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On Stationary Points of Multi-Valued Suzuki Mappings via New Approach in Metric Spaces

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Abstract. We investigate stationary points of multivalued Suzuki maps within the framework of 2-uniformly convex hyperbolic spaces. Initially, we present key strong and Δ-convergence results, followed by an example that demonstrates the theoretical findings. Additionally, our results hold in uniformly convex Banach spaces, CAT(0) spaces, and certain CAT(κ) spaces. Furthermore, our findings encompass cases where the map is assumed to be nonexpansive.

1. Introduction

Assume that K is any nonempty subset of a general metric space $X = (X, p)$. Fixed $w \in K$ and set

$$p(w, K) = \inf_{w' \in K} p(w, w'),$$

and

$$R(w, K) = \sup_{w' \in K} p(w, w').$$

Also, we shall write $C(K)$ throughout in the current research to represent the set family of nonempty compact subsets of K. The real function $H : C(K) \times C(K) \to \mathbb{R}$ is called Hausdorff metric [1] if and only if

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$$H(U,W) = \max\left\{\sup_{u\in U} p(u,W), \sup_{v\in W} p(v,U)\right\}, \text{ for each } U,W \in C(K),$$

If $T : K \to C(K)$ is given multi-valued map. Then T is called Suzuki map (see, e.g., [2]) if any two given elements $w, w' \in K$, follows:

$$\frac{1}{2}p(w,Tw) \le p(w,w') \Rightarrow H(Tw,Tw') \le p(w,w').$$

Since the class of Suzki maps requires the condition of nonexpansiveness i.e., $H(Tw,Tw') \le \rho(w,w')$ only few elements and not for all, so it is obvious that the class of Suzuki maps is properly wider than the class of nonexpansive maps (for details on this topic, see [3,4] and others). Interestingly, the example provided below will show that the are many maps T, which are Suzuki but not nonexpansive.

Example 1.1. *Assume the closed convex subset $K = [2,5]$ of a metric space $\mathbb{R}$ and set a multivalued map T from $K = [2,5]$ to the $C(K = [2,5])$ by the formula given below:*

$$Tw = \begin{cases} \{2\} & when \quad w \neq 5 \\ [2.9,3] & when \quad w = 5. \end{cases}$$

If one select the value of $w \in (4,5)$ and $w' = 5$, then it is obvious that we have nothing to show. Because we have in this case the following:

$$\frac{1}{2}p(w,Tw) = \frac{w-2}{2} > 1 > p(w,w') \text{ and } \frac{1}{2}p(w',Tw') = 1 > p(w,w').$$

However, if we set $w < w'$ such that $(w,w') \in ([2,5]\times[2,5]) - ((4,5)\times\{5\})$ then T fulfils the condition of nonexpansiveness. Consequently, T is multi-valued Suzuki map.

Now, by selecting $w = 4.5$ and $w' = 5$, one has $H(Tw,Tw') = 1 > 0.5 = p(w,w')$. This proves that T is not nonexpansive.

Now we recall that a given element g in K is known as fixed point of T if it is in the set Tg and is called an endpoint of T if Tg contains g such that $Tg = \{g\}$. We shall use $F(T)$ and $E(T)$ to represent the fixed points and end points set of T respectively. The existence of endpoints for some kinds of multi-valued mappings achieved great attension of many researchers and many paper appeared in the literature (see, e.g., [5–12] and others). Now, for a given multi-valued mapping T, one has the following:

- $E(T) \subseteq F(T)$.
- $g \in F(T)$ if and only if $p(g,Tg) = 0$.
- $g \in E(T)$ if and only if $R(g,Tg) = 0$.

Fixed point findings of single-valued mappings and the extension of these results to the case of multivalued maps gained the attention of many researchers and many paper are now available in the current literature of fixed point theory (see, e.g., [13–17,19,21,22] and others). One of the widely studied iterative process is the three-steps Noor iteration [31] for finding fixed points of single-valued maps in linear and nonlinear spaces. The Noor iteration [31] is important because it includes Mann [29] and Ishikawa [30] iterative schemes as a special case. If T is selfmap of K, then Noor iteration process is given by:

$$\begin{cases} x_1 \in K, \\ z_t = (1-\gamma_t)x_t \oplus \gamma_t T x_t, \\ y_t = (1-\beta_t)x_t \oplus \beta_t T z_t, \\ x_{t+1} = (1-\alpha_t)x_t \oplus \alpha_t T x_t, t \geq 1, \end{cases} \tag{1.1}$$

where $\alpha_t, \beta_t, \gamma_t \in (0,1)$. In this paper, we shall study the modified version of this scheme in the case of multi-valued maps in some metric space setting.

