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Polynomiograph Comparison and Stability of a New Iteration Process

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Abstract. In this paper, we introduce a geometric version of the F iteration process. We establish some strong and weak convergence results for our proposed iteration process in the setting of generalized contractive mappings. We also prove stability of our proposed iteration process. Additionally, we support our analysis with polynomiographs generated by our proposed iteration process, compared with those from established iteration processes in the literature, showcasing the superiority and innovation of our approach.

1. Introduction

Fixed point theory continues to hold significant importance in the present era across various disciplines of science. It provides a framework for studying the existence, uniqueness and properties of solutions to equations or systems of equations. Existential and computation of fixed point of a mapping are two different dimensions. In numerical analysis, fixed point iteration is premier and convenient root finding algorithm. Modern research approach is primary focusing on cost effective as well as speedy computation. In this regards, fixed point theory is proved to be a simple and pre-eminent area of research. Fixed point theory reformulate problem as $\mathfrak{F}(x) = 0$ and allowing productive approach towards its solution. Fixed point theory plays a vital role in formulation of problems occurring in system of linear equations, differential equation, optimization theory or integral equations.

A mapping $\mathfrak{F}$ on a nonempty subset B of a Banach space E is called a contraction if for all $p, \mathfrak{z} \in B$, the following relation holds

$$\|\mathfrak{F}p - \mathfrak{F}\mathfrak{z}\| \leq \varsigma\|p - \mathfrak{z}\|, \ \varsigma \in [0, 1). \tag{1.1}$$

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$k \in B$ is called fixed point of $\mathfrak{F}$ if $\mathfrak{F}k = k$. Contraction mapping has been generalized in different directions, one of which is that of Zamfirescu [1]. An operator $\mathfrak{F} : B \to B$ defined on $\phi \neq B \subset E$ is called Zamfirescu operator if for each $p, \mathfrak{z} \in B$, at least one of the following conditions satisfied:

$$\begin{aligned} &(i)\ ||\mathfrak{F}p - \mathfrak{F}\mathfrak{z}|| \leq \varsigma||p - \mathfrak{z}||, \\ &(ii)\ ||\mathfrak{F}p - \mathfrak{F}\mathfrak{z}|| \leq \xi\{||p - \mathfrak{F}\mathfrak{z}|| + ||\mathfrak{z} - \mathfrak{F}p||\}, \\ &(iii)\ ||\mathfrak{F}p - \mathfrak{F}\mathfrak{z}|| \leq \psi\{||p - \mathfrak{F}p|| + ||\mathfrak{z} - \mathfrak{F}\mathfrak{z}||\}, \end{aligned} \tag{1.2}$$

where $\varsigma \in [0, 1)$ and $\xi,\ \psi \in [0, \frac{1}{2})$. Berinde [2] gave the idea of a novel class of operators which is given as;

$$||\mathfrak{F}p - \mathfrak{F}\mathfrak{z}|| \leq 2\delta||p - \mathfrak{F}p|| + \delta||p - \mathfrak{z}||,\ \forall p, \mathfrak{z} \in B, \tag{1.3}$$

where $\delta = max\{\varsigma, \frac{\xi}{1-\xi}, \frac{\psi}{1-\psi}\}$, $\varsigma \in [0, 1)$ and $\xi, \psi \in [0, 1.5)$ and $B \subseteq E$ is an arbitrary Banach space. Berinde has demonstrated that the class (1.3) is bigger than the class of Zamfirescu operator (1.2). A more general contractive condition than (1.3) is given by Oslike [3]:

$$||\mathfrak{F}p - \mathfrak{F}y|| \leq L||p - \mathfrak{F}p|| + \delta||p - \mathfrak{z}||\ , \forall p, \mathfrak{z} \in B\, , L\ \geq 0,\ \delta \in [0, 1). \tag{1.4}$$

Imoru and Olantiwo [4] extends the results of [3] by using the following contractive condition

$$||\mathfrak{F}p - \mathfrak{F}y|| \leq \rho\{||p - \mathfrak{F}p||\} + \delta||p - \mathfrak{z}||,\ \forall p, \mathfrak{z} \in B, \tag{1.5}$$

where the function $\rho : [0, \infty) \to [0, \infty)$ is continuous and monotone with $\rho(0) = 0$ and $\delta \in [0, 1)$.

Banach Contraction Principle [5] plays an important role in establishing the fixed point of the mappings described in (1.5). In fixed point theory, number of two steps, three steps and four steps iteration processes have been introduced in the literature to approximate fixed point of a mapping (see [6–9]). Let α_n be the sequence in $(0, 1)$ for all $n \in \mathbb{N}$. Picard iteration process [10] is known as very basic iterative algorithm for computation of fixed point, which is defined as;

$$x_{n+1} = \mathfrak{F}x_n. \tag{1.6}$$

Mann [11] introduced Mann iteration which is defined as;

$$\begin{aligned} u_0 &= u \in B, \\ u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n\mathfrak{F}u_n. \end{aligned} \tag{1.7}$$

Kanwar et al. coined the idea of simple fixed point iteration method [12] based on approximation through a straight line. Using that idea, Sharma et al. introduced SH iteration given in [13] that surpassed the existing of three step iterative schemes, defined as:

$$\begin{aligned} x_0 &= x \in B, \\ z_n &= \frac{\mathfrak{w}x_n + \mathfrak{F}x_n}{\mathfrak{w} + 1},\ \mathfrak{w} \in \mathbb{R} \text{ and } \mathfrak{w} > 0, \\ y_n &= \mathfrak{F}z_n, \\ x_{n+1} &= \mathfrak{F}y_n. \end{aligned} \tag{1.8}$$

Ali et al. in 2020 introduced F iterative scheme [14] given in for generalized contraction as follows:

$$\begin{aligned} v_0 &= v \in B, \\ w_n &= \mathfrak{F}((1-\alpha_n)v_n + \alpha_n \mathfrak{F} v_n), \\ y_n &= \mathfrak{F} w_n, \\ v_{n+1} &= \mathfrak{F} y_n. \end{aligned} \tag{1.9}$$

In this research work, we extend the idea of Kanwar et al. [12] to propose a new variant of the F iteration process (1.9). We prove some convergence results of fixed point theory for this new variant. We compare our prove results with the existing modified iteration process (1.8). We elaborate our findings using polynomiography and generate some polynomiographs for this new variant to showcase its superiority.

2. Preliminaries

In this section we will discuss some important results necessary for our analysis.

Lemma 2.1. *[1] Let B be a nonempty subset of a Banach space E and $\mathfrak{F} : B \to B$ be a mapping satisfying (1.2), then $\mathfrak{F}$ has a unique fixed point in E.*

Definition 2.1. *[15] Let $\mathfrak{f}_i$ be a sequence generated by some iteration process $\mathfrak{f}_{i+1} = f(\mathfrak{F}, \mathfrak{f}_i)$ converging to a fixed point $\mathfrak{q}$ of mapping $\mathfrak{F}$. Let $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ be arbitrary sequence and define $\epsilon_i = \|\mathfrak{g}_{i+1} - f(\mathfrak{F}, \mathfrak{g}_i)\|$. Then we say that iteration process is $\mathfrak{F}$-stable if*

$$\lim_{i\to\infty} \epsilon_i = 0 \Leftrightarrow \lim_{i\to\infty} \mathfrak{g}_i = \mathfrak{q},$$

where $i \in N$.

