}$$

The main advantages of the M-iterative scheme (1.1) over the other iterative schemes of the literatur are that it needs only one parameter ψ and secondly, Ullah and Arshad [28] proved that the M-iterativ scheme (1.1) in the broad class of mappings due to Suzuki [24] is faster than the many other iterative schemes of the literature. Keeping in view of these advantages of the M-iterative scheme, we use analyzed it with the class of α, β, γ–nonexpansive mappings and prove that our main outcome is applicable in a borad class of differential equations. The results are then we support by a new numerical example.

Definition 1.2. *Let $\mathcal{V}$ denotes a Banach space and $\{t_n\} \subseteq \mathcal{V}$ be bounded. If $\emptyset \neq E \subseteq \mathcal{V}$ is convex and closed. Then the asymptotic radius of $\{t_n\}$ corresponding to E is defined as*

$$r(E, \{t_n\}) = \inf\{\limsup_{n\to\infty} \|t_n - s\| : s \in E\}$$

Similarly, the asymptotic center of the sequence $\{t_n\}$ corresponding to E is explained by the formula

$$\mathcal{A}(E, \{t_n\}) = \{s \in E : \limsup_{n\to\infty} \|t_n - s\| = r(E, t_n)\}$$

Remark 1.1. *If $\mathcal{V}$ denotes a UCB space [10], then it is well-known that $\mathcal{A}(E, \{t_n\})$ contains an only element. Also note that when E is convex as well as weakly compact then $\mathcal{A}(E, \{t_n\})$ is convex(see e.g., [5, 25] and others).*

Definition 1.3. *[19] A Banach space $\mathcal{V}$ is said to be equipped with the Opial's condition iff $\{t_n\} \subseteq \mathcal{V}$ whenever converges in the weak sense to $s_0 \in E$, then the following condition must be valids:*

$$\limsup_{n\to\infty} \|t_n - s_0\| < \limsup_{n\to\infty} \|t_n - e_0\| \ \forall e_0 \in \mathcal{V} - \{s_0\}.$$

Every Hilbert space is equipped with the Opial's condition.

Definition 1.4. *[22] A mapping Q defined on a subset E of a Banach space $\mathcal{V}$ is said to be equipped with the condition (I) iff one has a function $q : [0, \infty) \to [0, \infty)$ such that $q(0) = 0$, $q(r) > 0$ for every $r \in [0, \infty) - \{0\}$ and $\|t - Qt\| \geq q(d(t, F_Q))$ whenever $t \in E$. Here, $d(t, F_Q)$ is the distance of t to F_Q.*

Lemma 1.1. *[27] Suppose Q is (α, β, γ)–nonexpansive mapping on a subset E of a Banach space with a fixed point, namely, s_0. Then $\|Qt - Qs_0\| \leq \|t - s_0\|$ holds for all $t \in E$ and $s_0 \in F_Q$.*

Lemma 1.2. *[27] Let Q be an (α, β, γ)–nonexpansive mapping defined on a subset E of a Banach space $\mathcal{V}$. The set F_Q is closed. Additionally, if E is convex and $\mathcal{V}$ is strictly convex, then F_Q is also convex*

Lemma 1.3. *[27] Suppose Q is (α,β,γ)–nonexpansive mappings on a subset E of a Banach space. Then for all $t, y \in E$, we have*

$$\|t - Qy\| \leq \frac{(1+\beta)}{(1-\gamma)}\|t - Qt\| + \frac{\alpha}{(1-\gamma)}\|t - y\|$$

Lemma 1.4. *[27] If Q is (α,β,γ)–nonexpansive mapping, $\{t_n\}$ is weakly convergent to s and $\lim_{n\to\infty}\|Qt_n - t_n\| = 0$, then $s \in F_Q$ provided that $\mathcal{V}$ is equipped with the Opial's condition.*

2. Main Results

Now we are in the position to connect the M iterative scheme (1.1) with (α,β,γ)–nonxpansive mapping. The first result of this section is the following basic lemma.