We suggest and present some well know definitions are results, which will be either used in the main results or to understand the given concept used herein.

Definition 1.1. *[23]* A given metric space (X, p) is called Kolenbach–hyperbolic space or simply hyperbolic space when there is a map $C : X \times X \times [0,1] \to X$ such that for all $w, w', u, q \in X$ and $i, s \in [0,1]$, we have

$(C1)$ $p(u, C(w, w', i)) \leq (1-i)p(u, w) + ip(u, w')$;
$(C2)$ $p(C(w, w', i), C(w, w', j) = |i-j|\rho(w, w')$;
$(C3)$ $C(w, w', i) = C(w', w, 1-i)$;
$(C4)$ $p(C(w, u, i), C(w', q, i)) \leq (1-i)p(w, w') + ip(u, q)$.

Notice that, for given two elements $w, w' \in X$ and given element $i \in [0,1]$, we shall use $(1-i)w \oplus iw'$ for $C(w, w', i)$. Also, the property $(C1)$ gives:

$$\rho(w, (1-i)w \oplus iw') = ip(w, w') \text{ and } p(w', (1-i)w \oplus iw') = (1-i)p(w, w').$$

Keep in mind that $\emptyset \neq K \subseteq X$ is said to be convex in X if for every two given elements $w, w' \in K$, follow $[w, w'] = \{(1-\sigma)w \oplus \sigma w' : \sigma \in [0,1]\} \subseteq K$.

Definition 1.2. *Suppose a triplet (X, p, C) be a given hyperbolic space. We regard (X, ρ, C) as a uniformly convex hyperbolic space whenever a real $r > 0$ and $\epsilon \in (0,2]$, are give, then one can find a real $\sigma \in (0,1]$ such that any given three elements $w, w', q \in X$ satisfying $\rho(w, q) \leq r$, $p(w', q) \leq r$ and $\rho(w, w') \geq 2\epsilon$, follow that:*

$$p\left(\frac{1}{2}w \oplus \frac{1}{2}w', q\right) \leq (1-\sigma)r.$$

Moreover, if a function $f : (0, \infty) \times (0,2] \to (0,1]$ is given such that $\sigma = f(r, \epsilon)$ for all $r \in (0, \infty)$ and $\epsilon \in (0,2]$ then f is called a modulus of uniform convexity. Also f is called monotone if and only if nonincreasing in the variable r for every chosen fixed ϵ.

Definition 1.3. *Suppose a pair (X, p) be a given hyperbolic space. For a real $r \in (0, \infty)$ and any $\epsilon \in (0,2]$, define:*

$$\varphi(r, \epsilon) = \inf\left\{\frac{1}{2}p^2(w, q) + \frac{1}{2}p^2(w', q) - p^2(\frac{1}{2}w \oplus \frac{1}{2}w', q)\right\}$$

and keeping in mind that infimum is taken over every $w, w', q \in X$ satisfying $\rho(w,q) \leq r$, $\rho(w',q) \leq r$, and $p(w,w') \geq r\epsilon$. The space (X,p) is called 2-uniformly convex provided that

$$c_M = \inf\left\{\frac{\varphi(r,\epsilon)}{r^2\epsilon^2} : r \in (0,\infty), \epsilon \in (0,2]\right\} > 0.$$

The following remark suggest that 2-uniformly convex hyperbolic space is more general than many other linear and nonlinear spaces.

Remark 1.1. *Any given uniformly convex Banach space, CAT(0) space and CAT(κ) spaces having $\kappa > 0$ and $diam(X) \leq \left(\frac{\frac{\pi}{2}-\epsilon}{\kappa^{\frac{1}{2}}}\right)$, $\epsilon \in (0, \frac{\pi}{2})$) can be considered as an 2-uniformly convex hyperbolic space (see [20,24,25]).*

Definition 1.4. *Suppose $\{x_t\}$ be a given bounded sequence in a complete 2-uniformly convex hyperbolic space and $\emptyset \neq K \subseteq X$. We often denote by $r(K,\{x_t\})$ the asymptotic radius of $\{x_t\}$ wrt K and defined by $r(K,\{x_t\}) = \inf\{\limsup_{t\to\infty} p(x_t,w) : w \in K\}$. Further, We often denote by $A(K,x_t\})$ the asymptotic center of $\{x_t\}$ wrt Y and is defined by $A(K,x_t\}) = \{w \in K : \limsup_{t\to\infty} p(x_t,w) = r(K,x_t)\}$.*

The Δ-convergence enjoys the roll of the analog of the weak convergence as weak convergence in linear setting.