Lemma 2.2. *[16] If $\sigma \in [0,1)$ is a real number and $\{\mathfrak{f}_i\}_{i=0}^{\infty}$ is a positive number sequence such that $\lim_{i\to\infty} \mathfrak{f}_i = 0$, then for any sequence of positive numbers $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ satisfying $\mathfrak{g}_{i+1} \le \sigma \mathfrak{g}_i + \mathfrak{f}_i$, $i = 0, 1, 2, 3 \ldots$, we have $\lim_{i\to\infty} \mathfrak{g}_i = 0$.*

Lemma 2.3. *[17] Let $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ and $\{\mathfrak{f}_i\}_{i=0}^{\infty}$ be two non-negative real sequences such that $\mathfrak{g}_{i+1} \le (1-\mu_i)\mathfrak{g}_i + \mathfrak{f}_i$, $\mu_i \in (0,1)$, $\forall\, i \in \mathbb{N}$, $\sum_{i=o}^{\infty} \mu_i = \infty$ and $\lim_{i\to\infty} \frac{\mathfrak{f}_i}{\mathfrak{g}_i} \to 0$ as $i \to \infty$, then $\lim_{i\to\infty} \mathfrak{g}_i = 0$.*

Definition 2.2. *[18] Let $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ be a real convergent sequence with limit $\mathfrak{g}$ and $\{\mathfrak{f}_i\}_{i=0}^{\infty}$ be another real convergent sequence with limit $\mathfrak{f}$. If $\lim_{i\to\infty} |\frac{\mathfrak{g}_i - \mathfrak{g}}{\mathfrak{f}_i - \mathfrak{f}}| = 0$, then $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ is said to converge faster than $\{\mathfrak{f}_i\}_{i=0}^{\infty}$.*

Definition 2.3. *[2] A sequence $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ in Banach space converges strongly to $\mathfrak{g}$ iff $\|\mathfrak{g}_i - \mathfrak{g}\| \to 0$ as $i \to \infty$.*

Definition 2.4. *[18] Let $\{\mathfrak{s}_i\}_{i=0}^{\infty}$ and $\{\mathfrak{t}_i\}_{i=0}^{\infty}$ be two non-negative number sequences converging to zero. If two iteration processes $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ and $\{\mathfrak{f}_i\}_{i=0}^{\infty}$ converge to same point p, then the errors estimates can be calculated* *as*

$$\|\mathfrak{g}_i - p\| \le \mathfrak{s}_i,$$

$$\|\mathfrak{f}_i - p\| \le \mathfrak{t}_i.$$

If $\{\mathfrak{s}_i\}_{i=0}^{\infty}$ *converges faster than* $\{\mathfrak{t}_i\}_{i=0}^{\infty}$, *then* $\{\mathfrak{g}_i\}_{i=0}^{\infty}$ *converges faster than* $\{\mathfrak{f}_i\}_{i=0}^{\infty}$.

3. Convergence Results

First, we give the definition of our newly proposed iteration process, namely, the BK iteration process. It generates the sequence $\{v_n\}$ for some initial point v_0 given as:

$$\begin{cases} v_0 \in B, \\ w_n = \frac{sv_n + \mathfrak{F}v_n}{s+1}, \ s \in \mathbb{R} \text{ and } s > 0 \\ l_n = \mathfrak{F}w_n, \\ q_n = \mathfrak{F}l_n, \\ v_{n+1} = \mathfrak{F}q_n. \end{cases} \tag{3.1}$$

Now, prove strong convergence theorem for our new proposed iteration process.

Theorem 3.1. *Let* $\mathfrak{F} : B \to B$ *be a mapping defined on a convex and closed subset* B *of a Banach space* E, *satisfying (1.5) with a fixed point* k. *Let* $\{v_n\}_{n=0}^{\infty}$ *be a sequence generated by the iteration process* (3.1). *Then* $\lim_{n\to\infty} v_n = k$.

Proof. We note that

$$\begin{aligned} \|w_n - k\| &= \left\| \frac{sv_n + \mathfrak{F}v_n}{s+1} - k \right\| = \left\| \frac{sv_n + \mathfrak{F}v_n - k(s+1)}{1+s} \right\| \\ &= \left\| \frac{s}{s+1}(v_n - k) + \frac{1}{s+1}(\mathfrak{F}v_n - k) \right\| \\ &\le \frac{s}{s+1}\|v_n - k\| + \frac{1}{1+s}\|\mathfrak{F}v_n - k\| \\ &= \frac{s}{1+s}\|v_n - k\| + \frac{1}{s+1}\|\mathfrak{F}v_n - \mathfrak{F}k\| \quad \because k = \mathfrak{F}k \\ &= \frac{s}{s+1}\|v_n - k\| + \frac{1}{1+s}\|\mathfrak{F}k - \mathfrak{F}v_n\| \\ &\le \frac{s}{1+s}\|v_n - k\| + \frac{1}{1+s}(\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - v_n\|) \\ &= \frac{s}{s+1}\|v_n - k\| + \frac{1}{s+1}(\zeta\|v_n - k\|) \\ &= \frac{s+\zeta}{s+1}\|v_n - k\|. \end{aligned} \tag{3.2}$$

Now,

$$\begin{aligned} \|l_n - k\| &= \|\mathfrak{F}w_n - k\| \\ &= \|\mathfrak{F}w_n - \mathfrak{F}k\| = \|\mathfrak{F}k - \mathfrak{F}w_n\| \\ &\le \rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - w_n\| \\ &= \zeta\|w_n - k\|. \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} \|q_n - k\| &= \|\mathfrak{F}l_n - k\| \\ &= \|\mathfrak{F}l_n - \mathfrak{F}k\| = \|\mathfrak{F}k - \mathfrak{F}l_n\| \\ &\le \rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - w_n\| \\ &= \zeta\|l_n - k\|. \end{aligned} \tag{3.4}$$

And,

$$\begin{aligned} \|v_{n+1} - k\| &= \|\mathfrak{F}q_n - k\| \\ &= \|\mathfrak{F}q_n - \mathfrak{F}k\| = \|\mathfrak{F}k - \mathfrak{F}q_n\| \\ &\le \rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - q_n\| \\ &= \zeta\|q_n - k\|. \end{aligned} \tag{3.5}$$

By using equation (3.4)

$$\|v_{n+1} - k\| \le \zeta^2\|l_n - k\|. \tag{3.6}$$

From equation (3.3)

$$\|v_{n+1} - k\| \le \zeta^3\|w_n - k\|. \tag{3.7}$$

Then by again using equation (3.2)

$$\|v_{n+1} - k\| \le \zeta^3\left(\frac{s+\zeta}{1+s}\right)\|v_n - k\|. \tag{3.8}$$

By repeating the above process again and again, we get

$$\begin{aligned} \|v_n - k\| &\le \zeta^3\left(\frac{s+\zeta}{s+1}\right)\|v_{n-1} - k\| \\ \|v_{n-1} - k\| &\le \zeta^3\left(\frac{s+\zeta}{s+1}\right)\|v_{n-2} - k\| \\ &\vdots \\ \|v_1 - k\| &\le \zeta^3\left(\frac{s+\zeta}{1+s}\right)\|v_0 - k\|. \end{aligned} \tag{3.9}$$