Lemma 2.1. *Let $\mathcal{V}$ be a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $Q : E \to E$ is (α,β,γ)–nonxpansive mapping satisfying $F_Q \neq \emptyset$ and $\{t_n\}$ a sequence of M iterates (1.1). Then for each $s_0 \in F_Q$, it follows that, $\lim_{n\to\infty}\|t_n - s_0\|$ exists.*

Proof. If $s_0 \in F_Q$ is any element, then applying Lemma 1.1 on the (1.1), we have

$$\begin{aligned} \|z_n - s_0| &= \|(1-\psi_n)t_n + \psi_n Qt_n - s_0\| \\ &= \|t_n - t_n\psi_n + \psi_n Qt_n - s_0\| \\ &\leq (1-\psi_n)\|t_n - s_0\| + \psi_n\|t_n - s_0\| \\ &\leq \|t_n - s_0\|. \end{aligned} \tag{2.1}$$

Using (2.1) and Lemma1.1, we have

$$\begin{aligned} \|y_n - s_0\| &= \|Qz_r - s_0\| \\ &\leq \|z_n - s_0\|. \end{aligned} \tag{2.2}$$

From (2.2) and (2.1)

$$\begin{aligned} \|t_{n+1} - s_0\| &= \|Qy_n - s_0\| \\ &\leq \|y_n - s_0\| \\ &\leq \|z_n - s_0\|. \end{aligned} \tag{2.3}$$

It can be observed from (2.1), (2.2) and (2.3) that $\|t_{n+1} - s_0\| \leq \|t_n - s_0\|$ i.e., $\{\|t_n - s_0\|\}$ is essentially bounded and also non-increasing. This means that $\lim_{n\to\infty}\|t_n - s_0\|$ exists for each element s_0 of F_Q. □

For existence of fixed points, this theorem elaborate necessary and sufficient assumptions.

Theorem 2.1. *Let $\mathcal{V}$ represents a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $Q : E \to E$ is (α,β,γ)–nonxpansive mapping satisfying $F_Q \neq \emptyset$ and $\{t_n\}$ is a sequence of M iterates (1.1). Then, $F_Q \neq \emptyset$ $\iff$ $\{t_n\}$ is bounded and satisfies $\lim_{n\to\infty}\|t_n - Qt_n\| = 0$.*

Proof. To prove this, we first assume that $F_Q \neq \emptyset$. Threfore, for any $s_0 \in F_Q$, Lemma 2.1 suggests that $\{t_n\}$ is bounded and $\lim_{r\to\infty} \|t_n - s_0\|$ exists. Consider

$$\lim_{r\to\infty} \|t_n - s_0\| = e \tag{2.4}$$

we need to prove $\lim_{n\to\infty} \|t_n - Qt_n\| = 0$ Now from (2.1)

$$\begin{aligned} \|z_n - s_0\| &\leq \|t_n - s_0\| \\ \Rightarrow \limsup_{n\to\infty} \|z_n - s_0\| &\leq \limsup_{n\to\infty} \|t_n - s_0\| = e \end{aligned} \tag{2.5}$$

Since $s_0 \in F_Q$, we can apply Lemma 1.1 to get

$$\begin{aligned} \|Qt_n - s_0\| &\leq \|t_n - s_0\| \\ \Rightarrow \limsup_{n\to\infty} \|Qt_n - s_0\| &\leq \Rightarrow \limsup_{n\to\infty} \|t_n - s_0\| \end{aligned} \tag{2.6}$$

Now from (2.3), we have

$$\|t_{n+1} - s_0\| \leq \|z_n - s_0\|$$

Using this together with (2.4), we obtain

$$e \leq \liminf_{n\to\infty} \|z_n - s_0\|. \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$\lim_{n\to\infty} \|z_n - s_0\| = e \tag{2.8}$$

Since $\|z_n - s_0\| = \|(1-\psi_n)(t_n - s_0) + \psi_n(Qt_n - s_0)\|$, so using this together with (2.8), we get

$$e = \lim_{n\to\infty} \|(1-\psi_n)(t_n - s_0) + \psi_n(Qt_n - s_0)\|. \tag{2.9}$$