Definition 1.5. *Suppose K be closed and convex in a complete 2-uniformly convex hyperbolic space X and fix $w \in K$. Assume that $\{x_t\}$ is a bounded sequence in M. Then $\{x_t\}$ is called Δ-convergent to w if and only if the asymptotic center $A(K,\{z_t\}) = \{w\}$ for all subsequence $\{z_t\}$ of $\{x_t\}$. When w is Δ limit of $\{x_t\}$ the we shall write $\Delta\text{-}\lim_{t\to\infty} x_t = w$.*

Suzuki maps enjoy the following useful properties which we have combined in a proposition form. For details we refer the reader to [26–28].

Proposition 1.1. *Suppose K be a nonempty subset of a complete 2-uniformly convex hyperbolic space and assume that $F : K \to C(K)$.*

- (a) *When T is Suzuki map having $F(T) \neq \emptyset$, then $H(Tw,Tq) \leq p(w,q)$ for every element $w \in K$ and $q \in F(T)$.*
- (b) *When F is Suzuki map, then T fulfils the following property:*

$$p(w,Tw') \leq 3p(w,Tw) + p(w,w') \text{ for any two elements } w,w' \in K.$$

We also need the following characterization of 2-uniformly convex hyperbolic spaces.

Lemma 1.1. *[20] Assume that a pair (X,p) is a given 2-uniformly convex hyperbolic space. Then*

$$p^2((1-\xi)w \oplus \xi w', e) \leq (1-\xi)p^2(w,e) + \xi p^2(w',e) - 4c_M\xi(1-\xi)p^2(w,w'),$$

for every scaler $\xi \in [0,1]$ and every three elements $w,w',e \in X$.

Lemma 1.2. *[20] Assume that a pair (X,ρ) is a given 2-uniformly convex complete hyperbolic space and $\emptyset \neq K \subseteq X$ be a convex and closed. If $T : K \to C(K)$ is a Suzuki map then any bounded sequence $\{x_t\}$ in K with $\lim_{t\to\infty} R(x_t,Tx_t) = 0$ and assume that the real sequence $\{\rho(w_t,e)\}$ is convergent to some $e \in E(T)$,* *then* $\omega_\omega(x_t) \subseteq E(T)$ *and* $\omega_\omega(x_t)$ *is singleton. Here,* $\omega_\omega(x_t) = \bigcup A(K, \{z_t\})$ *and the union is taken over all subsequences* $\{z_t\}$ *of* $\{x_t\}$.

2. Convergence theorems in 2-uniformly convex hyperbolic spaces

This section suggest some important Δ and strong convergence results of endpoint under the three-steps Noor type iteration (2.1) for the larger class of Suzuki maps. For the sake of simplicity, we denote simply by X a complete 2-uniformly convex hyperbolic space with monotone modulus of uniform convexity. For a given multi-valued map $T : K \to C(K)$, we define the three–steps Noor iteration [31] iteration in the generalized setting as follows, which is the modification of (1.1):

$$\begin{cases} x_1 \in K, \\ z_t = (1-\gamma_t)x_t \oplus \gamma_t u_t, \\ y_t = (1-\beta_t)x_t \oplus \beta_t v_t, \\ x_{t+1} = (1-\alpha_t)x_t \oplus \alpha_t w_t, t \geq 1, \end{cases} \tag{2.1}$$

where $u_t \in Tx_t$ such that $p(x_t, u_t) = R(x_t, Tx_t)$, $v_t \in Tz_t$ such that $p(z_t, v_t) = R(z_t, Tz_t)$and $w_t \in Ty_t$ such that $p(y_t, w_t) = R(y_t, Ty_t)$. One of the main purpose of this research is to establish some important Δ and strong convergence theorems of this scheme for a wider class of multi-valued maps so-called Suzuki multivalued maps. We also suggest an appropriate example to understand the usefulness of the established outcome. Our results can be regard as an improvement and extension of many single-valued and multi-valued maps.

The following lemma is crucial .