Finally we obtained

$$\|v_{n+1} - k\| \le \|v_0 - k\|\zeta^{3(n+1)}\prod_{i=0}^{n}\left(\frac{s+\zeta}{1+s}\right)^i.$$

By applying limit

$$\lim_{n\to\infty} \|v_{n+1} - k\| \le \|v_0 - k\| \lim_{n\to\infty} \zeta^{3(n+1)} \prod_{i=0}^{n} \left(\frac{s+\zeta}{1+s}\right)^i. \tag{3.10}$$

As $\zeta \in [0,1)$

$$\begin{aligned} &\Rightarrow s+\zeta < 1+s \\ &0 < 1 \Rightarrow \frac{s+\zeta}{s+1} < 1. \end{aligned} \tag{3.11}$$

Therefore

$$\left(\frac{s+\zeta}{1+s}\right)^{n+1} \to 0 \ as\ n \to \infty.$$

From equation (3.10), it implies that

$$\begin{aligned} &\lim_{n\to\infty} \|v_{n+1} - k\| \le 0 \\ \Rightarrow &\lim_{n\to\infty} \|v_{n+1} - k\| = 0. \end{aligned} \tag{3.12}$$

Hence $\{v_n\}$ converges strongly to k.

Uniqueness: Let $k \neq k^*$ be two fixed point of $\mathfrak{F}$, i.e., $\mathfrak{F}k = k$ and $\mathfrak{F}k^* = k^*$, then

$$\begin{aligned} \|k - k^*\| &= \|\mathfrak{F}k - \mathfrak{F}k^*\| \\ &\le \rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - k^*\| \\ &= \zeta\|k - k^*\|. \end{aligned}$$

Therefore

$$\|k - k^*\| < \|k - k^*\| \ \because \zeta < 1, \tag{3.13}$$

which is not possible, therefore $k = k^*$. □

Theorem 3.2. *Let B, E, $\mathfrak{F}$, and $\{v_n\}_{n=0}^{\infty}$ be defined as in Theorem (3.1). Then $\{v_n\}_{n=0}^{\infty}$, defined by (3.1), is $\mathfrak{F}$-stable.*

Proof. Consider an arbitrary sequence $\{\mathfrak{g}_n\}_{n=0}^{\infty}$ in B. Let the sequence generated by (3.1) be $v_{n+1} = f(\mathfrak{F}, v_n)$, which converges to some fixed point k. Define $\epsilon_n = \|k_{n+1} - f(\mathfrak{F}, \mathfrak{g}_n)\|$. For the $\mathfrak{F}$-stability proof, we will show that $\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} \mathfrak{g}_n = k$.
Now suppose that $\lim_{n\to\infty} \epsilon_n = 0$, then

$$\begin{aligned} \|\mathfrak{g}_{n+1} - k\| &= \|\mathfrak{g}_{n+1} - f(\mathfrak{F}, \mathfrak{g}_n) + f(\mathfrak{F}, \mathfrak{g}_n) - k\| \\ &\le \|\mathfrak{g}_{n+1} + f(\mathfrak{F}, \mathfrak{g}_n)\| - \|f(\mathfrak{F}, \mathfrak{g}_n) - k\| \\ &= \epsilon_n + \|f(\mathfrak{F}, \mathfrak{g}_n) - k\| \\ &= \epsilon_n + \left\|\mathfrak{F}\left(\mathfrak{F}\left(\mathfrak{F}\left(\frac{s\mathfrak{g}_n + \mathfrak{F}\mathfrak{g}_n}{1+s}\right)\right)\right) - k\right\| \\ &= \epsilon_n + \left\|k - \mathfrak{F}\left(\mathfrak{F}\left(\mathfrak{F}\left(\frac{s\mathfrak{g}_n + \mathfrak{F}\mathfrak{g}_n}{1+s}\right)\right)\right)\right\| \\ &= \epsilon_n + \left\|\mathfrak{F}k - \mathfrak{F}\left(\mathfrak{F}\left(\mathfrak{F}\left(\frac{s\mathfrak{g}_n + \mathfrak{F}\mathfrak{g}_n}{1+s}\right)\right)\right)\right\|. \end{aligned}$$

Using contractive condition (1.5), we have

$$\begin{aligned}\|\mathfrak{g}_{n+1}-k\| &\leq \epsilon_n+\zeta\left\|k-\mathfrak{F}\left(\mathfrak{F}\left(\frac{s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n}}{1+s}\right)\right)\right\| \\ &= \epsilon_n+\zeta\left\|\mathfrak{F}k-\mathfrak{F}\left(\mathfrak{F}\left(\frac{s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n}}{1+s}\right)\right)\right\|.\end{aligned}$$

Again by using (1.5), we get

$$\begin{aligned}\|\mathfrak{g}_{n+1}-k\| &\leq \epsilon_n+\zeta^2\left\|k-\mathfrak{F}\left(\frac{s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n}}{1+s}\right)\right\| \\ &= \epsilon_n+\zeta^2\left\|\mathfrak{F}k-\mathfrak{F}\left(\frac{s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n}}{1+s}\right)\right\|.\end{aligned}$$

From (1.5), we obtain

$$\begin{aligned}\|\mathfrak{g}_{n+1}-k\| &\leq \epsilon_n+\zeta^3\left\|k-\frac{s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n}}{1+s}\right\| \\ &= \epsilon_n+\zeta^3\left\|\frac{k(1+s)-(s\mathfrak{g}_\mathfrak{n}+\mathfrak{F}\mathfrak{g}_\mathfrak{n})}{1+s}\right\| \\ &\leq \epsilon_n+\zeta^3\left(\frac{s}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|+\frac{1}{1+s}\|\mathfrak{F}\mathfrak{g}_\mathfrak{n}-k\|\right) \\ &= \epsilon_n+\zeta^3\left(\frac{s}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|+\frac{1}{1+s}\|\mathfrak{F}\mathfrak{g}_\mathfrak{n}-\mathfrak{F}k\|\right) \ \because k=\mathfrak{F}k \\ &= \epsilon_n+\zeta^3\left(\frac{s}{s+1}\|\mathfrak{g}_\mathfrak{n}-k\|+\frac{1}{s+1}\|\mathfrak{F}k-\mathfrak{F}\mathfrak{g}_\mathfrak{n}\|\right) \\ &\leq \epsilon_n+\zeta^3\left(\frac{s}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|+\frac{1}{1+s}\{\rho\{\|k-\mathfrak{F}k\|\}+\zeta\|k-\mathfrak{g}_\mathfrak{n}\|\}\right) \\ &= \epsilon_n+\zeta^3\left(\frac{s}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|+\frac{1}{s+1}\{\zeta\|\mathfrak{g}_\mathfrak{n}-k\|\}\right).\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|\mathfrak{g}_{n+1}-k\| \leq \lim_{n\to\infty}\left(\epsilon_n+\zeta^3\left(\frac{s+\zeta}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|\right)\right)$$

$$\lim_{n\to\infty}\|\mathfrak{g}_{n+1}-k\| \leq \lim_{n\to\infty}\epsilon_n+\lim_{n\to\infty}\zeta^3\left(\frac{s+\zeta}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|\right).$$