Considering (2.4), (2.6) and (2.9) along with the Lemma 1.1, one gets

$$\lim_{n\to\infty} \|t_n - Qt_n\| = 0.$$

Conversely, we shall assume that $\{t_n\}$ is essentially bounded with the property $\lim_{n\to\infty} \|t_n - Qt_n\| = 0$ and prove that $F_Q \neq \emptyset$. To do this, we consider any $s_0 \in \mathcal{A}(E, \{t_n\})$. By Lemma 1.3, we have

$$\begin{aligned} r(Qs_0, \{t_n\}) &= \limsup_{n\to\infty} \|t_n - Qs_0\| \\ &\leq \frac{(1+\beta)}{(1-\gamma)} \|t_n - Qt_n\| + \frac{\alpha}{(1-\gamma)} \|t_n - s_0\| \\ &= \limsup_{n\to\infty} \|t_n - s_0\| \\ &= r(s_0, \{t_n\}). \end{aligned}$$

Thus $Qs_0 \in \mathcal{A}(E, \{t_n\}$. But the set $\mathcal{A}(E, \{t_n\})$ contains an only point,therefore $Qs_0 = s_0$. It implies $s_0 \in F_Q$ i.e $F_Q \neq \emptyset$. □

Now we will prove weak convergence theorem.

Theorem 2.2. *Let $\mathcal{V}$ represents a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $Q : E \to E$ is (α, β, γ)–nonxpansive mapping satisfying $F_Q \neq \emptyset$ and $\{t_n\}$ a sequence of M iterates (1.1). Then $\{t_n\}$ converges weakly to a point of F_Q provided that $\mathcal{V}$ is proclaimning Opial's condition.*

Proof. As given $\mathcal{V}$ is a UCB space and according to the Theorem 2.1, $\{t_n\}$ is bounded. It follows that there is a point, namely, $t_0 \in E$ such that a subsequence, namely, $\{t_{n_m}\}$ of $\{t_n\}$ weakly converges to it. From Theorem 2.1, it is clear that $\lim_{m\to\infty} \|t_{n_m} - Qt_{n_m}\| = 0$. Using Lemma 1.2, $t_0 \in F_Q$. We want to prove that the point t_0 is an only weak limit of $\{t_n\}$, contrary we suppose that t_0 cannot become a weak limit for $\{t_n\}$ i.e there exists another subsequence, namely, $\{t_{n_s}\}$ of $\{t_n\}$ with a weak limit, namely, $t_0' \neq t_0$. From Theorem 2.1, it is annotated that $\lim_{s\to\infty} \|t_{n_s} - Qt_{n_s}\| = 0$. Applying Lemma 1.2 $t_0' \in F_Q$. Using Opial's condition of $\mathcal{V}$ along with the Theorem 2.1, we get

$$\begin{aligned}
\lim_{n\to\infty} \|t_n - t_0\| &= \lim_{t\to\infty} \|t_{n_m} - t_0\| < \lim_{m\to\infty} \|t_{n_m} - t_0'\| \\
&= \lim_{r\to\infty} \|t_n - t_0'\| = \lim_{s\to\infty} \|t_{n_s} - t_0'\| \\
&< \lim_{s\to\infty} \|t_{n_s} - t_0\| = \lim_{n\to\infty} \|t_n - t_0\|.
\end{aligned}$$

Thus, we get $\lim_{n\to\infty} \|t_n - t_0\| < \lim_{n\to\infty} \|t_n - t_0\|$, which is a contradiction. Hence we must accept that t_0 is a weak limit of $\{t_n\}$ and so we have reached to the required target. □

Now proving strong convergence of M iterative scheme.