Lemma 2.1. *Suppose* $\emptyset \neq K$ *be closed and convex in* X *and the map* $T : K \to C(K)$ *be a Suzuki map such that* $E(T) \neq \emptyset$. *If* $\{x_t\}$ *is an iterative sequence produced from (2.1). Then limit* $\lim_{t\to\infty} \rho(x_t, g)$ *exists for every choice of* $g \in E(T)$.

Proof. Assume that $g \in E(T)$. By using Proposition 1.1(i), one has

$$\begin{aligned} \rho(z_t, g) &\leq (1-\gamma_t)p(x_t, g) + \gamma_n \rho(u_t, g) \\ &= (1-\gamma_t)p(x_t, g) + \gamma_t p(u_t, Tg) \\ &\leq (1-\gamma_t)p(x_t, g) + \gamma_t H(Tx_t, Tg) \\ &\leq (1-\gamma_t)p(x_t, g) + \gamma_t p(x_t, g) \\ &= p(x_t, g). \end{aligned}$$

and

$$\begin{aligned} p(y_t, g) &\leq (1-\beta_t)p(x_t, g) + \beta_t p(v_t, g) \\ &= (1-\beta_t)p(x_t, g) + \beta_t p(v_t, Tg) \\ &\leq (1-\gamma_t)p(x_t, g) + \beta_t H(Tz_t, Tg) \\ &\leq (1-\gamma_t)p(x_t, g) + \beta_t p(z_t, g) \end{aligned}$$

$$= \quad p(x_t, g).$$

Consequently

$$\begin{aligned} p(x_{t+1}, g) &\leq (1-\alpha_t)p(x_t, g) + \alpha_t p(w_t, g) \\ &= (1-\alpha_t)p(x_t, g) + \alpha_t p(w_t, Tg) \\ &\leq (1-\alpha_t)p(x_t, g) + \alpha_t H(Ty_t, Tg) \\ &\leq (1-\alpha_t)p(x_t, g) + \alpha_t p(y_t, g) \\ &= p(x_t, g). \end{aligned}$$

Hence for every $t \geq 1$, we have obtained $p(x_{t+1}, g) \leq p(x_t, g)$. Consequently, the real sequence $\left\{p(x_t, g)\right\}$ is a non-increasing and so the limit $\lim_{t\to\infty} \rho(x_t, g)$ must be exist for every choice of $g \in E(T)$. □

First we establish our Δ-convergence theorem.

Theorem 2.1. *Suppose $\emptyset \neq K$ be closed and convex in X and the map $T : K \to C(K)$ be a Suzuki map such that $E(T) \neq \emptyset$. If $\{x_t\}$ is an iterative sequence produced from (2.1) with $\alpha_t, \beta_t, \gamma_t \in [a, b] \subset (0, 1)$. Then $\{x_t\}$ is Δ-convergent to a point of $E(T)$.*

Proof. Fix an element $g \in E(T)$. By using Lemma 1.1, one has

$$\begin{aligned} p^2(z_t, g) &\leq (1-\gamma_t)p^2(x_t, g) + \gamma_t p^2(u_t, g) - 4c_M\gamma_t(1-\beta_t)p^2(x_t, u_t) \\ &\leq (1-\gamma_t)p^2(x_t, g) + \gamma_t H^2(Tx_t, Tg) - 4c_M\gamma_t(1-\gamma_t)p^2(x_t, u_t) \\ &\leq (1-\gamma_t)p^2(x_t, g) + \gamma_t p^2(x_t, g) - 4c_M\gamma_t(1-\gamma_t)p^2(x_t, u_t) \\ &= p^2(x_t, g) - 4c_M\gamma_t(1-\gamma_t)p^2(x_t, u_t), \end{aligned}$$

and using the above, we have

$$\begin{aligned} p^2(y_t, g) &\leq (1-\beta_t)p^2(x_t, g) + \beta_t p^2(v_t, g) - 4c_M\beta_t(1-\beta_t)p^2(x_t, v_t) \\ &\leq (1-\beta_t)\rho^2(x_t, g) + \beta_t H^2(Tz_t, Tg) - 4c_M\beta_n(1-\beta_t)p^2(x_t, v_t) \\ &\leq (1-\beta_t)p^2(x_t, g) + \beta_t p^2(z_t, g) - 4c_M\beta_n(1-\beta_t)p^2(x_t, v_t) \\ &\leq (1-\beta_t)p^2(x_t, g) + \beta_t p^2(z_t, g) \\ &\leq p^2(x_t, g) - 4c_M\beta_t\gamma_t(1-\gamma_t)p^2(x_n, v_n). \end{aligned}$$