As,

$$\because \frac{s+\zeta}{1+s}\|\mathfrak{g}_\mathfrak{n}-k\|<1 \textit{ and } \lim_{n\to\infty}\epsilon_n=0. \tag{3.14}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\|\mathfrak{g}_{n+1}-k\|\leq 0 \\ \Rightarrow &\lim_{n\to\infty}\|\mathfrak{g}_{n+1}-k\|=0 \\ &\lim_{n\to\infty}\mathfrak{g}_\mathfrak{n}=k.\end{aligned}$$

Conversely, suppose that $\lim_{n\to\infty} \mathfrak{g}_n = k$, then

$$\begin{aligned} \epsilon_n &= \|\mathfrak{g}_{n+1} - f(\mathfrak{F}, \mathfrak{g}_n)\| \\ &= \|\mathfrak{g}_{n+1} - k + k - f(\mathfrak{F}, \mathfrak{g}_n)\| \\ &\leq \|\mathfrak{g}_{n+1} - k\| + \|k - f(\mathfrak{F}, \mathfrak{g}_n)\| \\ &= \|\mathfrak{g}_{n+1} - k\| + \|\mathfrak{F}k - f(\mathfrak{F}, \mathfrak{g}_n)\|. \end{aligned} \tag{3.15}$$

Applying limit on both sides, we get

$$\begin{aligned} &\lim_{n\to\infty} \epsilon_n \leq \lim_{n\to\infty} \|\mathfrak{g}_{n+1} - k\| + \lim_{n\to\infty} \zeta^3 \frac{s+\zeta}{1+s} \|\mathfrak{g}_n - k\| \\ \because\ &\lim_{n\to\infty} \mathfrak{g}_n = k \\ &\lim_{n\to\infty} \epsilon_n = 0, \end{aligned}$$

which proves that the iterative process defined by (3.1) is $\mathfrak{F}$-stable. □

Theorem 3.3. *Let $\emptyset \neq B \subseteq E$ be convex and closed and $\mathfrak{F} : B \to B$ be a self-mapping satisfying* (1.5) *with a fixed point k. Let $\alpha_n \in (0,1)$ with $\alpha_0 \leq \alpha_n < 1$, $\forall n \in \mathbb{N}$ and $0 < s \in \mathbb{R}$ such that $\alpha_0(1+s) < 1$. Let $\{v_n\}_{n=0}^{\infty}$ be a sequence generated by the iteration process* (3.1), *and $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iteration process* (1.9), *then $\{v_n\}_{n=0}^{\infty}$ converges faster than $\{x_n\}_{n=0}^{\infty}$ to k.*

Proof. For $\{v_n\}_{n=0}^{\infty}$, using theorem (3.1), we get

$$\|v_{n+1} - k\| \leq \zeta^{3(n+1)} \left(\frac{s+\zeta}{1+s}\right)^{n+1} \|v_0 - k\|. \tag{3.16}$$

After performing similar calculation by using Theorem (3.1) as we did for $\{v_n\}_{n=0}^{\infty}$, we obtain the following relation for the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iteration process (1.9):

$$\|x_{n+1} - k\| \leq \|x_0 - k\| \zeta^{3(n+1)} \prod_{i=0}^{n} (1 - (1-\zeta)\alpha_i). \tag{3.17}$$

As $\alpha_0 \leq \alpha_i < 1$, so we can have

$$\begin{aligned} 1 - (1-\zeta)\alpha_0 &\geq 1 - (1-\zeta)\alpha_i \\ (1 - (1-\zeta)\alpha_0)^{n+1} &\geq \prod_{i=0}^{n} (1 - (1-\zeta)\alpha_i) \end{aligned} \tag{3.18}$$

Using (3.18) in equation (3.17), we have

$$\|x_{n+1} - k\| \leq \zeta^{3(n+1)} (1 - (1-\zeta)\alpha_0)^{n+1} \|x_0 - k\|.$$

Let suppose

$$\begin{aligned} a_n &= \zeta^{3(n+1)} \left(\frac{s+\zeta}{1+s}\right)^{n+1} \|v_0 - k\| \\ b_n &= \zeta^{3(n+1)} (1 - (1-\zeta)\alpha_0)^{n+1} \|x_0 - k\|. \end{aligned}$$

Also

$$
\begin{aligned}
\Upsilon &= \frac{a_n}{b_n} \\
&= \frac{\zeta^{3(n+1)}\left(\frac{s+\zeta}{1+s}\right)^{n+1}\|v_0 - k\|}{\zeta^{3(n+1)}(1-(1-\zeta)\alpha_0)^{n+1}\|x_0 - k\|} \\
\because v_o &= x_0 \\
&= \frac{(\frac{s+\zeta}{1+s})^{n+1}}{(1-(1-\zeta)\alpha_0)^{n+1}} \\
&= \left(\frac{\frac{s+\zeta}{1+s}}{1-(1-\zeta)\alpha_0}\right)^{n+1}.
\end{aligned}
$$

As $\alpha_0(1-\zeta) < 1$, so we have

$$
\begin{aligned}
1-\alpha_0(1-\zeta) &> \frac{s+\zeta}{1+s} \\
\Rightarrow \frac{\frac{s+\zeta}{1+s}}{1-\alpha_0(1+s)} &< 1.
\end{aligned}
$$

Therefore $\lim_{n\to\infty} \Upsilon = 0$.
From (2.2) and (2.4), we can conclude that $\{v_n\}_{n=0}^{\infty}$ converges faster than $\{x_n\}_{n=0}^{\infty}$. □

Theorem 3.4. *Let $\emptyset \neq B \subseteq E$ be convex and closed, and let $\mathfrak{F} : B \to B$ be a self-mapping satisfying* (1.5) *with a fixed point k. Let $\{v_n\}_{n=0}^{\infty}$ be a new proposed iterative sequence* (3.1) *and $\{u_n\}_{n=0}^{\infty}$ be a Mann iterative sequence* (1.7)*, where $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following statements are equivalent*
(i) The new iterative sequence converges to the fixed point k.
(ii) The Mann iteration sequence converges to the fixed point k.

Proof. First, suppose that our newly proposed iteration process (3.1) converges to the fixed point k i.e., $\lim_{n\to\infty} \|v_n - k\| = 0$.
Now,

$$
\begin{aligned}
\|v_{n+1} - u_{n+1}\| &= \|\mathfrak{F}q_n - (1-\alpha_n)u_n - \alpha_n \mathfrak{F}u_n\| \\
&= \|(1-\alpha_n+\alpha_n)\mathfrak{F}q_n - (1-\alpha_n)u_n - \alpha_n\mathfrak{F}u_n\| \\
&= \|(1-\alpha_n)\mathfrak{F}q_n + \alpha_n\mathfrak{F}q_n - (1-\alpha_n)u_n - \alpha_n\mathfrak{F}u_n\| \\
&\leq (1-\alpha_n)\|\mathfrak{F}q_n - u_n\| + (\alpha_n)\|\mathfrak{F}q_n - \mathfrak{F}u_n\| \\
&\leq (1-\alpha_n)\|\mathfrak{F}q_n - u_n + q_n - q_n\| + \alpha_n\{\rho\|\mathfrak{F}q_n - q_n\| + \zeta\|q_n - u_n\|\} \\
&\leq (1-\alpha_n)\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n)\|q_n - u_n\| + \alpha_n\{\rho\|\mathfrak{F}q_n - q_n\| + \zeta\|q_n - u_n\|\} \\
&= (1-\alpha_n)\|q_n - u_n\| + \alpha_n\zeta\|q_n - u_n\|\alpha_n\rho\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n)\|\mathfrak{F}q_n - q_n\| \\
&= (1-\alpha_n(1-\zeta))\|q_n - u_n\| + \alpha_n\rho\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n)\|\mathfrak{F}q_n - q_n\|.
\end{aligned}
$$