Theorem 2.3. *Suppose E is convex and compact subset of a UCB space $\mathcal{V}$ and $Q : E \to E$ is (α, β, γ)–nonxpansive mapping satisfying $F_Q \neq \emptyset$ and $\{t_n\}$ a sequence of M iterates (1.1). Then sequence $\{t_n\}$ converges strongly to some fixed point of F_Q.*

Proof. Since E is convex and compact, therefore sequence $\{t_n\} \subseteq E$ has a convergent subsequence. We donate this sequence by $\{t_{n_m}\}$ with a strong limit $s_0 \in E$ i.e., $\lim_{n_m\to\infty} \|t_{n_m} - s_0\| = 0$. Suppose $t = t_{n_m}$ and $y = s_0$, then applying Lemma 1.3, we have

$$\|t_{n_m} - Qs_0\| \leq \frac{(1+\beta)}{(1-\gamma)}\|t_{n_m} - Qt_{n_m}\| + \frac{\alpha}{(1-\gamma)}\|t_{n_m} - s_0\| \tag{2.10}$$

By Theorem 2.1, $\lim_{n_m\to\infty} \|t_{n_m} - Qt_{n_m}\| = 0$ and also $\lim_{n_m\to\infty} \|t_{n_m} - s_0\| = 0$. Accordingly (2.10) provides $\lim_{n_m\to\infty} t_{n_m} = Qs_0$. Since the limit of t_{n_m} is unique, we have $Qs_0 = s_0$, that is, $s_0 \in F_Q$. By Lemma 2.1 $\lim_{n\to\infty} \|t_n - s_0\|$ exist. Consequently we have proved that $s_0 \in F_Q$ and $t_n \to s_0$. □

Now removing compactness condition on E and proving strong convergence theorem as follows.

Theorem 2.4. *Suppose that E is closed and convex subset of UCB space $\mathcal{V}$. If $Q : E \to E$ is (α, β, γ)–nonxpansive mapping satisfying $F_Q \neq \emptyset$ and $\{t_n\}$ a sequence of M iterates (1.1). Then $\{t_n\}$ converges strongly to a point F_Q whenever* $\liminf_{r\to\infty} d(t_n, F_Q) = 0$

Proof. For any $s_0 \in E$, from Lemma 2.1 $\lim_{n\to\infty} \|t_n - s_0\|$ exist. It follows that $\liminf_{n\to\infty} d(t_n, F_Q)$ also exist. Accordingly $\liminf_{n\to\infty} d(t_n, F_Q) = 0$. Hence two subsequences of t_n namely $\{t_{n_m}\}$ and

$\{s_m\}$ exist in F_Q with property $\|t_{n_m} - s_m\| \le \frac{1}{2^m}$. We need to prove that $\{s_m\}$ is Cauchy in F_Q. To do this, using Lemma 2.1 to write that $\{t_n\}$ is nonincreasing. Thus, we have

$$\|s_{m+1} - s_m\| \quad \le \quad \|s_{m+1} - t_{n_{m+1}}\| + \|t_{n_{m+1}} - s_m\| \le \frac{1}{2^{m+1}} + \frac{1}{2^m}.$$

It follows that $\lim_{m\to\infty} \|s_{m+1} - s_m\| = 0$. Hence it is proved that $\{s_m\}$ is Cauchy in F_Q. According to the Lemma 1.2 that F_Q is closed, hence $\{s_m\}$ converges to some $q_0 \in F_Q$. By Lemma 2.1, $\lim_{n\to\infty} \|t_n - q_0\|$ exists and hence q_0 is the strong limit of $\{t_n\}$. □

Theorem 2.5. *[23] Consider E as convex and closed set in any given UCBS $\mathcal{V}$. Set $Q : E \to E$ is an (α, β, γ)-nonexpansive nonlinear operator satisfying the condition $F_Q \neq \emptyset$. If the sequence $\{t_n\}$ is essentially obtained from M fixed point scheme (1.1). Then $\{t_n\}$ converges is the strong sense to an element of F_Q when Q admits a condition (I).*

Proof. We establish this result by applying Theorem 2.4. For this, from the Theorem 2.1, we have $\liminf_{n\to\infty} \|Qt_n - t_n\| = 0$. By applying condition (I) of Q, we have $\liminf_{n\to\infty} d_s(t_n, F_Q) = 0$. It follows from Theorem 2.4 that $\{t_n\}$ has a strong limit in F_Q. This completes the proof. □

3. Application

Recently, some authors have solved FDEs using fixed point techniques (see, e.g., [13,15]) within the framework of nonexpansive nonlinear mapping. But we know that all nonexpansive mappings continuous functions, therefore, our alternative in this paper, we will solve a FDE in the class of (α, β, γ)–nonexpansive mappings that are not always continuous on their whole domain. Also, we suggest the M iterative scheme (1.1) that is more effective than the many iterative schemes for approximating the solution.