Consequently

$$\begin{aligned} p^2(x_{t+1}, g) &\leq (1-\alpha_t)p^2(x_t, g) + \alpha_t p^2(w_t, g) - 4c_M\alpha_t(1-\alpha_t)p^2(x_t, w_t) \\ &\leq (1-\alpha_t)p^2(x_t, g) + \alpha_t H^2(Ty_t, Tg) - 4c_M\alpha_t(1-\alpha_t)p^2(x_t, w_t) \\ &\leq (1-\alpha_t)p^2(x_t, g) + \alpha_t p^2(y_t, g) - 4c_M\alpha_t(1-alpha_t)p^2(x_t, w_t) \\ &\leq (1-\alpha_t)p^2(x_t, g) + \alpha_t p^2(y_t, g) \\ &\leq p^2(x_t, g) - 4c_M alpha_t\beta_t\gamma_t(1-\gamma_t)p^2(x_t, u_t). \end{aligned}$$

Since $c_M > 0$, one has

$$\sum_{t=1}^{\infty} a^3(1-b)p^2(x_t,u_t) \leq \sum_{t=1}^{\infty} \alpha_t\beta_t\gamma_t(1-\gamma_t)p^2(x_t,u_t) < \infty. \tag{2.2}$$

From (2.2), we can write $\lim_{t\to\infty} p^2(x_t,u_t) = 0$. Thus

$$\lim_{t\to\infty} R(x_t,Tx_t) = \lim_{t\to\infty} p(x_t,u_t) = 0. \tag{2.3}$$

By Lemma 2.1, the real sequence $\{p(x_t,e)\}$ is convergent to for all $e \in E(T)$. By Lemma 1.2, $\omega_\omega(x_t)$ is singleton and contained in the set $E(T)$. Consequently, $\{x_t\}$ Δ-converges to an element of $E(T)$. □

Now we recall the following definition from [18].

Definition 2.1. *Suppose $\emptyset \neq K$ be a subset of X and assume that $T : K \to C(K)$. Then T is said to be fulfils J if and only if there is a nondecreasing function $S : [0,\infty) \to [0,\infty)$ having the properties $S(0) = 0$, $S(r) > 0$ for $r > 0$ and $R(w,Tw) \geq S(\rho(w,E(T)))$ for every element w of K. Moreover, the map T is called semi-compact if and only if any $\{x_t\}$ in Y with $\lim_{t\to\infty} R(x_t,Tx_t) = 0$, follows that there is a subsequence $\{x_{t_k}\}$ of $\{x_t\}$ and some $z \in K$ such that $x_{t_k} \to z$. Finally, a given sequence $\{x_t\}$ in X is said to be Fejer-monotone wrt K if and only if $p(x_{t+1},z) \leq p(x_t,z)$, for every element z of K and $t \geq 1$.*

We now give the statement of the following proposition from [28].

Proposition 2.1. *Suppose $\emptyset \neq K$ be a closed subset in X and assume that $\{x_t\}$ is a Fejer monotone sequence wrt K. Then $\lim_{t\to\infty} p(x_t,K) = 0$ if and only if $\{x_t\}$ converges strongly to a point of K.*

Now we suggest a strong convergence result under the assumption of semi-compactness.

Theorem 2.2. *Suppose $\emptyset \neq K$ be closed and convex in X and the map $T : K \to C(K)$ be a Suzuki map such that $E(T) \neq \emptyset$. If $\{x_t\}$ is an iterative sequence produced from (2.1) with $\alpha_t,\beta_t,\gamma_t \in [a,b] \subset (0,1)$. Then $\{x_t\}$ is strongly convergent to a point of $E(T)$ whenever T is semi-compact.*

Proof. In view of (2.2),

$$\sum_{t=1}^{\infty} \alpha_t\beta_t\gamma_t(1-\gamma_t)p^2(x_t,u_t) < \infty.$$

It follows that $\lim_{t\to\infty} p^2(x_t,u_t) = 0$. Hence

$$\lim_{t\to\infty} R(x_t,Tx_t) = \lim_{t\to\infty} p(x_t,u_t) = 0. \tag{2.4}$$

Since T is semi-compact, so we assume a subsequence $\{x_{t_k}\}$ of $\{x_t\}$ and $z \in K$ such that $x_{t_k} \to z$. We shall prove that $z \in E(T)$. By Proposition 1.1(ii), we have

$$\begin{aligned} p(z,Tz) &\leq p(z,x_{t_k}) + p(x_{t_k},Tz) \\ &\leq p(x_{t_k},z) + 3p(x_{t_k},Tx_{t_k}) + p(x_{t_k},z) \\ &= 2p(x_{t_k},z) + 3p(x_{t_k},Tx_{t_k}) \longrightarrow 0. \end{aligned}$$