Also,

$$\begin{aligned} \|q_n - u_n\| &= \|\mathfrak{F}l_n - u_n\| \\ &\leq \|\mathfrak{F}l_n - l_n\| + \|l_n - u_n\|. \end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned} \|l_n - u_n\| &= \|\mathfrak{F}w_n - u_n\| \\ &\leq \|\mathfrak{F}w_n - w_n\| + \|w_n - u_n\|. \end{aligned} \tag{3.20}$$

$$\begin{aligned} \|w_n - u_n\| &= \left\|\frac{sv_n + \mathfrak{F}v_n}{1+s} - u_n\right\| \\ &= \left\|\frac{sv_n + \mathfrak{F}v_n - (1+s)u_n}{s+1}\right\| \\ &= \left\|\frac{s}{s+1}(v_n - u_n) + \frac{1}{s+1}(\mathfrak{F}v_n - u_n)\right\| \\ &\leq \frac{s}{s+1}\|v_n - u_n\| + \frac{1}{s+1}\|\mathfrak{F}v_n - u_n\| \\ &\leq \frac{s}{s+1}\|v_n - u_n\| + \frac{1}{s+1}\|\mathfrak{F}v_n - v_n\| + \frac{1}{s+1}\|v_n - u_n\| \\ &\leq \frac{1}{s+1}\|v_n - u_n\| + \frac{1}{s+1}\|\mathfrak{F}v_n - v_n\|. \end{aligned} \tag{3.21}$$

Using equation (3.21) in equation (3.20), we have

$$\|l_n - u_n\| \leq \|\mathfrak{F}w_n - l_n\| + \frac{1}{1+s}\|v_n - u_n\| + \frac{1}{1+s}\|\mathfrak{F}v_n - v_n\|.$$

Substituting equation (3) in equation (3.19), we can get

$$\|q_n - u_n\| \leq \|\mathfrak{F}l_n - l_n\| + \|\mathfrak{F}w_n - l_n\| + \frac{1}{1+s}\|v_n - u_n\| + \frac{1}{1+s}\|\mathfrak{F}v_n - v_n\|. \tag{3.22}$$

Using (3.22) in (3.19), we have

$$\begin{aligned} \|v_{n+1} - u_{n+1}\| \leq\ & \alpha_n\rho\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n)\|\mathfrak{F}q_n - q_n\| \\ & + (1-\alpha_n(1-\zeta))\|\mathfrak{F}l_n - l_n\| + (1-\alpha_n(1-\zeta))\|\mathfrak{F}w_n - l_n\| + \\ & (1-\alpha_n(1-\zeta))\frac{1}{s+1}\|v_n - u_n\| + \\ & \frac{1}{s+1}(1-\alpha_n(1-\zeta))\|\mathfrak{F}v_n - v_n\|. \end{aligned} \tag{3.23}$$

Let

$$\begin{aligned} &\mu = (1-\alpha_n(1-\zeta)) \in (0,1), \\ &\mathfrak{a}_n = \|v_n - u_n\|, \\ &\mathfrak{b}_n = \alpha_n\rho\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n)\|\mathfrak{F}q_n - q_n\| + (1-\alpha_n(1-\zeta))\|\mathfrak{F}l_n - l_n\| \\ &\quad + (1-\alpha_n(1-\zeta))\|\mathfrak{F}w_n - l_n\| + \frac{1}{1+s}(1-\alpha_n(1-\zeta))\|\mathfrak{F}v_n - v_n\|. \end{aligned}$$

Now

$$\begin{aligned}
\|\mathfrak{F}v_n - v_n\| &= \|\mathfrak{F}v_n - v_n - k + k\| \\
&\leq \|\mathfrak{F}v_n - k\| + \|k - v_n\| \\
&= \|\mathfrak{F}k - \mathfrak{F}v_n\| + \|k - v_n\| \\
&\leq \rho\{\{\|k - \mathfrak{F}k\|\}\} + \zeta\|k - v_n\|\} + \|k - v_n\| \\
&= (1+\zeta)\|v_n - k\| \to 0 \, as \ n \to \infty.
\end{aligned}$$

Also,

$$\begin{aligned}
\|\mathfrak{F}l_n - l_n\| &\leq \|\mathfrak{F}l_n - k\| + \|k - l_n\| \\
&\leq \|\mathfrak{F}l_n - k\| + \|k - l_n\| \\
&\leq \|\mathfrak{F}k - \mathfrak{F}l_n\| + \|k - l_n\| \\
&\leq \{\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - l_n\|\} + \|k - l_n\| \\
&= (1+\zeta)\|l_n - k\| \\
&= (1+\zeta)\|\mathfrak{F}w_n - \mathfrak{F}k\| \\
&\leq (1+\zeta)\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|w_n - k\| \\
&= (1+\zeta)\zeta\|w_n - k\|.
\end{aligned}$$

$$\begin{aligned}
\|\mathfrak{F}q_n - q_n\| &= \|\mathfrak{F}q_n - k\| + \|k - q_n\| \\
&\leq \|\mathfrak{F}q_k\| + \|k - l_n\| \\
&\leq \|\mathfrak{F}k - \mathfrak{F}q_n\| + \|k - q_n\| \\
&\leq \{\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - q_n\|\} + \|k - q_n\| \\
&= (1+\zeta)\|q_n - k\| \\
&= (1+\zeta)\|\mathfrak{F}l_n - \mathfrak{F}k\| \\
&\leq (1+\zeta)\{\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - l_n\|\} \\
&= (1+\zeta)\zeta\|l_n - k\| \\
&= (1+\zeta)\|\mathfrak{F}w_n - \mathfrak{F}k\| \\
&\leq (1+\zeta)\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|w_n - k\| \\
&= (1+\zeta)\zeta\|w_n - k\|.
\end{aligned}$$

We also have

$$\begin{aligned}
\|w_n - k\| &= \left\| \frac{sv_n + \mathfrak{F}v_n}{s+1} - k \right\| \\
&\leq \frac{s}{s+1}\|v_n - k\| + \frac{1}{s+1}\|\mathfrak{F}v_n - k\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s}{s+1}\|v_n - k\| + \frac{1}{s+1}\|\mathfrak{F}v_n - \mathfrak{F}k\| \\
&= \frac{s}{s+1}\|v_n - k\| + \frac{1}{s+1}\|\mathfrak{F}k - \mathfrak{F}v_n\| \\
&\leq \frac{s}{s+1}\|v_n - k\| + \frac{1}{s+1}\{\rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - v_n\|\} \\
&= \frac{\zeta + s}{s+1}\|v_n - k\| \to 0\, as\ n \to \infty.
\end{aligned} \tag{3.24}
$$

Using (3.24), we can observe that $\|\mathfrak{F}q_n - q_n\| \Rightarrow 0$ and $\|\mathfrak{F}l_n - l_n\| \Rightarrow 0$ as $n \to \infty$. Similarly

$$\lim_{n\to\infty} \alpha_n\, \rho\{\|\mathfrak{F}q_n - q_n\|\} = \alpha_n\, \rho\{\lim_{n\to\infty} \|\mathfrak{F}q_n - q_n\|\} = 0.$$

Hence, we obtain $\mathfrak{b}_\mathfrak{n} \to 0$. Now, by using Lemma (2.3), we get

$$\|v_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Therefore, we get

$$\|u_n - k\| \leq \|u_n - v_n\| + \|v_n - k\| \to 0\, as\, n \to \infty,$$

which shows convergence of Mann iteration (1.7) to a fixed point k of $\mathfrak{F}$.