The following FDE is well-known and appears in many areas of applied sciences.

$$\left.\begin{array}{l} D^{\zeta} w(x) + L(x, w(x)) = 0, \\ w(0) = w(1) = 0, \end{array}\right\} \tag{3.1}$$

where, $x \in [0,1]$, $(1 < \zeta < 2)$ and ζ is order of caputo fractional derivative D^{ζ} and $L : [0,1] \times \mathbb{R} \to \mathbb{R}$.

Now we consider $\mathcal{V} = C[0,1]$, where $C[0,1]$ is the Banach space of continuous maps on $[0,1]$ to $\mathbb{R}$ equiped with the maximum norm. The corresponding Green's function with (3.1) is defined by

$$G(u,v) = \begin{cases} \frac{1}{\Gamma(\zeta)}(u(1-v)^{(\zeta-1)} - (u-v)^{(\zeta-1)} & \text{when } \ 0 \le v \le u \le 1 \\ \frac{u(1-v)^{(\zeta-1)}}{\Gamma(\zeta)} & \text{when } \ 0 \le u \le v \le 1. \end{cases}$$

The main result is provided in the following way.

Theorem 3.1. *If $\mathcal{V} = C[0,1]$, then set an operator $Q : E \to E$ by the formula*

$$Q(w(x)) = \int_0^1 G(x,y)L(y, w(y))dy, \textit{ for each } \ w(x) \in \mathcal{V}.$$

If

$$|L(y,w(y)) - L(y,z(y))| \leq \alpha\|w(y) - z(y)\| + \beta\|w(y) - Qw(y)\| + \gamma\|w(y) - Qz(y)\|,$$

then, the M iterates (1.1) assoicated with the Q (as defined above) essentially converges to some solution of (3.1), namely, $w_0 \in \mathcal{V}$.

Proof. $w \in \mathcal{V}$ will solve (3.1) $\Leftrightarrow$ it solves

$$w(x) = \int_0^1 G(x,y)L(y,w(y))dy.$$

Let $w, z \in \mathcal{V}$ and $0 \leq x \leq 1$, it follows that

$$\begin{aligned}
\|Q(w(x)) - Q(z(x))\| &\leq \left|\int_0^1 G(x,y)L(y,w(y)))dy - \int_0^1 G(x,y)L(y,z(y))dy\right| \\
&= \left|\int_0^1 G(x,y)[L(y,w(y)) - L(y,z(y))]dy\right| \\
&\leq \int_0^1 G(x,y)\left|L(y,w(y)) - L(y,z(y))\right|dy \\
&\leq \int_0^1 G(x,y)(\alpha\|w(y) - z(y)\| + \beta\|w(y) - Qw(y)\| + \gamma\|w(y) - Qz(y)\|)dy \\
&\leq (\alpha\|w(y) - z(y)\| + \beta\|w(y) - Qw(y)\| + \gamma\|w(y) - Qz(y)\|) \\
&\quad \left(\int_0^1 G(x,y)dy\right) \\
&\leq \alpha\|w(y) - z(y)\| + \beta\|w(y) - Qw(y)\| + \gamma\|w(y) - Qz(y)\|
\end{aligned}$$

Finally, we have

$$\|Q(w(x)) - Q(z(x))\| \leq \alpha\|w(y) - z(y)\| + \beta\|w(y) - Qw(y)\| + \gamma\|w(y) - Qz(y)\|.$$

As Q is (α, β, γ)–nonxpansive mapping and according to our main results, M iterates sequence converges to a fixed point of Q and hence to the solution of the given equation. □

4. Numerical Example

M iteration scheme indubitably exhibit faster convergence rate as compare to other iterative scheme using in connection with (α, β, γ)–nonxpansive mapping. Observation are given below with the help of numerical example.