Hence $z \in Tz$. By Proposition 1.1(i),

$$H(Tx_{t_k}, Tz) \leq p(x_{t_k}, z) \longrightarrow 0. \tag{2.5}$$

Select $v \in Tz$ and choose $y_{t_k} \in Tx_{t_k}$ such that $p(v, y_{t_k}) = p(v, Tx_{t_k})$. From (2.4) and (2.5) one has

$$\begin{aligned} p(z,v) &\leq p(z, x_{t_k}) + p(x_{t_k}, y_{t_k}) + p(y_{t_k}, v) \\ &= p(z, x_{t_k}) + p(x_{t_k}, y_{t_k}) + p(v, Tx_{t_k}) \\ &\leq p(z, x_{t_k}) + R(x_{t_k}, Tx_{t_k}) + H(Tx_{t_k}, Tx_{t_k}) \longrightarrow 0. \end{aligned}$$

Thus for every choice of $v \in Tz$, follows $v = z$ that is, $\{z\} = Tz$. Therefore $z \in E(T)$. By Lemma 2.1, $\lim_{t\to\infty} \rho(x_t, z)$ exists. Hence z is the strong limit of $\{x_t\}$. □

Example 2.1. *Choose $K = [2,5]$ and T a map given in Example 1.1. Then T is Suzuki map such that $E(T) = \{2\}$. Since K is compact, it follows that T is Semicompactness of F. By Theorem 2.2, the iterative sequence produced by (2.1) is strongly convergent to 2. However, we cannot directly apply any result in [19, 21, 22] because, in this situation, T is not nonexpansive.*

We now provide a strong convergence under the assumption of condition Jb of the map T.

Theorem 2.3. *Suppose $\emptyset \neq K$ be closed and convex in X and the map $T : K \to C(K)$ be a Suzuki map such that $E(T) \neq \emptyset$. If $\{x_t\}$ is an iterative sequence produced from (2.1) with $\alpha_t, \beta_t, \gamma_t \in [a,b] \subset (0,1)$. Then $\{x_t\}$ is strongly convergent to a point of $E(T)$ whenever T fulfils the condition J.*

Proof. From (2.3), one can see that

$$\lim_{t\to\infty} R(x_t, Tx_t) = 0. \tag{2.6}$$

Applying condition J of T, one has

$$R(x_t, Tx_t) \geq S(p(x_t, E(T))).$$

Using (2.6), we obtain the following

$$\lim_{t\to\infty} S(p(x_t, E(T))) = 0.$$

Now the map S is nondecreasing with $S(0) = 0$ and $S(r) > 0$ for every choice of $r > 0$. Thus

$$\lim_{t\to\infty} p(x_t, E(T))) = 0.$$

Now Closeness of the set $E(T)$ follow from the quasi-nonexpansiveness of T. The proof of Lemma 2.1, provides that $\{x_t\}$ is Fejer-monotone wrt $E(T)$. By Proposition 2.1, the sequence $\{x_t\}$ converges strongly convergent to the some point of $E(T)$. □

Example 2.2. *Choose $K = [2,5]$ and F a map given in Example 1.1. Then T is Suzuki map such that $E(T) = \{2\}$. It is easy to show that T fulfils the condition J. By Theorem 2.3, the iterative sequence produced by (2.1) is strongly convergent to 2. However, we cannot directly apply any result in [19, 21, 22] because, in this situation, T is not nonexpansive.*

3. Conclusions

We have studied a modified three–steps Noor type iteration for finding endpoints of multivalued Suzuki mappings in the ground setting of 2-uniformly convex hyperbolic spaces. Under some apprperiate situations and assumptions, we have prove some important strong and Δ-convergence results of endpoints for the larger class of multi-valued maps so–called Suzuki maps. We have provide an an example to illustraite the theoratical results. In the view of the earlier discussion, our established results simaltensiously applicable in the uniformly convex Banach, CAT(0) and some CAT(κ) spaces. Moreover, our results inclueds the results when the mapping is assumed to be nonexpansive. The presented results improve the corresponding outcomes due to Ullah et al. [32] and others.

Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

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