Now we will prove that convergence of the Mann iteration (1.7) to a fixed point k of $\mathfrak{F}$ implies the convergence of the newly proposed iteration process (3.1) to the fixed point k of $\mathfrak{F}$.
Let the Mann iteration (1.7) converges to a fixed point k i.e., $\lim_{n\to\infty} u_n = k$ as $n \to \infty$.

$$
\begin{aligned}
\|u_{n+1} - v_{n+1}\| &= \|(1-\alpha_n)u_n + \alpha_n\mathfrak{F}u_n - \mathfrak{F}q_n\| \\
&= \|(1-\alpha_n)u_n - \alpha_n\mathfrak{F}u_n - (1 - \alpha_n + \alpha_n)\mathfrak{F}q_n\| \\
&\leq (1-\alpha_n)\|u_n - \mathfrak{F}q_n\| + \alpha_n\|\mathfrak{F}u_n - \mathfrak{F}q_n\| \\
&\leq (1-\alpha_n)\|u_n - \mathfrak{F}u_n\| + (1-\alpha_n)\|\mathfrak{F}u_n - \mathfrak{F}q_n\| + \alpha_n\|\mathfrak{F}u_n - \mathfrak{F}q_n\| \\
&= (1-\alpha_n)\|u_n - \mathfrak{F}u_n\| + \|\mathfrak{F}u_n - \mathfrak{F}q_n\| \\
&\leq (1-\alpha_n)\|u_n - \mathfrak{F}u_n\| + \zeta\|u_n - q_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\}.
\end{aligned} \tag{3.26}
$$

And,

$$
\begin{aligned}
\|u_n - q_n\| &= \|u_n - \mathfrak{F}l_n\| \\
&= \|u_n - \mathfrak{F}u_n\| + |\mathfrak{F}u_n - \mathfrak{F}l_n\| \\
&\leq \|u_n - \mathfrak{F}u_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\} + \zeta\|u_n - l_n\|.
\end{aligned} \tag{3.27}
$$

Also,

$$
\begin{aligned}
\|u_n - l_n\| &= \|u_n - \mathfrak{F}w_n\| \\
&= \|u_n - \mathfrak{F}u_n\| + |\mathfrak{F}u_n - \mathfrak{F}w_n\| \\
&\leq \|u_n - \mathfrak{F}u_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\} + \zeta\|u_n - w_n\|.
\end{aligned} \tag{3.28}
$$

$$\begin{aligned}
\|u_n - w_n\| &= \left\|u_n - \frac{sv_n + \mathfrak{F}v_n}{s+1}\right\| \\
&= \left\|\frac{s}{s+1}(u_n - v_n) + \frac{1}{s+1}(u_n - \mathfrak{F}v_n)\right\| \\
&\leq \frac{s}{s+1}\|u_n - v_n\| + \frac{1}{s+1}\|u_n - \mathfrak{F}u_n\| + \|\mathfrak{F}u_n - \mathfrak{F}v_n\| \\
&\leq \frac{s}{s+1}\|u_n - v_n\| + \frac{1}{s+1}\|u_n - \mathfrak{F}u_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\} + \zeta\|u_n - v_n\| \\
&\leq \frac{\zeta + s}{s+1}\|u_n - v_n\| + \frac{1}{s+1}\|u_n - \mathfrak{F}u_n\| + \frac{1}{s+1}\rho\{\|u_n - \mathfrak{F}u_n\|\}.
\end{aligned} \tag{3.29}$$

Using equation (3.29) in equation (3.28), we have

$$\begin{aligned}
\|u_n - l_n\| \leq{}& \|u_n - \mathfrak{F}u_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\} + \\
& \zeta\{\frac{s+\zeta}{s+1}\|u_n - v_n\| + \frac{1}{s+1}\|u_n - \mathfrak{F}u_n\| + \frac{1}{s+1}\rho\{\|u_n - \mathfrak{F}u_n\|\}\} \\
={}& \zeta(\frac{s+\zeta}{s+1})\|u_n - v_n\| + (1 + \frac{\zeta}{s+1})\|u_n - \mathfrak{F}u_n\| + \\
& (1 + \frac{\zeta}{s+1})\rho\{\|u_n - \mathfrak{F}u_n\|\}.
\end{aligned} \tag{3.30}$$

Replacing equation (3.30) in equation (3.27), we get

$$\begin{aligned}
\|u_n - q_n\| \leq{}& \|u_n - \mathfrak{F}u_n\| + \rho\{\|u_n - \mathfrak{F}u_n\|\} + \\
& \zeta^2(\frac{s+\zeta}{s+1})\|u_n - v_n\| + \zeta(1 + \frac{\zeta}{s+1})\|u_n - \mathfrak{F}u_n\| + \\
& \zeta(1 + \frac{\zeta}{s+1})\rho\{\|u_n - \mathfrak{F}u_n\|\}.
\end{aligned} \tag{3.31}$$

By using equation (3.31) in equation(3.26), we have

$$\begin{aligned}
\|u_{n+1} - v_{n+1}\| \leq{}& (1-\alpha_n)\|u_n - \mathfrak{F}u_n\| + \zeta\|u_n - \mathfrak{F}u_n\| + \zeta\rho\{\|u_n - \mathfrak{F}u_n\|\} + \\
& \zeta^3(\frac{s+\zeta}{s+1})\|u_n - v_n\| + \zeta^2(1 + \frac{\zeta}{s+1})\|u_n - \mathfrak{F}u_n\| + \\
& \zeta^2(1 + \frac{\zeta}{s+1})\rho\{\|u_n - \mathfrak{F}u_n\|\} + \rho\{\|u_n - \mathfrak{F}u_n\|\} \\
={}& \zeta^3(\frac{s+\zeta}{s+1})\|u_n - v_n\| + \{(1-\alpha_n) + \zeta + \zeta^2(1 + \frac{\zeta}{s+1})\}\|u_n - \mathfrak{F}u_n\| + \\
& \{1 + \zeta + \zeta^2(1 + \frac{\zeta}{s+1})\}\rho\{\|u_n - \mathfrak{F}u_n\|\} \\
\leq{}& (\frac{s+\zeta}{s+1})\|u_n - v_n\| + \{(1-\alpha_n) + \zeta + \zeta^2(1 + \frac{\zeta}{s+1})\}\|u_n - \mathfrak{F}u_n\| + \\
& \{1 + \zeta + \zeta^2(1 + \frac{\zeta}{s+1})\}\rho\{\|u_n - \mathfrak{F}u_n\|\}.
\end{aligned} \tag{3.32}$$

Let

$$\mu = \frac{s+\zeta}{s+1} \in (0,1),$$

$$\mathfrak{a}_\mathfrak{n} = \|u_n - v_n\|,$$

$$\mathfrak{b}_\mathfrak{n} = \{(1-\alpha_n) + \zeta + \zeta^2(1+\frac{\zeta}{s+1})\}\|u_n - \mathfrak{F}u_n\| + \{1+\zeta+\zeta^2(1+\frac{\zeta}{s+1})\}\rho\{\|u_n - \mathfrak{F}u_n\|\}.$$