Example 4.1. *Let $E = [0,3] \subset \mathcal{V}$ and Norm on E be defined as $\|.\| = |.|$, $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{3}$. Defined function $Q : E \to E$ as*

$$Qt = \begin{cases} \frac{t+3}{3} & \text{if } \ t \in [0,2] \\ 1 & \text{if } \ t \in (2,3]. \end{cases}$$

is $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$–nonexpansive but not non-expnasive mapping.

Proof. Observe that Q is not continuous on 2, it follows that the mapping Q is not nonexpansive. Now we prove that the given mapping is $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$–nonexpansive and thus we proceed as follows.

Case(I): If $t, y \in [0,2]$, then

$$\begin{aligned} \alpha\|t-y\| + \beta\|t-Qt\| + \gamma\|t-Qy\| &= \frac{2}{3}\|t-y\| + \frac{1}{3}\|t - \frac{t+3}{3}\| + \frac{1}{4}\|t - \frac{y+3}{3}\| \\ &= \frac{2}{3}|t-y| + \frac{1}{3}|t - \frac{t+3}{3}| + \frac{1}{4}|t - \frac{y+3}{3}| \\ &= \frac{2}{3}|t-y| + \frac{1}{9}|2t+3| + \frac{1}{12}|3t-y-3| \\ &\geq \|Qt - Qy\| \end{aligned}$$

Case(II): If $t, y \in (2,3]$, then

$$\begin{aligned} \alpha\|t-y\| + \beta\|t-Qt\| + \gamma\|t-Qy\| &= \frac{2}{3}\|t-y\| + \frac{1}{3}\|t-1\| + \frac{1}{4}\|t-1\| \\ &= \frac{2}{3}|t-y| + \frac{1}{3}|t-1| + \frac{1}{4}|t-1| \\ &\geq |Qt - Qy| \end{aligned}$$

Case(III): If $t \in [0,2]$ and $y \in (2,3]$, then

$$\begin{aligned} \alpha\|t-y\| + \beta\|t-Qt\| + \gamma\|t-Qy\| &= \frac{2}{3}\|t-y\| + \frac{1}{3}\|t - \frac{t+3}{3}\| + \frac{1}{4}\|t-1\| \\ &= \frac{2}{3}|t-y| + \frac{1}{3}|t - \frac{t+3}{3}| + \frac{1}{4}|t-1| \\ &= \frac{2}{3}|t-y| + \frac{1}{9}|2t-3| + \frac{1}{4}|t-1| \\ &\geq |Qt - Qy| \end{aligned}$$

Case(IV): If $y \in [0,2]$ and $t \in (2,3]$ then

$$\begin{aligned} \alpha\|t-y\| + \beta\|t-Qt\| + \gamma\|t-Qy\| &= \frac{2}{3}\|t-y\| + \frac{1}{3}\|t-1\| + \frac{1}{4}\|t - \frac{y+3}{3}\| \\ &= \frac{2}{3}|t-y| + \frac{1}{3}|t-1| + \frac{1}{4}|t - \frac{y+3}{3}| \\ &= \frac{2}{3}|t-y| + \frac{1}{3}|t-1| + \frac{1}{12}|3t-y-3| \\ &\geq |Qt - Qy| \end{aligned}$$

□

Now we connect Mann [17], Ishikawa [12], Noor [18], Agarwal [6], Abbas [4] and M [28] with this example. The observations are listed in the Table 1 and Figure 1.