Now,

$$\begin{aligned}\|u_n - \mathfrak{F}u_n\| &\leq \|u_n - k\| + \|k - \mathfrak{F}u_n\| \\ &= \|u_n - k\| + \|\mathfrak{F}k - \mathfrak{F}u_n\| \\ &= \|u_n - k\| + \rho\{\|k - \mathfrak{F}k\|\} + \zeta\|k - u_n\| \\ &= (1+\zeta)\|u_n - k\| \to 0 \, as \, n \to \infty.\end{aligned}$$

Similarly,

$$\lim_{n\to\infty} \alpha_n \rho\{\|u_n - \mathfrak{F}u_n\|\} = \alpha_n \rho\{\lim_{n\to\infty} \|u_n - \mathfrak{F}u_n\|\} = 0. \tag{3.33}$$

which implies $\mathfrak{b}_\mathfrak{n} \to 0$. By using Lemma (2.3), we get

$$\|u_n - v_n\| \to 0 \, as \, n \to \infty.$$

Therefore, we get

$$\|v_n - k\| \leq \|v_n - v_n\| + \|v_n - k\| \to 0 \, as \, n \to \infty.$$

Which shows that the newly proposed iteration process (3.1) converges to a fixed point k of $\mathfrak{F}$. □

4. Numerical Example

Example 4.1. *Define a self mapping $\mathfrak{F}$ on $B = [2,4]$ as follows*

$$\mathfrak{F}v = \begin{cases} \frac{v+6}{4}, & if\, v \in [2,4) \\ 1, & if\, v = 4. \end{cases} \tag{4.1}$$

Clearly, $\mathfrak{F}$ satisfies contractive condition (1.5) with the fixed point 2. We now create a table and graph in comparing the BK, F and SH iteration processes to our newly proposed BK iteration process (3.1), which converges more rapidly to the fixed point 2 of $\mathfrak{F}$ presented in the preceding example. We have selected $\alpha_n = 0.70$, along with the stopping criteria $\|v_n - v_{n+1}\| < 10^{-14}$. The results obtained are shown in Fig. 1 and Tab. 1.

Table 1. Some iterates for Example 4.1 generated by BK (3.1) ($s = 0.01$), F (1.9), SH (1.8) ($s = 0.01$) iteration processes for initial point $v_0 = 3$.

n	**BK Iteration**	**F Iteration**	**SH Iteration**
0	3.0	3.0	3.0
1	2.00402227722772	2.00742187500000	2.01608910891089
2	2.00001617871410	2.00005508422852	2.00025885942555
3	2.00000006507527	2.00000040882826	2.00000416481749
4	2.00000000026175	2.00000000303427	2.00000006700820
5	2.00000000000105	2.00000000002252	2.00000000107810
6	**2**	2.00000000000017	2.00000000001735
7	2	**2**	2.00000000000028
8	2	2	**2**

The acquired findings show that, in comparison to the first iteration of the other iteration processes, the value calculated using the BK (2.00402227722772) is closer to the fixed point, or 2, following the first iteration than the F (2.00742187500000) and the SH (2.01608910891089). We see that in the following iterations, each iteration method approaches the fixed point at a different rate. The newly proposed, BK iteration, which discovered the fixed point in 6 iterations, is the quickest approach. It took 7 iterations for the F and 8 iterations for the SH iteration process to locate the fixed point. This comparison can also be seen in the graphical representation of all three iterations converging to the fixed point 2 of mapping $\mathfrak{F}$ in Fig. (1).

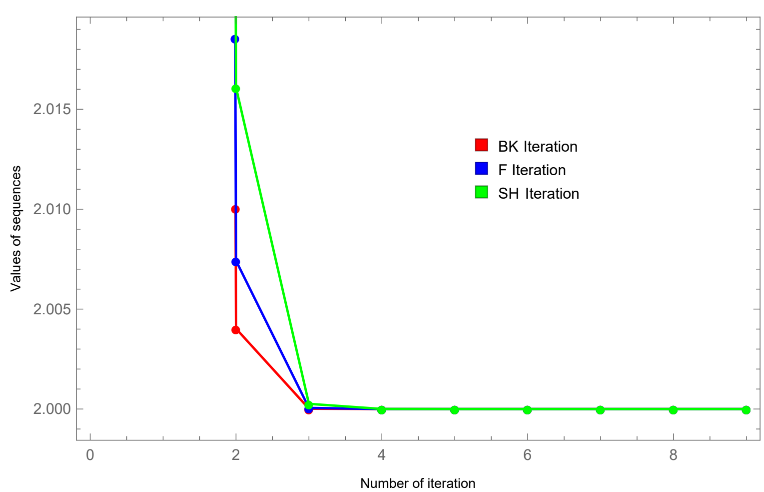


Figure 1. Graphical comparison of BK (3.1), F (1.9) and SH (1.8) iteration processes.

5. Comparison via Polynomiography

Mathematician and computer scientist Bahman Kalantari elaborated polynomiography, which is a digital art form and a visual analytic method for root-finding [19,20]. This technique visualizes complex polynomials utilizing mathematical principles and iterative approximation procedures. The methods of polynomiography are widely used for comparing and analyzing various types of iteration processes [21, 22]. Polynomiography generates graphical images by analyzing the convergence of the iteration process used to approximate polynomial roots. The Newton's iteration method is a well known root-finding method and it is also known as the Newton-Raphson method. For some polynomial $q(x_n)$, it is define as

$$x_{n+1} = x_n - \frac{q(x_n)}{q'(x_n)} \text{ for } (n = 0, 1, 2, 3...).$$

Here, $q'(x_n)$ stands for the first derivative of $q(x_n)$. Now, Newton's iterative process can be expressed in the form of a fixed point iterative process as follows:

$$x_{n+1} = \mathfrak{F}(x_n).$$

If the above iterative converges to any fixed point, namely, x of $\mathfrak{F}$, then one has

$$x = \mathfrak{F}(x) = x - \frac{q(x)}{q'(x)}. \tag{5.1}$$

If $\frac{q(x)}{q'(x)} = 0$ then $q(x) = 0$. Equation (5.1) implies that $x = \mathfrak{F}(x)$ which means x is a root is of $\zeta(x)$. The set of all x_0 that converges to the same root forms a basin of attraction. Now, instead of the Picard iteration, we can use other iteration processes, e.g., the introduced BK iteration or other iteration processes defined in Section 1 for different values of α_n. We choose grid lengths $B = [-5.0, 5.0]^2$ and $K = 30$, where K indicates the number of iterations. Using Newton's operator into BK, SH and F-iteration processes, we obtain a complex sequence, namely, $\{x_n\}$ that starts at every grid point x_0. Suppose that x_0 is a starting guess, then if the sequence of iterations $\{x_n\}$ essentially converges to any root with accuracy of 0.001, then we assign a color to x_0, and if $\{x_n\}$ does not converge to any root, then we assign a red color to $\{x_n\}$. The set of all x_0 that converges to the same root, forms a basin of attraction. We use the color map presented in Fig. 2.

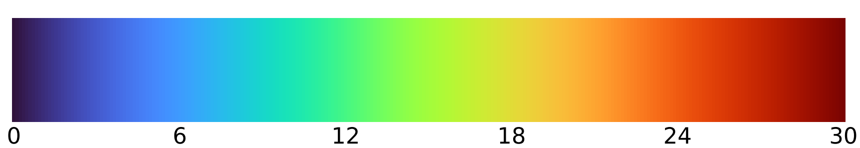


Figure 2. Color map used in the examples.

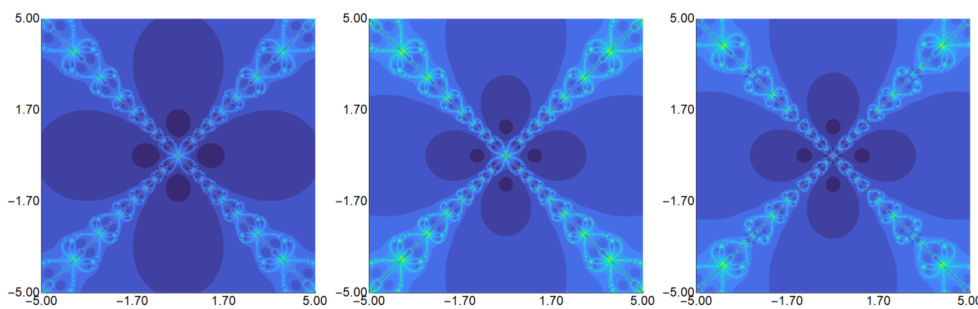
Now, we apply the algorithm given as a pseudocode in Algorithm 1 to produce a polynomiograph. We color the points in the algorithm using a technique known as "iteration coloring" [23]. In this kind of coloring, the color assigned to each starting point is determined by the number of iterations completed. As a result, this kind of polynomiograph displays the iteration process's speed of convergence, which is determined by the number of iterations completed. Additionally, we can compute an average number of iterations (ANI) [24] using the polynomiograph produced by Algorithm 1.

Algorithm 1: Generation of a polynomiograpgh.

Input: $q \in \mathbb{C}[Z]$, $\deg q \geq 2$ – polynomial; $\gimel$ – iteration process; $B \subset \mathbb{C}$ – area; K – the maximum number of iterations; ε – accuracy; *colors* – color map.

Output: Polynomiograph for the complex-valued polynomial q within the area B.

```
for x_0 ∈ B do
    n = 0
    while |q(x_n)| > ε and n < K do
        x_{n+1} = ℷ(x_n, q)
        n = n + 1
    Map n to a color from the color map colors and color x_0
```

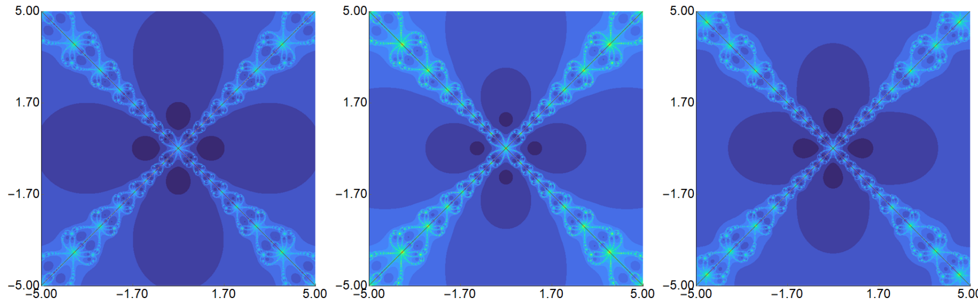


(a) BK iteration (b) SH iteration (c) F iteration

FIGURE 3. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.05$.

For our numerical experiment, we consider a polynomial $q(x) = x^4 - 1$ and proposed three settings of a parameter α_n i.e., $\alpha_n = 0.05$, $\alpha_n = 0.5$ and $\alpha_n = 0.8$. The obtained graphs for the BK, SH and F-iteration processes using these parameter settings are shown in the Figs. 3–5. The measured values of average number of iterations (ANI) are shown in the Tab. 2. We can notice from the Tab. 2 that obtained ANI values are very close to each other, so all graphics look like similar. We can only make a difference on the basis of color tone.

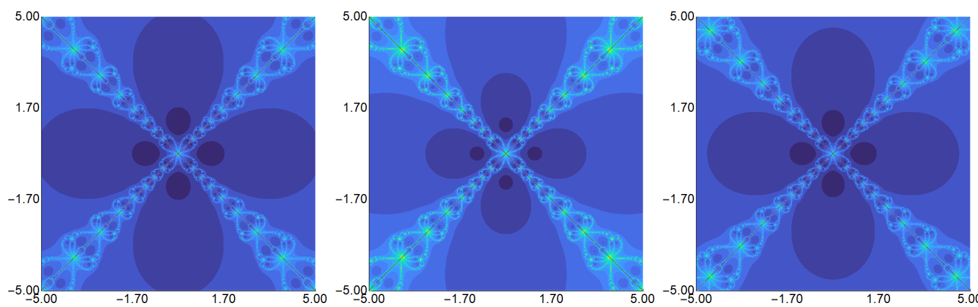
We can notice that a blue color for $\alpha_n = 0.05$ (Fig. 3). Visual examination reveals that the proposed BK iteration achieves the fastest speed of convergence (2.93469), followed by the F iteration (3.60703) and the SH (3.70724).



(a) BK iteration (b) SH iteration (c) F iteration

Figure 4. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.5$.

We can notice from the polynomiographs for $\alpha_n = 0.5$ shown in Fig. 4 a slight dark color as compared to the graphs for $\alpha_n = 0.05$. For this setting of the parameter, the BK iteration is again the quickest of the iterations, with a value (2.77184) followed by the F iteration and SH iteration.



(a) BK iteration (b) SH iteration (c) F iteration

Figure 5. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.8$.

We use a high value of α_n for our third parameter setting. This high parameter value produces a dark blue color. This demonstrates that all iterations require fewer iterations to reach the roots of the polynomial. These facts are indicated by the recorded values provided in the Tab. 2. We notice that for the third parameter setting, we obtained the lowest ANI value for our proposed BK iteration (2.68107) and the highest value for the SH iteration (3.28732). It is evident that for high parameter values, the BK iteration yields the lowest ANI value, which is 2.68107. High parameter values likewise yield the lowest values for the subsequent iterations.

Table 2. ANI values calculated from polynomiographs presented in Figures 3, 4 and 5.

Iteration	$\alpha_n = 0.05$	$\alpha_n = 0.5$	$\alpha_n = 0.8$
BK	2.93469	2.77184	2.68107
SH	3.70724	3.34814	3.28732
F	3.60703	2.98872	2.73034

6. Conclusions

We have successfully analyzed our newly proposed variant of the F iteration process, namely the BK iteration process. We proved weak and strong convergence results for the newly proposed iteration process. The effectiveness of the proposed iteration processes is demonstrated through a numerical example. We also provided some polynomiographs generated by this new iteration process to support our results. From our proved results, it is obvious that our newly proposed iteration process shows better convergence than the other two iteration process under the discussion. Furthermore, the BK iteration process exhibits robustness and efficiency in reaching the fixed point with fewer iterations, highlighting its potential for practical applications where computational efficiency is critical. The comparative analysis with other well-known iterative methods underscores its superior performance in terms of convergence speed and stability

Data Availability: All data supporting the findings of this study are available within the paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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