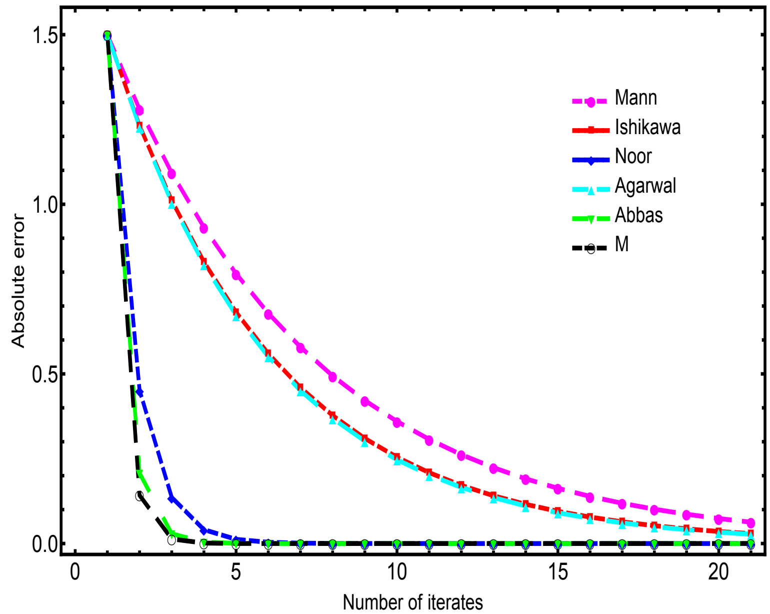

FIGURE 1. Analysis of behaviors using graphs of various fixed point iterations.

TABLE 1. Numerical results produced by M, Abbas, Agarwal, Noor, Ishikawa and Mann iterates.

n	M	Abbas	Agarwal	Noor	Ishikawa	Mann
1	0.000000000	0.000000000	0.000000000	0.000000000	0.00000000	0.000000000
2	1.357777778	1.292633333	1.048400000	0.272433333	0.268400000	0.220000000
3	1.486515226	1.471332710	1.364038293	0.495386719	0.488774293	0.407733333
4	1.498721444	1.496036906	1.459066462	0.677846815	0.669716279	0.567932444
5	1.499878774	1.499452124	1.487676276	0.827168104	0.818281713	0.704635685
6	1.499988506	1.499924259	1.496289738	0.949369328	0.940263838	0.821289118
7	1.499998910	1.499989529	1.498882964	1.049376094	1.040419296	0.920833381
8	1.499999897	1.499998552	1.499663698	1.131219410	1.122653603	1.005777819
9	1.499999990	1.499999800	1.499898751	1.198198160	1.190173452	1.078263739
10	1.499999990	1.499999972	1.499969517	1.253012081	1.245611749	1.140118300
11	1.500000000	1.499999996	1.499990823	1.297870576	1.291130287	1.192901026
12	1.500000000	1.499999999	1.499997237	1.334581771	1.328504041	1.237942209
13		1.500000000	1.499999168	1.364625397	1.359190384	1.276377352
14		1.500000000	1.499999750	1.389212433	1.384385918	1.309175340
15			1.499999925	1.409333917	1.405073131	1.337162957
16			1.499999977	1.425800893	1.422058712	1.361045723
17			1.499999993	1.43927710	1.436005007	1.381425684
18			1.499999998	1.450305728	1.447455844	1.398816584
19			1.499999999	1.459331312	1.456857745	1.413656818
20			1.500000000	1.466717649	1.464577333	1.426320485
21			1.500000000	1.472762464	1.470915629	1.437126814

5. Conclusions

The article produced the following new results.

(i) The three-step M-iteration procedure is connected with (α, β, γ)-nonexpansive nonlinear mappings and several convergence (weak and strong) results are obtained.

(ii) We provided a novel example which has the property of (α, β, γ)-nonexpansive mappings and proved that unlike nonexpansive mappings, a mapping in the class of (α, β, γ)-nonexpansive mappings may not thoughout continuous on its whole domain.

(iii) We connect several iterative schemes with this example including our three-step M-iterative scheme, and proved numerical that the effectiveness of M-iterative scheme is essentially high accurate than the other iterative schemes.

(iv) An application to solve a FDE is provided based on our main outcome.

(v) Thus, our findings consolidate the primary outcome of Ullah and Arshad [28] within the context of Suzuki mappings, extending it to encompass (α, β, γ)-nonexpansive mappings. Similarly, our results represent enhancements and refinements of the findings by Agarwal [6], Abbas [4], and Thakur [26], transitioning from nonexpansive and Suzuki mappings to the realm of (α, β, γ)-nonexpansive mappings, while also addressing the issue of accelerated convergence.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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